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CHAPTER

1

NUMBER SYSTEMS : N, Z, Q, R, AND C— AN OUTLINE

The *natural numbers* 1, 2, 3, 4, ... *n*, ... have been with each one of us since childhood. Almost all the important properties of this number set, which we shall call **N**, have been accepted by us intuitively from experience. These properties may be listed now as follows. We add a few comments where necessary.

1. The set **N** is an endless set. That is, there is no last number. The sequence of natural numbers goes on and on.
2. There is a built-in order in the set in the way we write it:

   $$1, 2, 3, 4, ..., a, ..., b, ...$$

   If b appears later in the sequence than a then b is said to be greater than a. We write this: $b > a$; or, what is the same thing, $a < b$, *i.e.*, a is less than b.
3. Every number has a successor number and, except for 1, every number has a predecessor number.
4. Any two numbers in the set can be 'added' to produce another number in the set. Recall that after one learns to count, the next thing that is learnt is to 'add'.
5. Whether one adds a to b or b to a it is the same thing—in the sense the result is the same. In other words, addition '+' is a commutative process; *i.e.*,

   $$a + b = b + a \qquad \text{for all } a, b \in \mathbf{N} \qquad (1)$$
6. Repeated addition of the same number to itself is known as 'multiplication'. Thus, for instance, 4 added to itself 5 times is nothing but 4 × 5, that is, 20.
7. This multiplication is also commutative. That is,

   $$a \times b = b \times a \qquad \text{for all } a, b \in \mathbf{N} \qquad (2)$$
8. Both the operations, addition and multiplication, have another property, called 'associativity'. This means : $a + b$ added to c and a added to $b + c$ are both the same. Symbolically,

   $$(a + b) + c = a + (b + c) \qquad \text{for all } a, b, c \in \mathbf{N} \qquad (3)$$

   In the same way, we have, for multiplication,

   $$(a \times b) \times c = a \times (b \times c) \qquad \text{for all } a, b, c \in \mathbf{N} \qquad (4)$$
9. Further, there is a 'compatibility' between the two processes 'addition' and 'multiplication'; namely,

   $$a \times (b + c) = (a \times b) + (a \times c)$$

   and $$(a + b) \times c = (a \times c) + (b \times c) \qquad \text{for all } a, b, c \in \mathbf{N} \qquad (5)$$

This property is called 'distributivity' of multiplication with respect to addition.

These nine properties of the set **N** shall now be assumed without any further justification. Higher mathematics may require the construction of natural numbers from scratch and the derivation of these properties thereof. We do not have either the luxury of time or the necessity of logic to get into all that now, at this level.

One of the first things that we learn as we grow learning mathematics is that the system **N** of natural numbers has several deficiencies. For instance, we can solve for x, the equation: $2+x=3$ within the system **N**. The answer is $x=1$. Whereas, an equation like: $3+x=2$ is not solvable in **N**. In other words, there is no value of x in **N** satisfying $3+x=2$. We know the answer is -1 but -1 is not a natural number. Thus the system **N** of natural numbers does not have solutions of the equation $a+x=b$ *i.e.*, this equation has no solution for x in **N** unless $a<b$. Mathematics develops by concerning itself with such questions and resolving the issue. In the above situation, the resolution comes like this. Mathematics invents new numbers, namely, $0, -1, -2, -3, \ldots$ expressly to satisfy the need to solve the equations $a+x=b$ even when $a \geq b$. For instance, if $a=b$, the equation is $a+x=a$. We invent the new number "0" (= zero) to be the solution of

$$a+x=a=x+a.$$

Once we include a new number "0" to the system **N** we want also to solve equations like

$$1+x=0;\ 2+x=0;\ 3+x=0;\ \ldots$$

The solutions of these are called the negatives of 1, 2, 3, ... and are written

$$-1, -2, -3, \ldots.$$

Thus the enlarged system now contains zero and all the negative integers and **N**. This new system is denoted by **Z** and is called *the set of all integers*. Thus

$$\mathbf{Z}=\{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

It can also be written as below, where we bring out the 'order' relation in **Z**. In other words, in the following style of listing the elements of **Z**, if a precedes b then $a<b$, or what is the same thing, $b>a$.

$$\mathbf{Z}=\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

There are several points we have to note about this enlargement of **N** to **Z**. In enlarging **N** to **Z** we have been able to 'protect' or 'preserve' as many properties of **N** as possible. Precisely we mean the following:

1. **Z** is an infinite (= endless) sequence as **N** was (and is !).
2. The built-in order in **N** is still preserved. It has in fact been extended to **Z**. In other words '$a>b$' has a meaning in **Z** for every a and b in **Z** and further, if $a>b$ in **N** for two elements $a, b \in$ **N**, it is so in **Z**, even as elements of **Z**.
3. Every number in **Z** has a successor and a predecessor. Recall that in **N** the number 1 does not have a predecessor. Also any number in **N** whether considered as a member in **N** or a member in **Z** has the same successor. Similarly, any number $\neq 1$ in **N** has the same predecessor in **N** or **Z**. We express this by saying that the 'successor-predecessor' concept has been extended to **Z** without damage to the concept already existing in **N**.

4. The operation of addition already available in **N** carries over to **Z**. If $x = -a$ where $a \in \mathbf{N}$, $y = -b$ where $b \in \mathbf{N}$, we may define $x + y$ $= -(a + b)$ where + in the R.H.S. is the addition in **N**. Since $(a + b) \in \mathbf{N}$, $-(a + b) \in \mathbf{Z}$. Thus we get the familiar equality.

   $$(-a) + (-b) = -(a + b)$$

   Again, if $x = -a$, $a \in \mathbf{N}$, is 'added' to $c \in \mathbf{N}$ we will have $x + c = (-a) + c$. This is to be taken as

   $$-(a - c) \text{ if } a > c$$

   and as $$c - a \text{ if } c > a \text{ or } c = a.$$

   Proceeding in this way and carefully going through every new situation we get a thorough definition of addition in **Z**. We see that 'addition' is closed in **Z** — by which, we mean, two numbers in **Z** always lead to a number in **Z** by the addition process. If two numbers are already in **N** their sum is what it is in the system **N**. Thus the extension of **N** and the addition therein to **Z** has been achieved without 'damaging' the addition in **N**. This process of enlarging a number system, preserving its algebraic structure is called an *extension* of the system. Addition of zero to any number, again satisfies,

   $$a + 0 = a = 0 + a \qquad \text{for all } a \in \mathbf{Z}.$$

5. Addition in **Z** continues to be commutative. In other words,

   $$a + b = b + a \qquad \text{for all } a, b \in Z \qquad (1')$$

6. Multiplication in **N** can be extended to a multiplication in **Z**, without damaging the meaning of multiplication in **N** — except that, we have to make proper rules for handling the negative sign, thus: If $a, b \in \mathbf{N}$, then

   $$a \times b = ab \text{ (as in N)}$$
   $$(-a) \times (-b) = ab$$
   $$(-a) \times (b) = -(ab)$$
   $$a \times (-b) = -(ab).$$

   Multiplication by zero however has to be controlled by a new rule, *viz.*,

   $$a \times 0 = 0 = 0 \times a \qquad \text{for all } a \in \mathbf{Z}.$$

7. Multiplication in **Z** is commutative. In other words,

   $$a \times b = b \times a \text{ for all } a, b \in \mathbf{Z}. \qquad (2')$$

8. The associative properties of both addition and multiplication continue to be valid in **Z**. In other words

   $$a + (b + c) = (a + b) + c \qquad \text{for all } a, b, c \in \mathbf{Z} \qquad (3')$$

   and $$a \times (b \times c) = (a \times b) \times c \qquad \text{for all } a, b, c \in \mathbf{Z}. \qquad (4')$$

9. The distributive property

   $$a(b + c) = ab + ac$$
   $$(a + b)c = ac + bc \qquad \text{for all } a, b, c \in \mathbf{Z} \qquad (5')$$

   holds, as it holds in **N**.

10. Finally, we record, at one place, the special roles of the numbers 0 and 1 in **Z** as follows:

    (*i*) In **Z**, 0 is the unique number which has the property:

    $$0 + a = a = a + 0 \qquad \text{for all } a \in \mathbf{Z} \qquad (6')$$

(*ii*) In **Z**, 1 is the unique number which has the property:

$$1 \times a = a = a \times 1 \qquad \text{for all } a \in \mathbf{Z} \tag{7'}$$

Note that this property (*ii*) of 1 is already present in **N** for all $a \in \mathbf{N}$ and it is now valid for all $a \in \mathbf{Z}$, as well.

While the number 'zero' plays a unique role as far as addition is concerned in **Z**, the number 'one' plays an exactly analogous role with respect to multiplication in **Z**. We call '0' the additive identity in **Z** and call '1' the multiplicative identity in **Z**.

If we now compare the two systems **N** and **Z** we find that **Z** is a meaningful extension of **N**. The extension protects the properties already existing in **N** as listed above. Further, **Z** has the extra property of solvability of equations of the form

$$a + x = b,\ a, b \in \mathbf{Z}. \tag{8'}$$

A beauty of the extension from **N** to **Z** is the following. We invented new numbers to solve $a + x = b$ with a, b, in **N**; but in the enlarged set **Z** we are able to solve $a + x = b$ for any two a, b in **Z** !

But there is one property, viz., the following, which is true in **N** but is not true in **Z**:

If $a > b$, for any three $a, b, x \in \mathbf{N}$, then $xa > xb$ (9′)

In other words, so long as we are in **N**, multiplication by the same number of both sides of an inequality preserves the same inequality. But in **Z** this property will fail for multiplication by a negative quantity. For instance, if $a < b$ in **Z** then $-a > -b$. Multiplication by a negative integer reverses the inequality.

Thus in carrying out the extension of the number system from **N** to the larger system **Z**, we could preserve many properties, we were able to solve extra equations but we had to lose something, as if we had to pay a price for the extension !

We are now going to carry this extension process through three more stages. Each time we will have the general situation (not unlike the above extension from **N** to **Z**) between the smaller and the larger systems.

(1) We preserve most of the properties of the smaller system;

(2) We achieve something extra in the larger system — something which was not available in the smaller system; and

(3) We 'pay a price' for this extension by losing some property which was present in the smaller system.

In this series of extensions there is an enormous amount of detail to be taken care of for the purpose of mathematical rigour and completion of the argument. We shall not go through all that detail. They will be duly taken up at the level of university education. Here we shall only indicate the lines of these extensions in a broad manner. The extension that we have just completed namely,

from **N** to **Z**

may be called *the first stage*. The *second stage* of the extensions is

from **Z** to **Q**

where **Q** is the set of all rational numbers. A *rational number* is a number of the form

$$\frac{p}{q},\ p, q \in \mathbf{Z}, \text{ with } q \neq 0. \tag{10}$$

Most often, for printing convenience, we write $\frac{p}{q}$ as p/q.

This extension is needed for solving equations of the form

$$ax = b,\ a, b \in \mathbf{Z}. \tag{11}$$

For instance, we cannot solve in **Z** the equation

$$2x = 1 \text{ or } 3x = -2. \tag{*}$$

We know from our knowledge of lower class arithmetic that $x = 1/2$ is the solution of $2x = 1$ and $x = (-2)/3$ is the solution of $3x = -2$. But these numbers $1/2$, $(-2)/3$ etc. are not in **Z**. In other words, neither of the equations in (*) is solvable in **Z**. In fact (11) does not have a solution in **Z** whenever a is not a factor (= divisor) of b. Therefore we create numbers like $1/2$, $(-2)/3$ ... as solutions for $2x = 1$, $3x = -2$, ... In general, we create the rational numbers (10). But once we create them we have to merge them in the additive and multiplicative structures already existing in **Z**.

First we define the equality of two rational numbers as follows.

Definition 1. Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if $ad = bc$. We also agree to write every $n \in \mathbf{Z}$ as $\frac{n}{1}$ in **Q**.

Note. We now have the reason for writing $\frac{4}{6} = \frac{2}{3}$. In fact, always $\frac{a}{b} = \frac{pa}{pb}$ where p is any nonzero integer. Also $\frac{-5}{9} = \frac{5}{-9}$ because $(-5) \times (-9) = 45 = 5 \times 9$. So hereafter **we can safely assume that the denominators of rational numbers are positive.**

Now we define, thereby ensuring that the sum and product of two rational numbers is again a rational number,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \tag{12}$$

and

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \tag{13}$$

It is an interesting routine to verify the following:

(*i*) The addition and multiplication defined in (12) and (13) are both meaningful *i.e.*, if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, $\frac{c_1}{d_1} = \frac{c_2}{d_2}$ then $\frac{a_1}{b_1} + \frac{c_1}{d_1} = \frac{a_2}{b_2} + \frac{c_2}{d_2}$ and $\frac{a_1}{b_1} \times \frac{c_1}{d_1} = \frac{a_2}{b_2} \times \frac{c_2}{d_2}$.

(*ii*) Addition and multiplication in **Q** are both associatitve as well as commutative.

(*iii*) The distributive property (5′) holds in **Q** as well

(*iv*) For all $a \in \mathbf{Q}$,

$$a + 0 = a = 0 + a$$
$$1 \times a = a = a \times 1.$$

All these properties are thus present both in the smaller system **Z** and in the larger system **Q**. An additional property that is present in **Q**, but not in **Z** is the solvability of equations of the form

$$ax = b,\ a, b \in \mathbf{Q} \text{ and } a \neq 0 \tag{14}$$

The corresponding equation (11) is not always solvable in **Z**. But (14) is always solvable in **Q**. This is the advantage of the extension from **Z** to **Q**. But there is a 'price' that we pay for this extension. Let us describe it now. If we write the numbers of **Z** in their natural order, namely,

$$\{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

we note that for every number there is a 'next greater' number. But this fails in **Q** because in **Q** there is no 'next greater' number. We shall explain what this means.

First of all note that there is a natural order in the system **Q**. It is an extension of the natural order in **Z**.

Definition 2. For any two numbers $\frac{a}{b}$ and $\frac{c}{d}$ in **Q** with a, b, c, d in **Z** and $b, d > 0$,

$$\frac{a}{b} \geq \frac{c}{d} \text{ if and only if } ad \geq bc \tag{15}$$

Illustration $\frac{2}{3} > \frac{5}{9}$, because $2 \times 9 = 18 > 15 = 3 \times 5$

$$\frac{-7}{5} < \frac{-4}{3}, \text{ because } (-7) \times 3 = -21 < -20 = (-4) \times 5$$

Now let us observe what it means to say that there is no next greater number in **Q**. Consider any two rational numbers x and y with $x < y$. *Then we have* $x = \frac{x}{2} + \frac{x}{2} < \frac{x}{2} + \frac{y}{2}$ $= \frac{x+y}{2} < \frac{y}{2} + \frac{y}{2} = y$. *Thus, whenever* $x < y \in Q$ *we have* $\frac{x+y}{2} \in Q$ *such that*

$$x < \frac{x+y}{2} < y.$$

This says that between any two rational numbers x and y, there is a rational number and hence *an infinity of rational numbers are there between x and y*.

In **N** as well as in **Z** this concept of 'next greater' number is valid whereas in **Q** it is not. This is the price we pay for the extra advantage we achieve in extending to **Q**, viz., the solvability of the equations of the form (14).

Now we shall proceed to *the third stage* of this series of extensions. This stage is 'from **Q** to **R**' where **R** is the set of all *real numbers*. To explain what **R** is precisely, we have to take several steps. We shall not, in this book, be able to mathematically justify all these steps. First note, the necessity for the extension arises as follows. Suppose a is a positive rational number (*i.e.*, $a \in$ **Q**, $a > 0$) but not a perfect square. Then there is no rational number x such that $x^2 = a$. For instance $x^2 = 2$ has no solution for $x \in$ **Q**. One might say what about $x = \pm\sqrt{2}$? But what exactly is $\sqrt{2}$? We prove below, in Theorem 1, that such a number $\sqrt{2}$, if it exists, cannot be in **Q**. This is the reason for looking beyond **Q**, and obtaining an extension of **Q**. In fact, by one stroke of an extension from **Q** to **R**, mathematics solves not only the problem of solutions for $x^2 = 2$ but also $x^2 = 3$ and many other such equations which are not solvable in **Q**. But first let us take up the promised theorem.

Theorem 1. There is no rational number x such that $x^2 = 2$.

The proof is based on divisibility by 2. Recall that a natural number m is even iff $m = 2p$ for some natural number p and m is odd iff $m = 2q - 1$ for some natural number q. Each natural number is either odd or even but not both; for, if $2p = 2q - 1$ then $2(q - p) = 1$. This is impossible because $q - p$ is an integer. We also note that the square of an even number is even, since

$$(2p)^2 = 4p^2 = 2 \times 2p^2$$

and the square of an odd number is odd, since

$$\begin{aligned}(2q-1)^2 &= 4q^2-4q+1\\ &= 2(2q^2-2q+1)-1.\end{aligned}$$

Now suppose there is a rational number $x = m/n$ whose square is 2. We may suppose that m and n are natural numbers such that they do not have a common factor, otherwise we may cancel the common factor and reduce the fraction. So in particular m and n cannot both be even. Since $x^2 = (m/n)^2 = 2$ we have $m^2 = 2n^2$. This means m^2 is even and so m is even *i.e.*, $m = 2p$ for some $p \in \mathbf{N}$. Then $m^2 = 4p^2$; therefore $4p^2 = 2n^2$, so that $2p^2 = n^2$. This means n^2 is even and so n is also even. Thus both m and n are even, contradicting our assumption on m and n. This proves the theorem. ☐

Let us continue our description of the extension from **Q** to **R**. A number like $\sqrt{2}$ which is not a rational number is called an *irrational number*. Thus $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \ldots$ are all irrational numbers. A rational number is expressible as either a terminating decimal or a decimal with a recurring portion. On the other hand an irrational number when expressed as a decimal is neither terminating nor recurrent. These are deep statements which can be proved only with the tools of Higher Mathematics. It is therefore impossible to express irrational numbers — even such apparently simple ones as $\sqrt{2}$, $\sqrt{3}$ etc. — precisely in terms of terminating decimals or decimals with recurring parts. One can only approximate their actual value as closely as desired, by means of rational numbers.

Further it is not easy to perform the operations of addition and multiplication with irrational numbers. Of course we have been taught, for example, that

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}$$

and we shall certainly be using such relations all the time. But the proofs of these statements need precise definitions of irrational numbers. These precise definitions were given first by Dedekind in the 19th century. Just to give a broad picture of what his methods are, the definition of $\sqrt{2}$ goes as follows. Divide all the rational numbers (*i.e.*, the elements of **Q**) into two classes: the lower class L, and the upper class U. The lower class consists of all negative rationals and also those non-negative rationals whose squares are less than 2. The upper class consists of all those non-negative rationals whose squares are greater than 2. Both the classes are non-empty; because $1 \in L$ and $2 \in U$. Further their intersection is empty. Also every number in L is less than every number in U. So if we represent all the numbers on a geometric line, the above division of all rational numbers would be represented somewhat as follows:

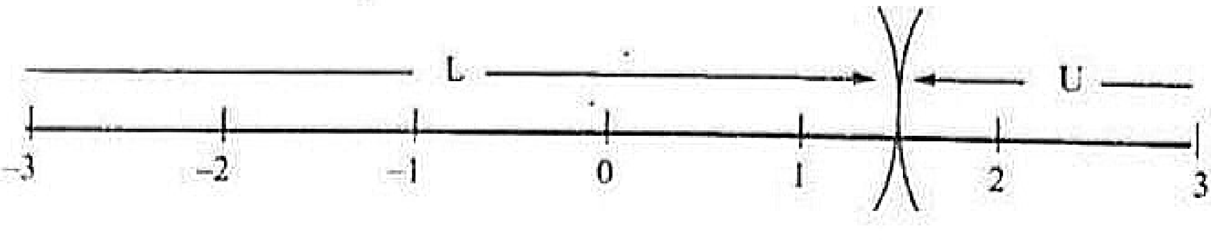


Fig. 1.1

Thus there is a gap between L and U. This gap was defined by Dedekind as the irrational number $\sqrt{2}$. The most proper way of establishing these results is to go with Dedekind and define irrational numbers by such 'cuts' of the geometric line. Dedekind not only defined them but also enunciated methods of addition and multiplication of irrational numbers such that the class **R** consisting of all rational numbers and irrational numbers became a proper extension of **Q**. The *real number system* **R** is thus the geometric line, which is continuous and without gaps. This is the major advantage over the smaller system **Q**. The continuity of the real line (as it is called) means that every point on the line represents a real number and every real number has a positional representation on the line.

But again there is a 'price' for this extension. The elements of **Q** can be sequenced in the following sense. The whole set of rational numbers can be put into one-one correspondence with **N** that is, with

$$\{1, 2, 3, \ldots, n, \ldots\}.$$

This property is called 'countability'. But this fails in **R**. The fact that **Q** is countable whereas **R** is not is a major result which will form one of the interesting, foundational results in Advanced Mathematics. It is enough to say at this point that **R** has so many numbers in it that even n repetitions of **Q** (however large the number n may be) would still not come up in terms of size, to the number of elements of **R**. We say **R** is "uncountable".

Now we come to the *fourth* (and last) *stage* of this series of extensions. It is from **R** to **C** where **C** is the set of all *complex numbers* $x + iy$, $x, y \in$ **R**. Here i is the so-called 'imaginary' square root of -1. Note that one of the major deficiencies of **R** is that, within **R** we cannot solve several algebraic equations, the simplest of them being

$$x^2 + 1 = 0.$$

There is no $x \in$ **R** which is a solution of this equation, because $x^2 = -1$ is an impossible relation for any $x \in$ **R**; since the square of any real number is non-negative. So we invent a new number called i such that $i^2 = -1$. It is called 'imaginary' because it is not in the real number system and so it is not real!

There is nothing 'imaginary' about it in the English sense of the word. It has as much of an existence in the mind as any other number in mathematics. The number "2" for example is itself only a mental construct. There are two apples, two fingers, etc. in the concrete visual world, there are symbols for '2' which can be written and seen, but the number '2' by itself is only in the mind. **The number 'i' also is as much of a mental construct and no more, as the number '2'.** The new number i is defined in such a way that it not only satifies $i^2 = -1$, but when adjoined to the system of real numbers it extends (**R**, +, .) into the bigger system (**C**, +, .) of complex numbers satisfying the associative, commutative, distributive properties already present in (**R**, +, .). Having defined i in this way we arrive at the general complex number $z = x + iy$ where x and y are real numbers. We note that (1) $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ and $y_1 = y_2$. (2) $(a + ib) + (c + id) = (a + c) + i(b + d)$. (3) $(a + ib) \times (c + id) = (ac - bd) + i(ad + bc)$. (4) **R** $\subseteq$ **C** in the sense that if $x \in$ **R** then $x = x + i0 \in$ **C**.

If $y = 0$, the number is x and therefore a (pure) real number. If $x = 0$, the number is iy and is called a wholly imaginary number. x is called the real part of the complex number z and is written as $\mathrm{Re}(z)$ or $\mathrm{Re}(x + iy)$. The real number y is called the imaginary part of z and is written as $\mathrm{Im}(z)$ or $\mathrm{Im}(x + iy)$. Thus we have, for every complex number $z = x + iy$

$$x = \mathrm{Re}(z) = \mathrm{Re}(x + iy)$$

and

$$y = \mathrm{Im}(z) = \mathrm{Im}(x + iy)$$

Also $x + iy$ and $x - iy$ are called conjugate complex numbers. Note that

$$(x + iy)(x - iy) = x^2 + ixy - ixy - i^2y^2 = x^2 + y^2.$$

By definition the sum of two complex numbers is a complex number and the product of two complex numbers is also a complex number. It is now a routine exercise to verify laws (1′), (2′), (3′), (4′) and (5′) for all $z \in$ **C** and also the special laws (6′) and (7′) for '0' and '1' in **C**. Note that the same 'zero' and the same '1' which worked for **Z**, **Q** and **R** also works here. Also at each stage of the extensions **N** to **Z**, **Z** to **Q**, **Q** to **R** whatever extra advantages we got, they are all present in **C**. In fact **C** not only extends (**R**, +, .) but extends them in a substantial manner. Any algebraic equation such as

$$a_0z^n + a_1z^{n-1} + ... + a_n = 0$$

with all a_i's in **C** has all its roots in **C**. This complete solvability of all algebraic equations is the major advantage of the extension from **R** to **C**. The result stating this complete solvability is known as the Fundamental Theorem of Algebra, whose proof requires quite a lot of higher mathematics.

But as before, we pay a price, and this time a big price for the extension. There is no order relation in **C** which extends the order relation in **R**. This is the loss we are prepared to put up with for the advantage gained in this last extension. But before we see the full force of this loss, we have to make an important observation.

The real line, as we know, has been designed in such a way that there are no gaps, in the sense we explained earlier. In this sense therefore, the real line is 'complete'. So when we introduce new numbers like the complex numbers for purposes of being able to solve more algebraic equations, we cannot expect to have geometrical representations of these numbers on the real line itself, keeping the earlier representation of numbers on the real line. So mathematics invented two 'copies' of the real line, one perpendicular to the other, the first one representing the real part of the complex number and the second one representing the imaginary part of the complex number. Such a representation of a complex number $z = x + iy$ as a point (x, y) on the coordinate plane is called an *Argand Diagram*. A point on the x-axis (now called the *Real axis*) is $(x, 0)$ and so represents the purely real numbers x; as a complex number it is nothing but $x + i0$. A point on the y-axis (now called the *Imaginary axis*) is $(0, y)$ and so represents the purely imaginary number iy. Note that, throughout in this discussion, x and y are real numbers.

If P is the point (x, y), it represents the complex number $z = (x + iy)$. We have the

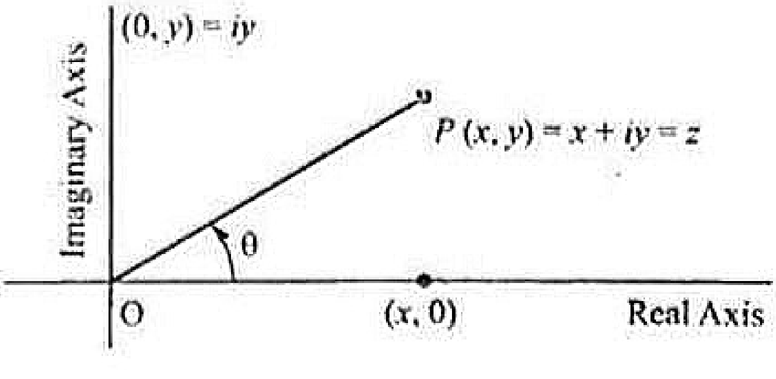


Fig. 1.2

following two fundamental definitions regarding $z = x + iy$ which is geometrically the same as $P = (x, y)$.

Definition 3. The distance $OP = \sqrt{x^2 + y^2}$ from the origin is called the *modulus of z*. It is denoted by $|z|$ or $|x + iy|$. It is always non-negative.

For example $|2 + 3i| = 5$;

$|1 - i| = \sqrt{2}$ and so on.

Note that in particular the modulus of a real number $x\ (= x + i0)$ reduces to the following:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus $|-7| = -(-7) = 7 = |7|$. The modulus is also called the *absolute value*.

Definition 4. If θ is the angle which OP makes with the positive direction of the x-axis, θ is called the '*argument*' of z and is written arg z. We usually take $\theta = \arg z$ to lie between $-180°$ and $180°$, *i.e.*, such that $-180° < \theta° \leq 180°$.

Now since the complex numbers are geometrically all over the plane, there is no natural order among them, which will coincide with the natural order of the real numbers (= complex numbers $x + i0$) on the real line (= x-axis of the Argand Diagram), and which is properly compatible with the addition and multiplication in **C**. For instance, whatever way we design the order, either the number i has to be greater than the number zero or has to be less. Either way we get an incongruity with multiplication, since $i^2 = -1$ and this would mean that the L.H.S. here is a product of two quantities which are either both greater than 0 or both less than zero. In both the cases the R.H.S. being negative, we face a contradiction. Thus it is impossible to introduce an order in C which is compatible with multiplication in the above sense and which reduces to the natural order on the real line. Hence two complex numbers are either equal or unequal; there is no concept of greater or less.

This completes our outline of the five number systems **N**, **Z**, **Q**, **R** and **C**. We summarise in Table 1.1 the information about what is gained by each extension and what is lost. As we go from **N** to **Z** to **Q** to **R** to **C**, note that what is once gained remains a gain through all the further stages and what is once lost remains a loss in all the further stages.

EXAMPLE 1. *If $(a/b) < (c/d)$ with $b > 0$, $d > 0$ show that $(a + c)/(b + d)$ lies between a/b and c/d. (where a, b, c, d are real numbers).*

SOLUTION. If $(a/b) < (c/d)$ and b, d are positive then $ad < bc$ and hence

$$ab + ad < ab + bc.$$

This means that $a(b + d) < b(a + c)$ or $(a/b) < (a + c)/(b + d)$.

Table 1.1. Gain and Loss in the Extensions From N To Z To Q To R To C

Properties which are not present in N but gained in an extension	**(1′)** $a + x = b$ **is solvable**	**(2′)** $ax = b$, $(a \neq 0)$ **is solvable**	**(3′) No gaps in the geometrical construct which represents the numbers**	**(4′) All algebraic equations are solvable**
	Z	**Q**	**R**	**C**
Properties which are present in N but lost in an extension				
1. If $a > b$ then $xa > xb$ for all x.	**1′ is true**			
	1 is not true			
2. There exists a next greater number for every number. (Induction)		**2′ is true**		
		2 is not true		
3. It is possible to arrange the entire system as an infinite sequence. (countability)			**3′ is true**	
			3 is not true	
4. There exists a natural order relation compatible with addition and multiplication				**4′ is true**
				4 is not true

Similarly $ad < bc$ means $ad + cd < bc + cd$ or $(a + c)d < (b + d)c$. This means that $(a + c)/(b + d) < (c/d)$. Thus $(a/b) < (a + c)/(b + d) < (c/d)$.

EXAMPLE 2. *Let a and b be positive integers. Show that $\sqrt{2}$ always lies between (a/b) and $(a + 2b)/(a + b)$.*

SOLUTION. Suppose $\sqrt{2} < (a/b)$. Then $2 < (a^2/b^2)$ or $2b^2 < a^2$. Therefore, we get

$$a^2 + 4b^2 < a^2 + a^2 + 2b^2 = 2a^2 + 2b^2$$

$$(a + 2b)^2 = a^2 + 4b^2 + 4ab < 2a^2 + 2b^2 + 4ab = 2(a + b)^2$$

$$\therefore \quad [(a + 2b)/(a + b)]^2 < 2 \text{ or } (a + 2b)/(a + b) < \sqrt{2}$$

On the other hand if $\sqrt{2} > (a/b)$ then $a^2 < 2b^2$.

$2(a + b)^2 = 2(a^2 + 2ab + b^2) = a^2 + a^2 + 2b^2 + 4ab < a^2 + 2b^2 + 2b^2 + 4ab = (a + 2b)^2$ or $\sqrt{2} < (a + 2b)/(a + b)$. Thus $\sqrt{2}$ always lies between a/b and $(a + 2b)/(a + b)$.

EXAMPLE 3. *Given any real number $x > 0$, show that there exists an irrational number ξ, such that $0 < \xi < x$.*

SOLUTION. If x is irrational, then choose $\xi = x/2$. Clearly $0 < \xi < x$.

If x is rational, then choose $\xi = x/\sqrt{2}$. Since $\sqrt{2} > 1$, we have $0 < \xi < x$.

(In fact there are infinitely many irrational numbers between any two real numbers.)

EXAMPLE 4. *Show that $\sqrt{2} + \sqrt{5}$ is irrational.*

SOLUTION. Suppose $\sqrt{2} + \sqrt{5} = x = p/q$ is a rational number with $p, q \in \mathbf{Z}$.

Then $(x - \sqrt{2})^2 = 5$ *i.e.* $x^2 - 2\sqrt{2}x + 2 = 5$.

Hence $x^2 - 3 = 2\sqrt{2}x$, which gives $\sqrt{2} = (x^2 - 3)/(2x)$, a rational number.

This contradicts the fact that $\sqrt{2}$ is rational. So $\sqrt{2} + \sqrt{5}$ is irrational.

EXAMPLE 5. *Express the rational number whose decimal expansion is 0.1234545454545 ... as ratio of two integers.*

SOLUTION. Let $x = 0.123454545\ldots$

Then $10^3x = 123.454545\ldots$

$10^5x = 12345.4545\ldots.$

Subtracting, we get $(10^5 - 10^3)x = 12222.$

$x = 12222/99000.$

EXAMPLE 6. *Find all positive integers n for which $\sqrt{(n-1)} + \sqrt{(n+1)}$ is rational.*

SOLUTION. Let $x_n = \sqrt{(n-1)} + \sqrt{(n+1)}$ be rational. Then $1/x_n$ is also rational.

But $1/x_n = 1/(\sqrt{(n-1)} + \sqrt{(n+1)})$

$= (\sqrt{(n+1)} - \sqrt{(n-1)})/(\sqrt{(n-1)} + \sqrt{(n+1)})(\sqrt{(n+1)} - \sqrt{(n-1)})$

$= (\sqrt{(n+1)} + \sqrt{(n-1)})/2.$

This means $(\sqrt{(n+1)} - \sqrt{(n-1)})$ is also rational.

So $(\sqrt{(n-1)}$ and $\sqrt{(n+1)})$ are also rational.

i.e., $(n-1)$ and $(n+1)$ are perfect squares.

This is not possible as any two perfect squares differ at least by 3. Hence there is no positive integer n such that $(\sqrt{(n-1)} + \sqrt{(n+1)})$ is rational.

EXAMPLE 7. *If $a + \sqrt{b} = c + \sqrt{d}$, where a, b, c, d are rational then $a = c$ and $b = d$, unless b, d are squares of rationals.*

SOLUTION. Suppose $a \neq c$, let $a = c + x$. Then $a + \sqrt{b} = c + x + \sqrt{b} = c + \sqrt{d}$. So $x + \sqrt{b} = \sqrt{d}$. Squaring we get, $d - x^2 - b = 2x\sqrt{b}$. This implies that $\sqrt{b}$ is rational, hence $\sqrt{d}$ is also rational. Thus b and d are squares of rationals. Hence the result.

EXAMPLE 8. *If $a + b(\sqrt[3]{p}) + c(\sqrt[3]{p^2})$, where a, b, c, p are rational and p is not a perfect cube, then a, b, c are all zero.*

SOLUTION. We have

$$a + b(\sqrt[3]{p}) + c(\sqrt[3]{p^2}) = 0 \tag{1}$$

Therefore,

$$a(\sqrt[3]{p}) + b(\sqrt[3]{p^2}) + cp = 0 \tag{2}$$

Now $b \times (1) - c \times (2)$ gives

$$(b^2 - ac)\sqrt[3]{p} + ab - c^2p = 0 \tag{3}$$

$\sqrt[3]{p}$ is irrational and therefore from (3) we get $b^2 - ac = 0$ and $ab = c^2p$.

$\therefore$ $c^4p^2 = a^2b^2 = a^3c$. If $c \neq 0$, then we get $p^2 = a^3/c^3$ which is not true as $\sqrt[3]{p}$ is irrational. $\therefore$ $c = 0$ which in turn implies that $a = b = 0$.

EXAMPLE 9. *If a, b, c, d are all rational and, $\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$ then show that either (i) $a = c$ and $b = d$ or (ii) $a = d$ and $b = c$ or (iii) the quotients $\sqrt{(a/b)}$, $\sqrt{(a/c)}$, $\sqrt{(a/d)}$, $\sqrt{(b/c)}$, $\sqrt{(b/d)}$, $\sqrt{(c/d)}$, are all rational.*

SOLUTION. $\sqrt{a}+\sqrt{b}=\sqrt{c}+\sqrt{d}$ gives on squaring $a+b=c+d$ and $\sqrt{ab}=\sqrt{cd}$ unless $\sqrt{ab}$, $\sqrt{cd}$ are rational. (See Example 7.)

$\therefore$ $(\sqrt{a}-\sqrt{b})^2=(\sqrt{c}-\sqrt{d})^2$ unless $\sqrt{ab}$, $\sqrt{cd}$ are rational. This means that $|\sqrt{a}-\sqrt{b}|=|\sqrt{c}-\sqrt{d}|$ unless $\sqrt{ab}$ and $\sqrt{cd}$ are rational.

Case 1. $\sqrt{a}+\sqrt{b}=\sqrt{c}+\sqrt{d}$ and $\sqrt{a}-\sqrt{b}=\sqrt{c}-\sqrt{d}$ gives $a=c$ and $b=d$.

Case 2. $\sqrt{a}+\sqrt{b}=\sqrt{c}+\sqrt{d}$ and $\sqrt{a}-\sqrt{b}=\sqrt{d}-\sqrt{c}$ gives $a=d$ and $b=c$.

That $\sqrt{ab}$ and $\sqrt{cd}$ are rational implies that $\sqrt{(a/b)}$ and $\sqrt{(c/d)}$ are rational.

$\sqrt{a}+\sqrt{b}=\sqrt{c}+\sqrt{d}$ also implies $\sqrt{a}-\sqrt{c}=\sqrt{d}-\sqrt{b}$, which gives $a=d$ and $b=c$ or $\sqrt{ac}$, $\sqrt{bd}$ are rational. Finally we conclude that either

(*i*) $a=c$ and $b=c$ or (*ii*) $a=d$ and $b=c$ or (*iii*) the quotients $\sqrt{(a/b)}$, $\sqrt{(a/c)}$, $\sqrt{(a/d)}$, $\sqrt{(b/c)}$, $\sqrt{(b/d)}$, $\sqrt{(c/d)}$ are all rational.

EXAMPLE 10. *Find a polynomial equation of the lowest degree with rational coefficients of which one root is* $\sqrt[3]{2}+3\sqrt[3]{4}$.

SOLUTION. Let $x=\sqrt[3]{2}+3\sqrt[3]{4}$.

Then we have
$$x^3=2+108+18(\sqrt[3]{2}+3\sqrt[3]{4})$$
$$=110+18x.$$

$$\therefore\quad x^3-18x-110=0$$

(It is clear from Example 8 that no quadratic expression in x with rational coefficients becomes 0. So the least degree is 3.)

CHA

2

ARITHMETIC OF INTEGERS

2.1 THE PRINCIPLE OF INDUCTION

In this chapter we shall see certain fundamental properties valid in the number systems **N** and **Z**.

Consider a statement about the positive integers.

For example (1) $n(n + 1)(n + 2)$ is always divisible by 6

(2) The sum of the first n natural numbers is given by

$$S_n = \frac{n(n+1)}{2}$$

(3) $2^n > n$ for all natural numbers

(4) $(1 + x)^n$

$$= 1 + nx + \frac{n(n+1)}{1.2}x^2 + \dots$$

$$+ \frac{n(n-1)(n-1)(n-2)\dots(n-r_{n+1})}{1.2.3.\dots r}x^r + \dots + x^n$$

for all positive integers

are all statements about positive integers n. If one wants to check the validity of these statements, how should one go about it?

Take the problem of finding the sum S_n of the first n natural numbers. The statement (2) above says that $S_n = \dfrac{n(n+1)}{2}$ for all n. When we try verifying with the first few natural numbers, we observe that $S_1 = 1$, $S_2 = 3$, $S_3 = 6$, $S_4 = 10$, $S_5 = 15$ all satisfy $S_n = \dfrac{n(n+1)}{2}$. But from a few verifications, or for that matter, from any number of verifications, one cannot conclude that the result is always true. So, what do we do? We observe that

$$S_{k+1} = 1 + 2 + 3 + \dots + k + (k + 1) = S_k + (k + 1).$$

If we have checked and found, or if we assume that the formula

$S_n = \dfrac{n(n+1)}{2}$ is true for all $n \le k$, the equation $S_{k+1} = S_k + (k + 1)$ gives

$$S_{k+1} = \frac{k(k+1)}{2} + (k+1) = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

This says that if the formula is true for $n = k$, then it has to be true for the next integer $k+1$. We have checked that the result is true for $k = 1$. Therefore, by what we have just seen, it must be true for $k = 2$ and hence for the next integer 3. Proceeding inductively, we see that the formula has to be true for all positive integers. Thus we have essentially proved a very important formula, namely, the sum of the first n natural numbers is given by $S_n = \dfrac{n(n+1)}{2}$.

The underlying mathematical principle in the above argument is called the **'Principle of Mathematical Induction'**. It can be stated as follows.

Let $P(n)$ be a statement about the positive integers such that

(1) $P(1)$ is true, *i.e.*, the statement is true for $n = 1$.

(2) Whenever the statement is true for $n = k$, it is true for $n = k + 1$. Then $P(n)$ is true for all natural numbers n.

This principle is one of the fundamental principles in mathematics and it is a tool indispensable in most of the branches of mathematics.

REMARK The above principle works because of the fact that every nonempty subset of positive integers has a least element. In fact, it can be proved that the principle of mathematical induction is equivalent to the fact that every nonempty set of positive integers has a smallest element.

EXAMPLE 1. *If x is any real (complex) number such that $x \neq 1$*

then $1 + x + x^2 + \dots + x^{n-1} = S_n = \dfrac{1-x^n}{1-x}$ for every positive integer n.

SOLUTION. We prove this by induction on n. As a first step, we check that $S_1 = 1 = (1-x)/(1-x)$. Therefore the result is true for $n = 1$. Suppose we assume that it is true for $n = k$. Then $1 + x + x^2 + \dots + x^{k-1} = S_k = \dfrac{1-x^k}{1-x}$.

$$\therefore \qquad S_{k+1} = 1 + x + x^2 + \dots + x^k = S_k + x^k$$

$$= \frac{(1-x^k)}{(1-x)} + x^k = \frac{1-x^{k+1}}{1-x}.$$

Thus, whenever the formula is true for k, it is true for $k+1$; and it is found to be true for $k = 1$. Hence by the principle of induction $S_n = \dfrac{1-x^n}{1-x}$ for all n.

Note. This example is actually that of what is called a *geometric progression*. See Chapter 15 for more on this.

EXAMPLE 2. $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + xy^{n-2} + y^{n-1})$, *where n is any positive integer greater than one.*

SOLUTION. The formula for $n = 2$ reads $x^2 - y^2 = (x-y)(x+y)$ which is true. Assume now that the result is true for $n = k$ for some $k > 2$. This gives

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \dots + xy^{k-2} + y^{k-1}).$$

Therefore $x^{k+1} - y^{k+1}$

$$= x(x^k - y^k) + y^k(x - y)$$

$$= x(x - y)(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \dots + xy^{k-2} + y^{k-1}) + y^k(x - y)$$

(by our induction hypothesis)

$$= (x - y)\,\{x(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \dots + xy^{k-2} + y^{k-1}) + y^k\}$$

$$= (x - y)\,(x^k + x^{k-1}y + x^{k-2}y^2 + \dots + xy^{k-1} + y^k).$$

Thus the formula is true for $k + 1$ whenever it is true for any k. We have checked that it is true for $k = 2$. Therefore by the principle of induction the formula is true for all positive integers $n \geq 2$.

EXAMPLE 3. *Prove that*

$$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

SOLUTION. When $n = 1$, the left hand side reads $1/1.2 = 1/2$ which is the same as $n/n + 1$ when $n = 1$. Thus the result is true for $n = 1$. Assume now that the result is true for $n = k$. This gives

$$S_k = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Therefore $$S_{k+1} = \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= S_k + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$ (by the induction hypothesis)

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

(Note that the cancellation of $(k+1)$ is valid here as $k + 1 \neq 0$). We have now proved that whenever the result is true for $n = k$ it is also true for $n = k + 1$; and we have already checked it to be true for $n = 1$. Hence, by the principle of induction the result is true for all positive integers.

EXAMPLE 4. *If there are n participants in a knock-out tournament then prove that (n − 1) matches will be needed to declare the champion.*

SOLUTION. When $n = 1$, there is no match needed and the result is trivially true. This gives a base for induction. Suppose $n = 2k$ for some positive integer k. Then we have k matches played in the first round of the tournament. The k winners should play $(k - 1)$ matches to find the champion, if we make the induction hypothesis that the result is true for all integers $m < n$. Therefore in all we require $k + k - 1 = 2k - 1 = n - 1$ matches to be played. Suppose $n = 2k + 1$ with $k \geq 1$. Again in the first round of k matches we have k losers. The remaining $(k + 1)$ men have to play k matches to find the champion, again using the induction hypothesis made above. Thus we require in all

$$k + k = 2k = n - 1$$

matches to be played.

EXAMPLE 5. *Prove that $5^{2n} - 6n + 8$ is divisible by 9 for all positive integers n.*

SOLUTION. If $f(n) = 5^{2n} - 6n + 8$ then $f(1) = 5^2 - 6 + 8 = 27$ which is divisible by 9. Therefore the result is true for $n = 1$. Assume that $f(n)$ is divisible by 9 for some $n > 1$. Then we have

$$\begin{aligned} f(n+1) &= 5^{2(n+1)} - 6(n+1) + 8 = 5^{2n} \,.\, 5^2 - 6(n+1) + 8 \\ &= 5^2(5^{2n} - 6n + 8) + 144n - 198 \\ &= (\text{Multiple of } 9) + 9(16n - 22), \quad \text{(by induction hypothesis)} \\ &= \text{Multiple of } 9 \end{aligned}$$

Therefore by the principle of induction 9 divides $f(n)$ for all positive integers n.

EXAMPLE 6. *Let $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$. Show that*

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \qquad \textit{for integers } n \geq 1.$$

SOLUTION. For $n = 1, 2$ we have

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = 1 \qquad \text{(readily checked).}$$

Assume the result to be true for all integers k such that $1 \leq k \leq n$.

For $n \geq 2$ we have

$$\begin{aligned} u_{n+1} &= u_n + u_{n-1} \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] \\ &\qquad \text{(by induction hypothesis)} \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{3-\sqrt{5}}{2}\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \end{aligned}$$

Thus, whenever the result is true for $k \leq n$, we see that it is true for $k = n + 1$. Therefore, by the principle of mathematical induction, the result is true for all positive integers.

EXERCISE 2.1

1. Prove that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ using the induction principle.
2. Using induction prove that $1^3 + 2^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2$ for each positive integer n.
3. Prove that $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$.
4. Let $a_1 = a_2 = 1, a_3 = 2$ and $= a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. The sequence (a_n) is known as the Fibonacci sequence. Prove that
   (*i*) $a_1 + a_2 + ... + a_n = a_{n+2} - 1$.
   (*ii*) $a_1 + a_3 + a_5 + ... + a_{2n-1} = a_{2n}$.
   (*iii*) $a_2 + a_4 + a_6 + ... + a_{2n} = a_{2n+1} - 1$.
   (*iv*) $a_{n+1}^2 = a_n a_{n+2} = (-1)^n$.
   (*v*) $a_1 a_2 + a_2 a_3 + ... + a_{2n-1} a_{2n} = (a_{2n})^2$.
   (*vi*) $a_1 a_2 + a_2 a_3 + ... + a_{2n} a_{2n+1} = (a_{2n+1})^2 - 1$.
5. Define (b_n) by $b_1 = 1$, $b_n = a_{n+1} - a_n$ for $n \geq 2$. (b_n) is known as the sequence of Lucas numbers.

   Prove
   (*i*) $b_n = b_{n-1} + b_{n-2}$ for $n \geq 3$.
   (*ii*) $a_{2n} = a_n b_n$.
   (*iii*) $b_1 + 2b_2 + 4b_3 + 8b_4 + ... + 2^{n-1} b_n = 2^n a_{n+1} - 1$.

   where (a_n) is the Fibonacci sequence of numbers defined in exercise 4.
6. Prove by induction that the product $n(n + 1)(n + 2) ... (n + r - 1)$ of any consecutive r numbers is divisible by $r!$.
7. If $S_n = (3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ show that S_n is an integer and that $S_{n+1} = 6S_n - 4S_{n-1}$. Deduce that the next integer greater than $(3 + \sqrt{5})^n$ is divisible by 2^n.
8. Show that n^2 is the sum of the first n consecutive odd numbers.
9. Prove by induction that $2^n > n^3$ if $n > 9$.
10. Show that $2.7^n + 3.5^n - 5$ is divisible by 24 for all positive integers n.
11. If m, n, p, q are non negative integers prove that
$$\sum_{m=0}^{q} (n-m)\frac{(p+m)!}{m!} = \frac{(p+q+1)!}{q!}\left(\frac{n}{p+1} - \frac{q}{p+2}\right).$$
12. Prove $\frac{k^7}{7} + \frac{k^5}{5} + \frac{2k^3}{3} - \frac{k}{105}$ is an interger for every positive integer k.

2.2 DIVISIBILITY

The equation $ax = b$, $a \neq 0$ does not always have a solution in **Z** the set of integers. For example $3x = 12$ has a solution $x = 4$ but $5x = 12$ does not have any solution in **Z**. This observation prompts the following definition.

Definition 1. The integer a *divides* an integer b if there exists an integer c such that $ac = b$.

In other words a divides b in **Z** if $ax = b$ has a solution for x in **Z**. When a divides b we call b a multiple of a. For example, the multiples of 2 are $0, \pm 2, \pm 4, \pm 6, \pm 8$ ... which we call the set of even numbers. A number is even if it is a multiple of 2. Given any integer a, we always have $a.0 = 0$ and therefore 0 is a multiple of every integer, or every integer divides 0. In fact, 0 is the only such integer. When a divides b we also say that a is a divisor of b. For example, the divisors of 2 are just $\pm 1, \pm 2$; the divisors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$ and ± 12. At times we also say 'factors' in the place of 'divisors'. We observe that ± 1 are divisors of every other integer. Given any integer a, it has at least two factors 1 and a.

Notation. $a \mid b$ means that a divides b.

Proposition 1. If a divides b and b divides c then a divides c.

Proof. a divides b implies that there exists an integer k such that $ak = b$. Also b divides c implies that there exists an integer l such that $bl = c$. This gives $c = bl = (ak)l. = a.kl$ which implies that a divides c as kl is an integer whenever k and l are integers. □

Proposition 2. For any integer k let $k\mathbf{Z} = \{0, \pm k, \pm 2k, \pm 3k \ldots.\}$ denote the set of multiples of k. Then a divides b implies that $a\mathbf{Z} \supseteq b\mathbf{Z}$, *i.e.*, every multiple of b is also a, multiple of a.

Proof. If $c \in b\,\mathbf{Z}$ then b divides c; now we have, a divides b and b divides c. Therefore by Proposition 1, a divides c, which implies that $c \in a\mathbf{Z}$. Thus $b\mathbf{Z} \subseteq a\mathbf{Z}$ whenever a divides b. □

For example, 3 divides 6 and 6 divides 12. We have and $3\mathbf{Z} \supseteq 6\mathbf{Z} \supseteq 12\mathbf{Z}$. (*i.e.*, the set of multiples of 3 contains the set of multiples of 6 and which in turn contains the set of multiples of 12).

Proposition 3. $a\mathbf{Z} = \mathbf{Z}$ if and only if $a = \pm 1$.

Proof. The above statement means the following. $a\mathbf{Z} = \mathbf{Z}$ implies that $a = \pm 1$ and conversely $a = \pm 1$ implies that $a\mathbf{Z} = \mathbf{Z}$. In fact, this statement follows immediately from the fact that ± 1 are the only integers which are divisors of every other integer. If $a \neq \pm 1$ we have $a\mathbf{Z}$ as a proper subset of $\mathbf{Z}$. □

Now 7 divides 21 and 7 divides 35 and we have 7 dividing $21x + 35y$ for any two integers x and y. In fact, in general we have the following proposition.

Proposition 4. If a divides b and a divides c in $\mathbf{Z}$ then a divides $xb + yc$ for any integers x, y in $\mathbf{Z}$ (In other words a divides every integral linear combination of b and c).

Proof. a divides b and a divides c imply that there exist integers k and l such that $ak = b$ and $al = c$. Therefore $xb + yc = x(ak) + y(al) = a(xk + yl)$. This means that a divides $xb + yc$. □

We note that certain integers have a large number of factors, while some others have only a few factors. We have already noted that every integer $x > 0$ has at least two positive factors, namely 1 and x. Some positive integers x have just these two as their positive factors. These integers play an important role in Mathematics.

Definition 2. A positive integer p is a *prime* if $p \neq 1$ and the only positive divisors of p are 1 and p.

For example 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37 ..... are the first few primes.

If a and b are two integers then any integer c that divides both a and b is called a common divisor of a and b.

For example (*i*) the common divisors of 4 and 8 are $\pm 1, \pm 2, \pm 4$.
(*ii*) the common divisors of 8 and 12 are $\pm 1, \pm 2, \pm 4$.
(*iii*) the common divisors of 12 and 35 are ± 1.

(*iv*) the common divisors of 7 and 24 are ± 1.

(*v*) the common divisors of 24 and 60 are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12.

Definition 3. If a and b are integers such that not both of them are zero, then a positive integer d is called *the greatest common divisor* (written g.c.d.) of a and b if

(*i*) d is a common divisor of a and b

(*ii*) each integer c that divides both a and b also divides d.

The first question that comes to our mind is that given a pair of integers, not both zero, should there exist a greatest common divisor, and when it exists, should it be unique, justifying the definite article 'the' used in the above definition. We try to answer these questions, by studying some examples.

EXAMPLE 1. (*i*) *The positive common divisors of 4 and 8 are 1, 2 and 4. Therefore the g.c.d. of 4 and 8 is 4.*

(*ii*) *The only positive common divisor of 12 and 35 is 1 and therefore the g.c.d. of 12 and 35 is 1.*

(*iii*) *The positive common divisors of 24 and 60 are 1, 2, 3, 4, 6 and 12. Therefore the g.c.d. of 24 and 60 is 12.*

SOLUTION. In these examples, we could actually enumerate the common divisors. If the numbers are big, such an enumeration becomes extremely difficult. Given two integers a and b we look for smaller integers a_1 and b_1 which have the same g.c.d. Consider 138 and 1239. We have $1239 = (8 \times 138) + 135$. Now any common divisor of 1239 and 138 has to divide 135 as seen from the equation $135 = 1239 - 8(138)$. Also from $1239 = 8(138) + 135$ we see that any common divisor of 138 and 135 also divides 1239. Thus the set **S** of common divisors of 1239 and 138 is the same as the set of common divisors of 138 and 135. Now 138 and 135 are smaller compared to 1239 and 138. Again writing $138 = 1(135) + 3$ we note that S = the set of common divisors of 135 and 3, given by $S = \{\pm 1, \pm 3\}$.

Thus the g.c.d. of 1239 and 138 is 3. Such a simplification was possible because we could divide 1239 by 138 to get a quotient and a remainder smaller than 138. That such a division is possible for any two integers $a > 0$ and b is the following division algorithm due to Euclid.

Theorem 1. (*Euclid's division lemma or the division algorithm*)

For any integers $a > 0$ and b there exist unique integers q and r such that $b = aq + r$ with $0 \le r < a$.

Proof. Consider the representation of the integers on a geometric line. Plot the points

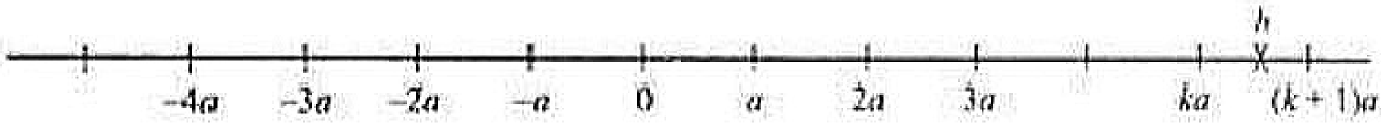


Fig. 2.1

corresponding to the multiples of a. If b is a multiple of a then it is of the form $b = ka$ for some integer k and we may take $q = k$ and $r = 0$. Otherwise, we can find two multiples ka and $(k + 1)a$ of a such that $ka < b < (k + 1)a$. Hence $0 \le b - ka < (k + 1)a - ka = a$. Take $q = k$, $r = b - ka$ to get $b = aq + r$ with q, r being integers and $0 \le r < a$.

Now we prove the uniqueness of q and r. Suppose $b = aq_1 + r_1 = aq_2 + r_2$ with q_1, q_2, r_1, r_2 being integers and $0 \le r_1, r_2 < a$. Then we should have $0 = (aq_1 + r_1) - (aq_2 + r_2)$ which implies that $a(q_1 - q_2) = r_2 - r_1$. We may assume $r_1 \le r_2$. This gives $a(q_1 - q_2) = r_2 - r_1 \le r_2 < a$ which means that a divides $r_2 - r_1$ where $r_2 - r_1$ is a non-negative integer less than a. This is possible only if $r_2 - r_1 = 0$ which in turn implies that $aq_1 = aq_2$. Now $aq_1 = aq_2$ gives $0 = a(q_1 - q_2)$ and this implies that $q_1 - q_2 = 0$ since by our assumption $a > 0$. This proves that $r_1 = r_2$ and $q_1 = q_2$, and the uniqueness is established. ❑

Note. Here in this proof we have used the fact that given $a > 0$ in $\mathbf{Z}$ and any integer b we can always find $k \in \mathbf{Z}$ such that $ka > b$, which is more or less evident from the geometric representation on a straight line. For a purely algebraic proof see the exercises.

Notation. We write (a, b) for the g.c.d of a and b. We note that if $a \neq 0$ and a divides b, then $(a, b) = |a|$.

EXAMPLE 2. *Find the g.c.d. of 341 and 403.*

SOLUTION. Dividing 403 by 341 we get 403 = 1(341) + 62. Now the common divisors of 341 and 403 are precisely the common divisors of 341 and 62 (as seen already in the discussion just preceding Euclid's algorithm). Again dividing 341 by 62 we get 341 = 5(62) + 31 and hence the common divisors of 341 and 62 are precisely the common divisors of 62 and 31. Now the common divisors of 31 and 62 are ± 1, ± 31.

Thus we have
$$403 = 1\,(341) + 62$$
$$341 = 5\,(62) + 31$$
$$62 = 2(31) + 0$$

Hence
$$(341, 403) = (341, 62) = (62, 31) = 31.$$

The above example tells us that there is nothing special about 341 and 403; we could have calculated the g.c.d. of any two integers a, b not both zero simultaneously, by the same algorithmic process. This observation leads to the following theorem.

Theorem 2. If a and b are integers, not both simultaneously zero, then (a, b) exists and is unique.

Proof. From the definition, it is clear that the g.c.d. of a and b is not affected by their signs, *i.e.*, $(a, b) = (a, -b) = (-a, b) = (-a, -b)$ whenever they exist. If one of them, say $b = 0$, then $(a, b) = |a|$ since every integer $a \neq 0$ divides 0. Therefore we assume that $a \ge b > 0$. We apply Euclid's division algorithm repeatedly to get

$$\begin{array}{ll} a = bq_1 + r_1 & 0 \le r_1 < b \\ b = r_1q_2 + r_2 & 0 \le r_2 < r_1 \\ r_1 = r_2q_3 + r_3 & 0 \le r_3 < r_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ r_{n-3} = r_{n-2}q_{n-1} + r_{n-1} & 0 \le r_{n-1} < r_{n-2} \\ r_{n-2} = r_{n-1}q_n + r_n & 0 \le r_n < r_{n-1} \end{array} \tag{1}$$

Now $b > r_1 > r_2 > r_3 > \ldots \ge 0$ says that this process has to end up in at most b steps; and we will obtain the corresponding $r_n = 0$. Therefore the last equation in the above sequence should read

$$r_{n-2} = r_{n-1}q_n + 0. \tag{2}$$

By construction, each $r_i > 0$ for $i = 1, 2, \ldots, (n-1)$ and therefore $r_{n-1} > 0$. Retracing back from the last equation (2) in the system of equations (1) we see that r_{n-1} divides

$a_4 \ldots a_n$. Thus by a repeated application of the theorem we see p divides a_i for some $i \in \{1, 2, 3, \ldots n\}$. □

EXAMPLE 5. *For any positive integer m we have*

$$(ma, mb) = m\,(a, b).$$

SOLUTION. We note that $(a, b) =$ least positive value of $\{ax + by \mid x, y \in \mathbf{Z}\}$. This follows from the Corollary to Theorem 3.

$$\begin{aligned}\therefore\quad (ma, mb) &= \text{least positive value of } \{max + mby \mid x, y \in \mathbf{Z}\}\\ &= m \,.\, \text{least positive value of } \{ax + by \mid x, y \text{ in } \mathbf{Z}\}\\ &= m\,(a, b).\end{aligned}$$

EXAMPLE 6. *If d divides a, d divides b and if d > 0 then*

$$\left(\frac{a}{d}, \frac{b}{d}\right) = \frac{1}{d}\,(a, b)$$

SOLUTION. Now $d > 0$, and hence by Example 5,

$$d \,.\, \left(\frac{a}{d}, \frac{b}{d}\right) = (a, b)$$

or,

$$\left(\frac{a}{d}, \frac{b}{d}\right) = \frac{1}{d}\,(a, b).$$

EXAMPLE 7. *If $(a, n) = (b, n) = 1$ then $(ab, n) = 1$ i.e., If a and b are relatively prime to n then so is ab.*

SOLUTION. We can find integers x_1, y_1, x_2, y_2 such that

$$ax_1 + ny_1 = 1 = bx_2 + ny_2, \text{ since } (a, n) = 1 = (b, n).$$

$$\therefore\quad (ax_1)(bx_2) = (1 - ny_1)(1 - ny_2) = 1 - n(y_1 + y_2 - ny_1y_2)$$

$$\therefore\quad ab\,x_1\,x_2 + n(y_1 + y_2 - ny_1y_2) = 1 \text{ and hence } (ab, n) = 1 \text{ (why?)}$$

EXAMPLE 8. *For any integer x we have*

$$(a, b) = (b, a) = (a, -b) = (a, b + ax).$$

SOLUTION. We have $\quad (a, b) = (b, a) = (a, -b) = (-a, -b)$.

Suppose $(a,b) = d_1$ and $(a, b + ax) = d_2$. Then d_1 divides a, d_1 divides b and hence d_1 divides $b + ax$. This means that d_1 divides d_2. Similarly we see that d_2 divides d_1. Therefore $d_1 = d_2$.

Definition 5. If $a_1, a_2, \ldots, a_n$ are all different from zero, the least of all the positive common multiples of $a_1, a_2, \ldots, a_n$ is *the least common multiple* or the *l.c.m.* of $a_1, a_2, \ldots, a_n$. We write the *l.c.m.* as $[a_1, a_2, \ldots, a_n]$.

Thus $[a_1, a_2, \ldots, a_n] = \min\{x > 0 \mid a_i \text{ divides } x \text{ for } i = 1, 2, \ldots, n\}$

For example $[6, 9] = 18$, $[5, 7] = 35$, $[12, 18] = 36$.

EXAMPLE 9. *If x is any common multiple of $a_1, a_2, \ldots, a_n$ all different from zero then $[a_1, a_2, \ldots, a_n]$ divides x.*

SOLUTION. Let $[a_1, a_2, \ldots, a_n] = a$. We write $x = aq + r$ with q, r being integers and $0 \le r < a$. Now a_i divides x and a_i divides a for each i. This means that a_i divides $x - aq = r$ for each i. But $0 \le r < a =$ *l.c.m.* of $(a_1, a_2 \ldots a_n)$ implies that $r = 0$. Therefore $x = aq$ or a divides x.

This example shows that if $a = [a_1, a_2, \ldots, a_n]$ then

$\{0, \pm a, \pm 2a, \ldots, \pm na, \ldots\}$ is the set of all common multiples of $a_1, a_2, \ldots, a_n$.

EXAMPLE 10. *For any $m > 0$, $[ma, mb] = m. [a, b]$*

SOLUTION. $[ma, mb]$ is clearly a multiple of m. Let $[ma, mb] = km$ and $[a, b] = l$. Now a divides l, and b divides l implies that ml is a common multiple of ma and mb. Therefore $[ma, mb] = mk$ divides ml or *equivalently* k divides l.

Also mk is a common multiple of ma, mb implies that k is a common multiple of a, b. Therefore l divides k. Thus we have k divides l and l divides k. Further both k and l are positive. Hence $k = l$. In other words $[ma, mb] = mk = ml = m[a, b]$.

EXAMPLE 11. *For any two non zero integers a and b we have $[a, b]\ (a, b) = |ab|$.*

SOLUTION. Without loss of generality we may assume that a and b are positive. First we take $a > 0$, $b > 0$ with $(a, b) = 1$. Suppose $[a, b] = ma = nb$. Then b divides ma and $(b, a) = 1$ implies that b divides m. Therefore $b \le m$ and $ba \le ma = [a, b]$. But ba itself is a common multiple of a, b and hence $ba = [a, b]$. Thus $ab = ab.\ 1 = [a, b]\ (a, b)$. Now assume $a > 0$, $b > 0$, $(a, b) = d > 1$.

Then $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ and hence by the first part of the proof

$$\left[\frac{a}{d}, \frac{b}{d}\right] = \frac{a}{d} \cdot \frac{b}{d}.$$

This gives $d^2\left[\frac{a}{d}, \frac{b}{d}\right] = ab$ or $d\left[d\frac{a}{d}, d\frac{b}{d}\right] = ab$

or $\qquad d[a, b] = ab \quad$ i.e., $\quad (a, b) . [a, b] = ab.$

EXAMPLE 12. *Let a, b, c be integers. The equation $ax + by = c$ has a solution if and only if (a, b) divides c. Also if (x_0, y_0) is a particular solution of $ax + by = c$ then a general solution is given by*

$$x = x_0 + t\frac{b}{(a,b)},\ y = y_0 - t\frac{a}{(a,b)} \quad \text{where} \quad t \in \mathbb{Z}.$$

SOLUTION. Let $(a, b) = d$. It has been already proved in the Corollary to Theorem 3 that $ax + by = c$ has a solution if and only if $d \mid c$. Suppose x_0, y_0 is a particular solution and x_1, y_1 is any other solution. Then $ax_1 + by_1 = c = ax_0 + by_0$.

$$\therefore \qquad \frac{a}{d}(x_1 - x_0) = \frac{b}{d}(y_0 - y_1).$$

Now $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ and $\frac{b}{d}$ divides $\frac{a}{d}(x_1 - x_0)$ implies that $\frac{b}{d}$ divides $x_1 - x_0$.

Hence there exists $t \in \mathbb{Z}$ such that $t\frac{b}{d} = x_1 - x_0$. This gives $\frac{a}{d}\ t\ \frac{b}{d} = \frac{b}{d}(y_0 - y_1)$ or $t\frac{a}{d} = y_0 - y_1$. Thus $x_1 = x_0 + t\frac{b}{d}$ and $y_1 = y_0 - t\frac{a}{d}$.

EXAMPLE 13. *Find all the integral solution of $93x - 27y = 6$.*

SOLUTION. We have $(93, 27) = 3$; and 3 divides 6. Therefore there exist solutions. To find one particular solution x_0, y_0 we first apply the Euclidean algorithm to find a and b such that $a\,93 + b\,27 = 3$. We have

$$93 = 3 \cdot 27 + 12$$

$$\begin{aligned} 27 &= 2 \cdot 12 + 3 \\ 12 &= 4 \cdot 3 + 0 \\ 3 &= 1 \cdot 27 - 2 \cdot 12 \\ &= 1 \cdot 27 - 2\,(1 \cdot 93 - 3 \cdot 27) \\ &= 7 \cdot 27 - 2 \cdot 93 \end{aligned}$$

Thus $\quad -2 \cdot 93 + 7 \cdot 27 = 3$

So $\quad -4 \cdot 93 + 14 \cdot 27 = 6$

In fact, $\quad -4 \cdot 93 - 14 \cdot (-27) = 6.$

Hence $x_0 = -4$ and $y_0 = -14$ constitute a particular solution. The general solution is now given by

$$x = -4 + k\left(\frac{-27}{3}\right), \quad y = -14 - k\left(\frac{93}{3}\right)$$

i.e., $\quad x = -4 - 9k, \qquad y = -14 - 31k$

The table of different solutions may be given by

k	0	1	-1	...
x	-4	-13	5	...
y	-14	-45	17	...

EXAMPLE 14. *If $f(x)$ is a nonconstant polynomial with integral coefficients then f takes some composite values. (i.e., $f(x)$ cannot be a prime for all $x \in \mathbf{Z}$).*

SOLUTION. Let $\quad f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ with $a_i \in \mathbf{Z}$.

Suppose $f(k) = a_0 + a_1k + a_2k^2 + \dots + a_nk^n = m \neq 1$ for some $k \in \mathbf{Z}$. Such a k exists since $f(x)$ is a nonconstant polynomial then

$$f(k+m) = a_0 + a_1\,(k+m) + a_2\,(k+m)^2 + \dots + a_n\,(k+m)^n.$$

Expanding the terms of $f(k+m)$ we find that

$$\begin{aligned} f(k+m) &= (\text{multiple of } m) + a_0 + a_1k + a_2k^2 + \dots + a_nk^n \\ &= (\text{multiple of } m) + f(k) \\ &= (\text{multiple of } m) + m \\ &= \text{multiple of } m = \text{a composite number.} \end{aligned}$$

Thus there exist no nonconstant polynomial $f(x)$ with integral coefficients which takes only prime values.

EXERCISE 2.2

1. Find the greatest common divisors of the following pairs of integers
   (*a*) 537, 765 (*b*) 801, 423 (*c*) 12321, 8658 (*d*) 138, 1740.
2. Compute the *l.c.m.* of (*a*) 27, 30 (*b*) $n, n+1$ (*c*) 1234, 3702.
3. Given that $d_1 = \dfrac{d}{(b,d)}$, $\quad d_2 = \dfrac{d}{(b,d)}$

   show that $\quad \dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad_1 + cd_2}{[b,d]}$.

4. Show that if a and b are nonzero integers

$$(a, b) \text{ divides } [a, b].$$

5. Prove that $(a + b, a - b) \geq (a, b)$ for any two integers.
6. Find integral values of x, y and z if

   (*i*) $243x + 198y = 9$ (*ii*) $61x - 40y = 1$

   (*iii*) $10x - 8y = 42$ (*iv*) $6x + 10y + 15z = -1$.
7. Prove that the product of any three consecutive integers is always divisible by 6.
8. Give an example of four integers which are relatively prime but not relatively prime in pairs.
9. Prove that any set of integers relatively prime in pairs is a set of relatively prime integers.
10. If $(a, 4) = 2$ and $(b, 4) = 2$ find $(a + b, 4)$.
11. For any n prove that $n^5 - n$ is divisible by 30.
12. Prove that if n is odd then $n^2 - 1$ is a multiple of 8.
13. Suppose $(a, b) = [a, b]$ for two integers, solve for a, b.
14. Find all integers n such that n^2 is of the form $3k + 2$.
15. Find all solutions of $(a, b) = 10$, $[a, b] = 100$.
16. Let $a > 0$ and $d > 0$ be two given integers. Prove that there exist integers x, y such that $(x, y) = d$, $xy = a$ if and only if d^2 divides a.
17. If $m > n$, prove that $a^{2^n} + 1$ is a divisor of $a^{2^m} - 1$. Find $(a^{2^m} + 1, a^{2^n} + 1)$ when a, m, n are positive integers and $m \neq n$.
18. If a is prime to b and y, b is prime to x then prove that $ax + by$ is prime to ab.
19. If $(a, b) = 1$ and n is a prime then prove that $(a^n + b^n)/(a + b)$ and $a + b$ have no common factors unless $a + b$ is a multiple of n.
20. If $m = a_1x + b_1y$, $n = a_2x + b_2y$ and $a_1b_2 - a_2b_1 = 1$ then prove that $(m, n) = (x, y)$.
21. If p is a prime and $a^2 - b^2 = p$ solve for a and b.

2.3 THE FUNDAMENTAL THEOREM OF ARITHMETIC

Every integer n can be factorised in at least one way namely $n = n \times 1$. If n is a prime integer then we see that the only factorisations possible are $n = n \times 1$ or $n = (-n)(-1)$. If n is a positive integer which is not a prime then we may write $n = ab$ with a, b being integers such that $1 < a, b < n$. Now consider $a > 1$. If a is not a prime then break it into factors as $a = a_1 a_2$ with $1 < a_1$ and $a_2 < a$. Similarly if b is not a prime, break it into factors as $b = b_1b_2$ with $1 < b_1, b_2 < b$. If we keep on doing this factorisation, ultimately we get $n = p_1 p_2 \ldots p_k$, a product of k prime numbers. Thus any positive n can be written as a product of prime numbers. (Of course this requires a formal proof which we give later).

For example

$$\begin{aligned} 120 &= 2 \times 2 \times 2 \times 3 \times 5 \\ &= 2 \times 3 \times 2 \times 2 \times 5 \\ &= 2 \times 5 \times 3 \times 2 \times 2 \\ &= 2 \times 3 \times 5 \times 2 \times 2 \text{ etc.} \end{aligned}$$

We observe that in all these factorisations, the primes appearing are the same, although the order in which they appear are different. Is this observation true for prime factorisations of all the positive integers bigger than 1?

Theorem 6 (Fundamental Theorem of Arithmetic)

For any integer $n > 1$ there exist primes $p_1 \le p_2 \le \dots \le p_k$ such that $n = p_1 p_2 \dots p_k$. Furthermore, such a factorisation is unique.

Proof. First we shall prove that each positive integer $n > 1$ has a prime factorisation which has already been observed in the discussion preceding the theorem. This we prove by induction on n. Clearly it is true for $n = 2$. Assume that the result is true for all n such that $2 \le n \le k$. This is our induction hypothesis. Consider now $n = k + 1$. If $k + 1$ itself is a prime, there is nothing to prove; its prime factorisation will just consist of $(k + 1)$ itself. Otherwise we write $k + 1 = ab$ with $1 < a, b < k + 1$. Now $1 < a \le k$, $1 < b \le k$ and therefore by our induction hypothesis both a and b have prime factorisations. Let $a = p_1 p_2 \dots p_m$ and $b = q_1 q_2 \dots q_l$ be the prime factorisations of a and b respectively. Then $k + 1 = ab = p_1 p_2 \dots p_m q_1 q_2 , \dots q_l$ is a Prime factorisation of $k + 1$. Thus, by the principle of mathematical induction, every integer $n > 1$ has a factorisation into primes.

Next we have to prove the uniqueness of prime factorisation. The uniqueness is true for $n = 2$. Assume the uniqueness of prime factorisation as stated in the theorem for $2 \le n \le k$. Suppose $k + 1 = p_1 p_2 \dots p_m = q_1 q_2 \dots q_l$ are two factorisations of $k + 1$ into primes such that $p_1 \le p_2 \dots \le p_m$ and $q_1 \le q_2 \le \dots \le q_l$. Since q_1 is a prime dividing the product $p_1 p_2 \dots p_m$ we must have q_1 dividing p_i for some $i \in \{1, 2, \dots m\}$. But p_i itself is a prime and therefore $q_1 = p_i$. Similarly $p_1 = q_j$ for some $j \in \{1, 2, \dots l\}$. We have $p_1 = q_j \ge q_1 = p_i \ge p_1$. Therefore $p_1 = q_j = q_1 = p_i$ or $p_1 = q_1$. Now consider

$$A = \frac{k+1}{p_1} = p_2 p_3 \dots p_m = q_2 q_3 \dots q_l.$$

If $A = 1$, then $k + 1 = p_1 = q_1$ and the uniqueness is verified. If $A > 1$, then $1 < A < k + 1$ and by induction hypothesis A has a unique factorisation into primes in ascending order. This means that $m - 1 = l - 1$ and $p_2 = q_2, p_3 = q_3, \dots, p_m = q_m$. This proves that $(k + 1)$ has a unique factorisation into primes in ascending order. Hence by the principle of mathematical induction, every integer $n > 1$ has a unique factorisation in the form $n = p_1 p_2 \dots p_k$ where each p_i is a prime and $p_1 \le p_2 \le \dots \le p_k$. This proves the Fundamental Theorem of Arithmetic. □

EXAMPLE 1. Let E be the set $\{2, 4, 6, 8, \dots\}$ of positive even integers. We say that $x \in E$ is a prime in E if x is not the product of two elements of E. For example 14 is a prime in E. Here again, every element in E can be written as a product of primes. However, such a prime factorisation in E is not unique! For example $60 = 2 \times 30 = 6 \times 10$ are two different prime factorisations of 60 in E. This example shows that the mere existence of prime factorisation does not imply the uniqueness of such a factorisation.

EXAMPLE 2. *Show that the number of primes in* $\mathbf{N}$ *is infinite.*

SOLUTION. Suppose the number of primes in $\mathbf{N}$ is finite. Let $\{p_1, p_2, \dots, p_n\}$ be the set of primes in $\mathbf{N}$ such that $p_1 < p_2 < \dots < p_n$. Consider $n = 1 + p_1 p_2 \dots p_n$. Clearly n is not divisible by any one of $p_1, p_2, \dots, p_n$. Hence n itself is a prime or n has a prime divisor other than $p_1, p_2 \dots, p_n$. This contradicts that the set of primes is $\{p_1, p_2, \dots, p_n\}$. Therefore the number of primes in $\mathbf{N}$ is infinite.

EXAMPLE 3. *Given any positive integer n, we can uniquely express n as a product of a non-negative power of 2 and an odd number.*

SOLUTION. Let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the unique factorisation of n into primes with $p_1 < p_2 < \dots < p_k$. Then either $p_1 = 2$ or each $p_i > 2$ and hence odd. Therefore $n = 2^{a_1} b$ or $n = c$ where $b = p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$ and $c = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$. When $n = 2^{a_1} b$, each p_i is odd for $i = 2, 3, \dots k$ and hence b is odd. Again when $n = c = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ each p_i is odd and hence c is odd. Thus $n = 2^{a_1} b$ or $n = 2^0.c$ = (a non-negative power of 2) × (an odd number). Uniqueness part is left as an exercise.

EXAMPLE 4. *Given any positive integer k, find k consecutive composite numbers.*

SOLUTION. Now all the positive integers $j \le k + 1$ divide $(k + 1)!$. Therefore $(k + 1)! + j$ is divisible by j for $j = 1, 2, 3, \dots, k + 1$. This gives k consecutive integers, viz., $2 + (k + 1)!, 3 + (k + 1)!, \dots, (k + 1) + (k + 1)!$ which are all composite.

EXAMPLE 5. *Use the unique factorisation theorem to find the l.c.m. and g.c.d. of 136 and 228.*

SOLUTION. We have

$$136 = 8 \times 17 = 2^3 \times 17 \text{ and}$$
$$228 = 2^2 \times 3 \times 19.$$
$$[136, 228] = 2^3 \times 3 \times 17 \times 19 = 7752 \text{ and}$$
$$(136, 228) = 2^2 = 4$$

If $a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ and

$b = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$ where $p_1, p_2, \dots, p_k$ are distinct primes, a_i and b_i are non-negative integers (one can always write two positive integers a and b in the above form) then

$$[a, b] = p_1^{\max(a_1, b_1)} p_2^{\max(a_2, b_2)} \dots p_k^{\max(a_k, b_k)}$$

$$(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \dots p_k^{\min(a_k, b_k)}.$$

EXAMPLE 6. *A positive integer n is a prime if (n, p) = 1 for every prime integer $p \le \sqrt{n}$. In other words a positive integer n is a prime if no prime $p \le \sqrt{n}$ divides n.*

SOLUTION. Let $(n, p) = 1$ for every prime $p \le \sqrt{n}$. Suppose n is not a prime, we may write $n = ab$ with $1 < a \le b$, then $a \le \sqrt{n}$. Any prime p dividing a also divides n and we have $p \le a \le \sqrt{n}$, contradicting our assumption on n. Hence n is a prime.

Note. The above Example 6 enables us to check whether a given number is prime or not. For example if one wants to check whether 187 is a prime, it is enough to check whether 2, 3, 5, 7, 11, 13 divide 187, since these are the only primes less than or equal to $\sqrt{187}$.

EXAMPLE 7. *Let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the unique factorisation of n into a product of distinct primes. Then the number of positive divisors of n is given by*

$$\tau(n) = (a_1 + 1)(a_2 + 1)(a_3 + 1) \dots (a_k + 1).$$

SOLUTION. Further the sum of the divisors $\sigma(n)$ is given by

$$\sigma(n) = \left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right) \dots \left(\frac{p_k^{a_k+1} - 1}{p_k - 1}\right).$$

Any positive divisor d of n must be of the form

$$d = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k} \text{ with } 0 \le b_i \le a_i \text{ for } i = 1, 2, \dots k.$$

Now consider the product

$$(1 + p_1 + p_1^2 + \dots + p_1^{a_1})(1 + p_2 + p_2^2 + \dots + p_2^{a_2}) \dots$$
$$(1 + p_k + p_k^2 + \dots + p_k^{a_k}) \quad (*)$$

Any typical term in this product is of the form

$$x = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k} \text{ with } 0 \le b_i \le a_i \text{ for } i \in \{1, 2, \dots k\}$$

This means that the number of positive divisors of n is the number of terms in the above product. This gives

$$\tau(n) = (a_1 + 1)(a_2 + 1) \dots (a_k + 1).$$

[If we multiply $(x_1 + x_2 + \dots + x_m)$ with $(y_1 + y_2 + \dots + y_n)$ then a typical term is $x_i y_j$ and the number of such terms is mn. In the above product (*), i^{th} bracket $(1 + p_i + p_i^2 + \dots + p_i^{a_i})$ contains $a_i + 1$ terms. Hence the total number of terms in the product (*) is

$$(a_1 + 1)(a_2 + 1) \dots (a_k + 1)].$$

Again, using our earlier observation, we get

$\sigma(n)$ = the sum of positive divisors of n

$$= (1 + p_1 + p_1^2 + \dots + p_1^{a_1})(1 + p_2 + p_2^2 + \dots + p_2^{a_2}) \dots (1 + p_k + p_k^2 + \dots + p_k^{a_k})$$

$$= \left(\frac{p_1^{a_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{a_2+1} - 1}{p_2 - 1}\right) \dots \left(\frac{p_k^{a_k+1} - 1}{p_k - 1}\right)$$

(see Example 1 of Section 2.1 for the sums of the form $(1 + p_i + p_i^2 + \dots + p_i^{a_i})$.)

For example $\tau(60) = \tau(2^2 . 3^1 . 5^1) = (2 + 1)(1 + 1)(1 + 1) = 12$

$$\sigma(60) = \left(\frac{2^3 - 1}{2 - 1}\right)\left(\frac{3^2 - 1}{3 - 1}\right)\left(\frac{5^2 - 1}{5 - 1}\right) = 7.4.6 = 168$$

In fact the positive divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 which are 12 in number; and their sum = 168.

EXAMPLE 8. *Find the number of ways in which a positive integer n can be written as a product of two positive integers, including n and 1.*

SOLUTION. Case (i) $n = p_1^{2a_1} p_2^{2a_2} \dots p_k^{2a_k}$ is a perfect square with the above prime factorisation. Then $\tau(n) = (2a_1 + 1)(2a_2 + 1) \dots (2a_k + 1)$ = an odd integer.

Let $1 = d_1 < d_2 < d_3 < \dots < d_{\tau(n)} = n$ be the distinct positive divisors of n in the ascending order. Suppose $\tau(n) = 2l - 1$, then the different factorisations are as follows $d_1 d_{\tau(n)}, d_2 d_{\tau(n)-1}, \dots, d_r d_{\tau(n)-r+1}, \dots, d_l d_l$.

There are l factorisations in all. We have $l = \dfrac{\tau(n) + 1}{2}$. Thus when n is a perfect square we have $\dfrac{\tau(n) + 1}{2}$ factorisations of n in the form $n = ab$, $1 \le a \le b \le n$.

Case (ii) n is not a perfect square.

Let $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ be the unique prime factorisation of n. Then at least one a_j is odd since n is not a perfect square. This implies that $\tau(n) = (a_1 + 1)(a_2 + 1) \ldots (a_n + 1)$ is an even number.

If $1 = d_1 < d_2 < d_3 < \ldots < d_{\tau(n)} = n$ are the positive divisors of n then the distinct factorisations of n are $d_1 d_{\tau(n)}, d_2 d_{\tau(n)-1}, d_3 d_{\tau(n)-2}, \ldots, d_l d_{l+1}$ where $\tau(n) = 2l$. Thus when n is not a perfect square we have $l = \dfrac{\tau(n)}{2}$ factorisations in the form $n = ab$.

For example if $n = 36 = 2^2 \times 3^2$, $\tau(n) = (2+1)(2+1) = 9$. The divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36. The distinct factorisations are $1 \times 36, 2 \times 18, 3 \times 12, 4 \times 9, 6 \times 6$ which are $5 = \dfrac{10}{2} = \dfrac{\tau(n)+1}{2}$ in number.

When $n = 60 = 2^2 \times 3 \times 5$, we have $\tau(n) = (2+1)(1+1)(1+1) = 12$. The divisors are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 and the distinct factorisations are $1 \times 60, 2 \times 30, 3 \times 20, 4 \times 15, 5 \times 12$ and 6×10 which are $6 = \dfrac{12}{2} = \dfrac{\tau(n)}{2}$ in number.

EXAMPLE 9. *In Example 8 find the number of ways in which n can be written as a product of two factors, which are relatively prime to each other.*

SOLUTION. Let $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ be the unique factorisation of n and $n = ab$ with $(a, b) = 1$. Then if p_j divides a then p_j does not divide b. This means that if P_j divides a then $p_j^{a_j}$ also divides a. Therefore any such factor of a or b must be a term in the expansion of $(1 + p_1^{a_1})(1 + p_2^{a_2}) \ldots (1 + p_k^{a_k})$ and vice versa. The number of terms in the above product is 2^k, and hence the number of factors a of n of the form $n = ab$ with $(a, b) = 1$ is 2^k. If $1 = d_1 < d_2 < d_3 < \ldots < d_{2^k} = n$ are these factors then the different factorisations of the required form are $d_1 d_{2^k}, d_2 d_{2^k - 1}, \ldots d_{2^{k-1}} d_{2^{k-1}+1}$ which are 2^{k-1} in number.

Thus, the number of ways in which n can be expressed as a product of two relatively prime factors is 2^{k-1}.

For example consider $n = 120 = 2^3 \times 3 \times 5$. Comparing with our calculations above, we have $k = 3$. Therefore, there are $2^{3-1} = 4$ ways in which 120 can be expressed in the desired form. They are $1 \times 120, 3 \times 40, 5 \times 24, 8 \times 15$.

Definition 6. Define $[x]$ = integral part of x

= the greatest integer less than or equal to x for $x \in \mathbf{R}$.

For example $\left[\dfrac{3}{2}\right] = 1, [7.893] = 7, [\sqrt{2}] = 1, \left[\dfrac{-3}{2}\right] = -2.$

If x, y are integers and $x = qy + r$ with $0 \le r < y$ then $\left[\dfrac{x}{y}\right] = q.$

EXAMPLE 10. *If $a_1, a_2, \ldots, a_n$ are integers with $s = a_1 + a_2 + \ldots + a_n$ then*

$$\left[\frac{s}{a}\right] \ge \left[\frac{a_1}{a}\right] + \left[\frac{a_2}{a}\right] + \ldots + \left[\frac{a_n}{a}\right] \textit{ for any integer } a > 0.$$

SOLUTION. We may write $a_j = aq_j + r_j$ with $0 \ r_j \leq a$ for $j = 1, 2, \ldots n$ using division algorithm. Then $s = a_1 + a_2 + \ldots + a_n$ gives $s = a(q_1 + q_2 + \ldots + q_n) + (r_1 + r_2 + \ldots + r_n)$.

$$\therefore \qquad \left[\frac{s}{a}\right] = \left[q_1 + q_2 + \cdots + q_n + \frac{(r_1 + r_2 + \cdots + r_n)}{a}\right]$$

$$\geq q_1 + q_2 + \cdots + q_n = \left[\frac{a_1}{a}\right] + \left[\frac{a_2}{a}\right] + \left[\frac{a_n}{a}\right]$$

EXAMPLE 11. *Find the largest power of a prime $p \leq n$, dividing $n!$.*

SOLUTION. We have $n! = 1 \times 2 \times 3 \times 4 \ldots . p \ldots 2p \ldots . p^2 \ldots . (p+1)p \ldots . p^3 \ldots . n$. In the above product there are $[n/p]$ terms which are divisible by p, and among these there are $[n/p^2]$ terms which are divisible by p^2 and so on. Therefore, the highest power of p that divides $n!$ is

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^k}\right]$$

where p^k is the largest power of p which is less than or equal to n.

EXAMPLE 12. *Find the highest power of 5 that divides 518!*

SOLUTION. The highest power of 5 less than 518 is $125 = 5^3$. Therefore the highest power of 5 that divides 518! is

$$\left[\frac{518}{5}\right] + \left[\frac{518}{25}\right] + \left[\frac{518}{125}\right] = 103 + 20 + 4 = 127.$$

EXAMPLE 13. *Find the number of zeros that appear at the end in the representation of 158! in base 10.*

SOLUTION. If 10^k is the highest power of 10 that divides 158!, then we will have k zeros at the end of the decimal representation of 158! Now $10^k = 2^k . 5^k$. The highest power of 2 that divides 158!

$$= \left[\frac{158}{2}\right] + \left[\frac{158}{4}\right] + \left[\frac{158}{8}\right] + \left[\frac{158}{16}\right] + \left[\frac{158}{32}\right] + \left[\frac{158}{64}\right] + \left[\frac{158}{128}\right]$$

$$= 79 + 39 + 19 + 9 + 4 + 2 + 1 = 153.$$

The highest power of 5 that divides 158! is given by

$$\left[\frac{158}{5}\right] + \left[\frac{158}{25}\right] + \left[\frac{158}{125}\right] = 31 + 6 + 1 = 38.$$

The highest power of 10 that divides 158! = min (153, 38) = 38.

Hence there are 38 zeros at the end of the decimal representation of 158!.

EXAMPLE 14. *Show that the product of any n consecutive integers is always divisible by n!.*

SOLUTION. Consider any such product $(k+1)(k+2) \ldots (k+n)$.

We have $$\frac{(k+1)(k+2)\cdots(k+n)}{n!} = \frac{(k+n)!}{k!\,n!}$$

Let p be any prime divisor of $k!\,n!$. Then the highest power of p that divides $k!\,n!$ is

$$\left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \cdots + \left[\frac{k}{p^a}\right] + \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^b}\right]$$

where p^a is the highest power of p less than or equal to k and p^b is the highest power of p less than or equal to n. We may assume that $a \le b$. As seen in Example 10, we have

$$\left[\frac{k}{p}\right]+\left[\frac{n}{p}\right]\le\left[\frac{k+n}{p}\right], \left[\frac{k}{p^2}\right]+\left[\frac{n}{p^2}\right]\le\left[\frac{k+n}{p^2}\right] \text{ and so on.}$$

$$\left[\frac{k}{p}\right]+\left[\frac{k}{p^2}\right]+\cdots+\left[\frac{k}{p^a}\right]+\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots\left[\frac{n}{p^b}\right]$$

$$\le\left[\frac{k+n}{p}\right]+\left[\frac{k+n}{p^2}\right]+\cdots+\left[\frac{k+n}{p^a}\right]+\cdots\left[\frac{k+n}{p^c}\right]$$

where p^c is the highest power of p less than or equal to $(k+n)$.

This means that if p^x divides $n!\,k!$ then p^x divides $(k+n)!$

Therefore $\dfrac{(k+n)!}{k!\,n!}$ is an integer.

EXAMPLE 15. *Prove that there are infinitely many primes of the form $4k + 3$ with $k \in \mathbf{Z}$.*

SOLUTION. Let $n = 4k + 3 > 0$ and $\mathbf{S} = \{d > 0 \mid d$ is a divisor of n of the form $4m + 3$ with $m \in \mathbf{Z}\}$. Then $n \in \mathbf{S}$ and therefore $\mathbf{S} \neq \varphi$. We can speak of the least integer in $\mathbf{S}$ and let $p = \min\{d \mid d \in \mathbf{S}\}$. It is clear that $p > 1$. Suppose $p = ab$ with $1 < a, b < p$, then $p = 4m + 3$ is odd and hence both a and b must be odd integers. If $a = 4k_1 + 1$, $b = 4k_2 + 1$, then $ab = 4(4k_1k_2 + k_1 + k_2) + 1$. But ab is of the form $4m + 3$. Therefore, either a or b is of the form $4l + 3$. We may assume that $a = 4l + 3$. Then $a < p$ and a divides p implies that a divides n, contradicting the minimality of p. Hence p must be a prime. This means that every positive integer n of the form $4k + 3$ has a prime divisor of the same form. Now, for any $n \in \mathbf{N}$, $4n! - 1$ is of the form $4k + 3$ and hence $4n! - 1$ has a prime divisor p of the same form. But j does not divide $4n! - 1$ for $2 \le y \le n$. Therefore $p > n$. In other words, given any positive integer n, there exists a prime number p of the form $4k + 3$ with $p > n$. Hence there are infinitely many primes of the form $4k + 3$.

EXAMPLE 16. If x and y are prime numbers which satisfy $x^2 - 2y^2 = 1$, solve for x and y.

SOLUTION. $x^2 - 2y^2 = 1$ gives $x^2 - 2y^2 + 1$ and hence x must be an odd number. If $x = 2n + 1$, then $x^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2y^2 + 1$. Therefore $y^2 = 2n(n + 1)$. This means that y^2 is even and hence y is an even integer. Now, y is also a prime implies that $y = 2$. This gives $x = 3$. Thus the only solution is $x = 3$, $y = 2$.

EXERCISE 2.3

1. An integer n whose prime factorisation is $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ is a perfect square if and only if each a_i is even and n is a perfect cube if and only if each a_i is a multiple of 3; and in general n is a perfect mth power if and only if each a_i is a multiple of m.
2. Find the smallest positive integer n such that $n/2$ is a perfect square, $n/3$ is a perfect cube, $n/5$ is a perfect fifth power.
3. Find all positive integers n such that $2^8 + 2^{11} + 2^n$ is a perfect square.
4. How many solutions are there in $\mathbf{N} \times \mathbf{N}$ to the equation $1/x + 1/y = 1/1995$?

5. If m, n, k are any three positive integers prove that
$(m, n)\,(m, k)\,(n, k)\,[m, n, k]^2 = [m, n]\,[m, k]\,[n, k]\,(m, n, k)^2$.
6. Find the number of zeroes at the end of 1000!
7. Let $X = \{x \mid x = 1 + 1/2 + 1/3 + \dots + 1/n, n \in N\}$. Find $X \cap N$.
8. Find the smallest number with 28 divisors?
9. If a and b are coprime integers then prove that $((a + b)^m, (a - b)^m) \leq 2^m$ and $(a^n + b^n, a^n - b^n) \leq 2$.
10. Prove that there are infinitely many primes of the form $6n - 1$.
11. Show that $N = 101010...101$ is not a prime except when N is 101.
12. Prove that there are infinitely many sets of five consecutive positive integers a, b, c, d, e such that $a + b + c + d + e$ is a perfect cube and $b + c + d$ is a perfect square.
13. Let $A = \{n \in N \mid n$ is the sum of seven consecutive integers $\}$
$B = \{n \in N \mid n$ is the sum of eight consecutive integers $\}$
$C = \{n \in N \mid n$ is the sum of nine consecutive integers $\}$
Find $A \cap B \cap C$.
14. If 'a' is not a multiple of a prime p, then prove that there is an integer b such that $p^b - 1$ is a multiple of a.

PROBLEMS

1. Prove that
$S = 1 - 2^2 + 3^2 - 4^2 + 5^2 \dots + (-1)^{n-1} n^2 = (-1)^{n-1} n(n + 1)/2$.
2. Prove that $1.1! + 2.2! + 3.3! + \dots + n.n! = (n + 1)! - 1$.
3. Prove that for any positive integer $n > 1$,

$$1/(n + 1) + 1/(n + 2) + \dots + 1/2n > 13/24.$$

4. Use induction to prove that

$$1 + 1\sqrt{2 + 1}/\sqrt{3} + \dots + 1/\sqrt{n} < 2\sqrt{n}.$$

5. Use induction to prove that $2!4!\dots(2n)! > ((n + 1)!)^n$.
6. Prove that in any party the number of people who have made an odd number of handshakes is always even.
7. Prove that every positive integer having 3^m equal digits is divisible by 3^m.
8. $(n + 1)$ numbers are picked at random from the $2n$ integers $1, 2 \dots 2n$. Prove that among the numbers picked we can find at least two, one of which is divisible by the other.
9. The plane is divided into regions by drawing a finite number of straight lines. Show that it is possible to colour each of these regions red or green in such a way that no two adjacent regions have the same colour.
10. If each person in a group of n people is a friend of at least half the people in the group, then prove that it is possible to seat them in a circle so that every one sits next to a friend of his/hers.
11. Prove that for a positive integer n,

$$11^{n+2} + 12^{2n+1} \text{ is divisible by } 133.$$

12. Prove by induction that $\sqrt{2}$ is irrational.
13. Prove that $x^2 + y^2 = z^n$ has a solution in N, for all $n \in N$.
14. Assuming that for every integer $n > 1$ there is a prime number between n and $2n$, prove that every positive integer can be written as a sum of distinct primes. (For this problem we treat 1 as a prime.)

15. A positive integer decreases an integral number of times when its last digit is deleted. Find all such numbers.

16. If the leading digit of a positive integer is deleted, the number gets reduced by 57 times. Find all such numbers. Find all positive integers such that if the leading digit is deleted the number gets reduced by 58 times.

17. Show that $3x^{10} - y^{10} = 1991$ has no integral solutions.

18. Find n if $2^{200} - 31.2^{192} + 2^n$ is a perfect square.

19. If a and b are integers and 3 divides $a^2 + b^2$, show that 3 divides a and b.

20. Find all integral solutions of $x^4 + y^4 + z^4 - w^4 = 1995$.

21. A positive integer gets reduced by nine times when one of its digits is deleted and the resultant number is divisible by 9. Prove that to divide the resultant number by 9, it is again sufficient to delete one of its digits. Find all such numbers.

22. Let n be a positive integer and m be a number having the same digits as that of n, but arranged in some other order. Prove that if $n + m = 10^{10}$, then n is divisible by 10.

23. Prove that for any integer n we have $n^k - n$ is divisible by k, where $k = 3, 5, 7, 11, 13$.

24. Prove that $5^{6n} - 3^{6n}$ is divisible by 152 for $n \in N$.

25. Prove that for any two integers a and b $ab(a^{60} - b^{60})$ is divisible by 56786730.

26. Prove that $n!/a!b! \ldots k!$ is an integer if $a + b + \ldots k \leq n$.

27. Prove that in $(n!)!$ is divisible by $(n!)^{(n-1)!}$.

28. Prove that for any positive integer n, $1^n + 2^n + 3^n + 4^n$ is divisible by five, if and only if, n is not divisible by 4.

29. Consider $S = \{a, a + d, a + 2d, \ldots\}$ where a and d are positive integers. Show that there are infinitely many composite numbers in S.

30. Prove that for any positive integers m and n, there exist a set of n consecutive positive integers each of which is divisible by a number of the form a^m, where a is some integer in N.

Chapter

3

GEOMETRY—STRAIGHT LINES AND TRIANGLES

3.1 STRAIGHT LINES

In the mathematical development of any branch of science, each definition of a general notion or concept involves other notions and relations. To avoid this logical hurdle we allow certain primitive concepts and relations as undefined concepts and relations. In fact all our theorems are of the form "Given that A is true it is implied that B is true". We have a hypothesis and we try to prove our proposition using certain axioms involving the undefined primitive concepts. Euclid, when he developed his geometry, did not clearly state what his primitive concepts were; but built his geometry based on certain axioms and postulates. Some of his primitive concepts were (1) the notion of a point, (2) the idea of intermediacy (*i.e.*, between any two points on a straight line there is another point. Compare this with the completeness axiom of the real numbers), (3) the idea of congruence. We do not intend to give any logical introduction to the Foundations of Geometry. Instead we give a reasonably systematic development of Euclidean Geometry and go on to prove some of the beautiful theorems of Euclid's Geometry. We fix a plane and study the geometry of straight lines, triangles and other geometric objects lying on this plane. Hereafter, all our geometric objects, unless otherwise stated, lie on this fixed plane.

By a line segment AB we mean the geometric figure consisting of two points A and B on a straight line and all the points lying between them. It is denoted by AB. We note that if three points A, B and C are taken on a straight line, exactly one of A, B, C lies between the other two. Also, a straight line l divides a plane into two disjoint half planes H_1 and H_2 such that the plane is the disjoint union of H_1, H_2 and l. If the ends of a line segment AB both belong to the same half plane, then AB does not intersect the line l. If the ends A and B are in different half planes demarkated by l then AB intersects the straight line l. We use the following notations:

$\overleftrightarrow{AB}$ – for the straight line determined by A and B

$\overrightarrow{AB}$ – for the ray AB with vertex at A

$\overline{AB}$ – for the line segment AB

AB – for the length of the line segment AB.

A point A on a straight line l (Fig. 1) divides the straight line into two rays, namely $\overrightarrow{Al_1}$ and $\overrightarrow{Al_2}$. (Note that $\overrightarrow{Al_1}$ is not a line segment, it is a half line with A as its initial point).

The rays of the straight line l into which it is broken by the point A will be called *supplementary* rays. In Fig. 3.1, the rays $\overrightarrow{Al_1}$ and $\overrightarrow{Al_2}$ are supplementary rays.

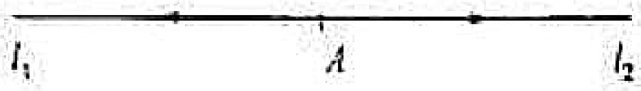


Fig. 3.1

For any three points A, B and C we have the distance $|AC| \leq$ distance $|AB|$ + distance $|BC|$. Here distance $|AB|$ stands for the distance between the points A and B. Two distinct points determine a straight line and two distinct straight lines have either one common point or none.

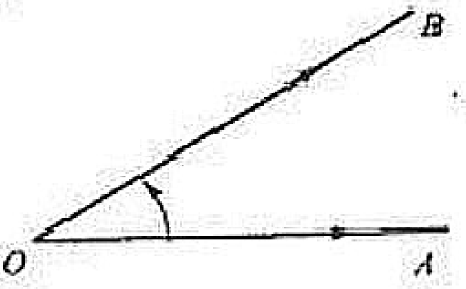


Fig. 3.2

An angle is a figure which consists of two different rays with a common origin, as in Fig. 3.2. For the angle AOB in Fig. 3.2, O is the vertex of the angle and the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ are known as the sides of the angle AOB. If the sides OA, OB are supplementary then the angle AOB is a straight angle and its measure is taken to be 180 degrees. We write angle AOB as $\angle AOB$. Other units in measuring angles are minutes and seconds. We have 60 minutes = 1 degree and 60 seconds = 1 minute. We use the symbols $\theta°$, θ', θ'' to mean θ degrees, θ minutes and θ seconds respectively.

While measuring lengths of line segments and angles we observe the following fundamental principles or rules.

(1) Every line segment has a positive length.

(2) If a point C on a straight line l lies between the points A and B on l, then the length of the line segment AB is equal to the sum of the lengths of the line segments AC and CB.

(3) Every angle has a certain magnitude. The measure of a straight angle is 180°.

(4) If a ray OC lies between the sides of the angle AOB as in Fig. 3.3, then $\angle AOB = \angle AOC + \angle COB$.

(5) In measuring the angles we follow the convention, namely the angles which are anticlockwise like $\angle AOB$, $\angle AOC$ and $\angle COB$ of Fig. 3.3, are taken to be positive; and the angles BOA, COA, BOC of Fig. 3.4, which are clockwise are taken to be negative.

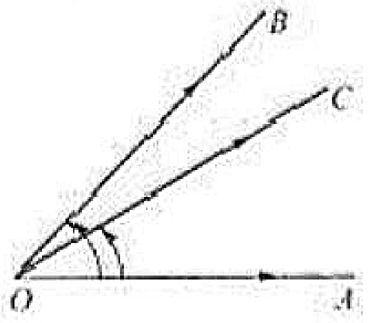


Fig. 3.3

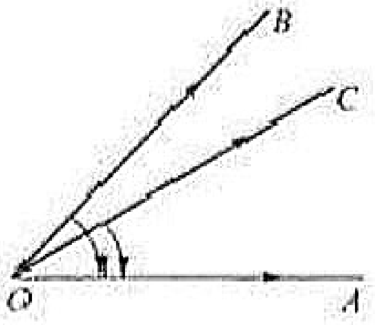


Fig. 3.4

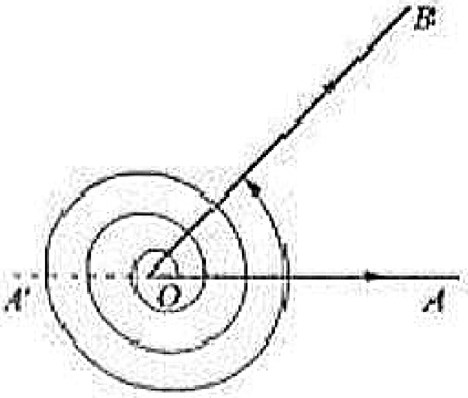


Fig. 3.5

Angles θ for which $0 < \theta^\circ < 90^\circ$ are called acute angles. Angles for which $90^\circ < \theta^\circ < 180^\circ$ are called obtuse angles. Two angles α, β with $0 \le \alpha^\circ, \beta^\circ \le 180^\circ$ are supplementary if $\alpha + \beta = 180^\circ$; are complementary if $\alpha + \beta = 90^\circ$.

Definition 1. If two angles AOB and BOC have a common vertex O, a common arm $\overrightarrow{OB}$ and if they are situated on the opposite sides of their common arm $\overrightarrow{OB}$, then they are called *adjacent angles.*

$\angle AOB$ and $\angle BOC$ are adjacent angles in Fig. 3.13.

Note. Recall that all our geometric objects lie on a fixed plane. This standing assumption also includes the geometric objects appearing in definitions.

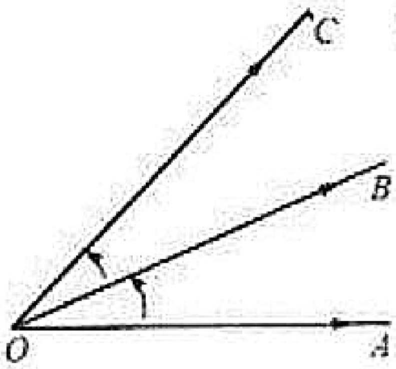


Fig. 3.13

Theorem 1. If a straight line stands on another straight line then the sum of the two adjacent angles is two right angles.

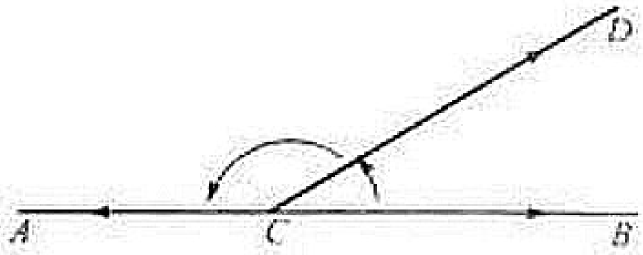


Fig. 3.14

Proof. The straight line CD stands on the straight line $\overleftrightarrow{ACB}$. It is required to prove that the sum of the adjacent angles BCD and DCA is 180°. By hypothesis, $\overleftrightarrow{ACB}$ is a straight line and hence $BCA = 180^\circ$; and the rays $\overrightarrow{CB}, \overrightarrow{CA}$ are supplementary rays. The ray $\overrightarrow{CD}$ lies between the sides $\overrightarrow{CB}, \overrightarrow{CA}$ of the straight angle BCA and hence $\angle BCD + \angle DCA = 180^\circ$. ❑

The converse of Theorem 1 is also true, which we state as follows.

Theorem 2. If the sum of two adjacent angles AOC and COB with the common arm $\overrightarrow{OC}$ is two right angles, then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are supplementary rays.

Proof. We are given that $\angle AOC + \angle COB = 2$ right angles $= 180^\circ$. It is required to prove that BOA is a straight line. Now, extend the line segment $\overrightarrow{BO}$ to D such that BOD is a straight line. By construction BOD is a straight line and the straight line OC stands on it. Therefore by Theorem 1, $\angle DOC + \angle COB = 180^\circ$. Also, by our hypothesis

$\angle AOC + \angle COB = 180°$. This gives $\angle AOC + \angle COB = \angle DOC + \angle COB$ or $\angle AOC = \angle DOC$. This means that the ray $\overrightarrow{OD}$ coincides with the ray $\overrightarrow{OA}$. In other words, BOA is a straight line. ☐

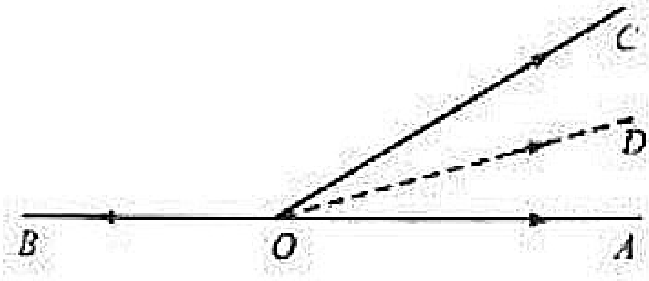


Fig. 3.15

Note. When we write $\angle AOC$ we follow our convention that anticlockwise angles are positive. Hence a possibility like Fig. 3.16 is ruled out.

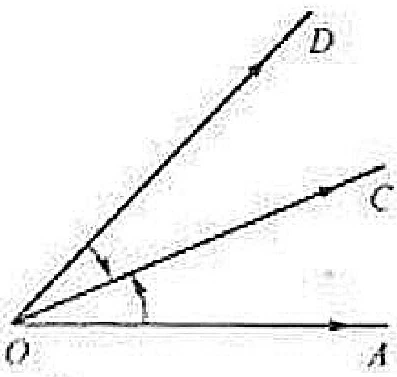


Fig. 3.16

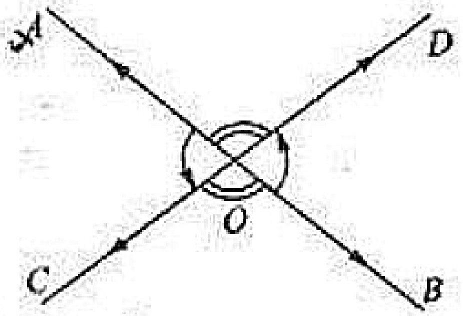


Fig. 3.17

Definition 2. Two angles are *vertically opposite angles* if the sides of one of them are supplementary rays of the sides of the other.

In Figure 3.17, $\angle BOD$ and $\angle AOC$ are vertically opposite angles. Also $\angle DOA$ and $\angle COB$ are vertically opposite angles.

Theorem 3. If two straight lines intersect the vertically opposite angles so formed are equal.

Proof. The straight lines AOB and COD intersect at the point O. $\angle BOD$ and $\angle AOC$ is one pair of vertically opposite angles so formed; and the other pair is $\angle DOA$ and $\angle COB$. It is required to prove that $\angle BOD = \angle AOC$ and $\angle DOA = \angle COB$. Now, AOB is a straight line and the straight line OD stands on it. Therefore by Theorem 1,

$$\angle BOD + \angle DOA = 180°. \tag{1}$$

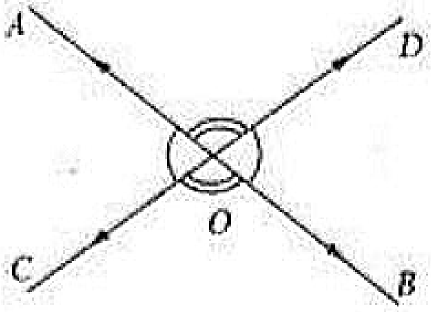


Fig. 3.18

Again by Theorem 1, applied to the line COD and the straight line OA standing on it we get

$$\angle DOA + \angle AOC = 180° \qquad (2)$$

From (1) and (2) we see that $\angle BOD = \angle AOC$.

Similarly, we observe that $\angle DOA = \angle COB$.

Hence the theorem. ❑

EXERCISE 3.1

1. What is the angle in degrees between the hands of a watch at (*i*) 4 O' clock (*ii*) 5 hrs and 45 mts.
2. What angles do (*i*) the minute hand (*ii*) the hour hand and (*iii*) the seconds hand turn through in 20 minutes.
3. A straight line segment $\overline{AB}$ is bisected at C and produced to D. Show that $AD + BD = 2CD$.
4. A straight line segment $\overline{AB}$ is bisected at C and D is any point on CB. Prove that $AD - DB = 2CD$.
5. In Fig. 3.17 prove that
   (*i*) the bisectors of the angles DOA and DOB are at right angles.
   (*ii*) the bisector of $\angle DOA$ when produced also bisects $\angle COB$.
6. $\angle XOA$ and $\angle XOB$ are angles on the same side of $\overrightarrow{OX}$ and $\overrightarrow{OC}$ bisects $\angle AOB$. Prove that $\angle XOA + \angle XOB = 2\angle XOC$.
7. AOX, XOB are adjacent angles, in which $\angle AOX > \angle XOB$; $\overrightarrow{OC}$ bisects $\angle AOB$. Prove that $\angle AOX - \angle XOB = 2\angle COX$. (Compare the problems 3, 4 with problems 6,7)
8. If the bisectors of two adjacent angles are perpendicular to one another, then prove that the adjacent angles are formed by two intersecting straight lines.

3.2 CONGRUENCE OF TRIANGLES

Definition 3. If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, then the two triangles are *congruent*.

In other words two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are congruent if $A_1B_1 = A_2B_2$, $B_1C_1 = B_2C_2$, $C_1A_1 = C_2A_2$, $\angle A_1 = \angle A_2$, $\angle B_1 = \angle B_2$ and $\angle C_1 = \angle C_2$. We write $\Delta A_1B_1C_1 \equiv A_2B_2C_2$ to mean that the two triangles are congruent. We observe that

(1) $\Delta ABC \equiv \Delta ABC$ for any triangle ABC

(2) If $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$, then $\Delta A_2B_2C_2 \equiv \Delta A_1B_1C_1$

(3) If $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$ and $\Delta A_2B_2C_2 \equiv \Delta A_3B_3C_3$ then $\Delta A_1B_1C_1 \equiv \Delta A_3B_3C_3$.

We take the following test for congruence of two triangles as an axiom.

SAS test: Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.

This says that, if in two triangles, $A_1B_1C_1$ and $A_2B_2C_2$ $A_1B_1 = A_2B_2$, $A_1C_1 = A_2C_2$ and $\angle A_1 = \angle A_2$, then the two triangles are congruent. This is known as the "*Side Angle Side*" test.

Theorem 4. The sum of any two angles of a triangle is less than a straight angle.

Proof. Let D be the mid point of the side BC of a triangle ABC. Produce AD to E such that $AD = DE$ (Fig. 3.19). Then by the SAS test the triangles ADB and EDC are congruent. Therefore, $\angle ABC = \angle ECD$ and hence $\angle ABC + \angle BCA = \angle ECD + \angle BCA = \angle ECA < 180°$ (note that E cannot lie on AC since the two distinct lines AD and AC have only one common point, namely A). Thus, the sum $\angle ABC + \angle BCA$ is less than two right angles. Similar arguments show that $\angle C + \angle A$ and $\angle A + \angle B$ are also each less than $180°$. □

Corollary 1. Any exterior angle of a triangle is greater than any of the two non-adjacent interior angles.

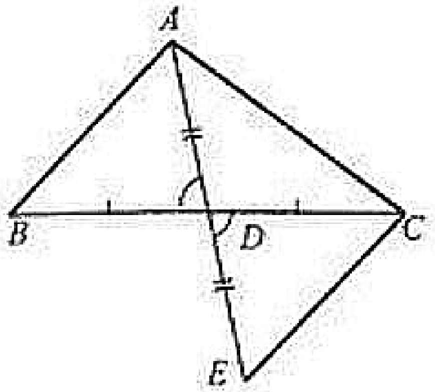


Fig. 3.19

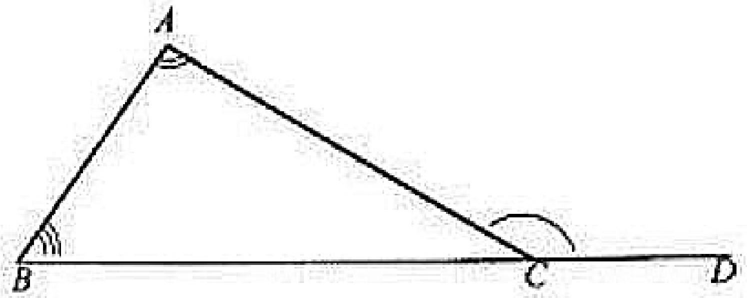


Fig. 3.20

Proof. Let ABC be any triangle and D be a point on BC produced as shown in Fig. 3.20. We want to prove that the exterior angle DCA is bigger than each of the non-adjacent interior angles A and B. By Theorem 1, $\angle DCA + \angle ACB = 180°$. By the theorem (Theorem 4) $\angle BCA + \angle CAB < 180° = \angle DCA + \angle BCA$. Therefore $\angle CAB < \angle DCA$. Again $\angle ABC + \angle BCA < 180° = \angle DCA + \angle BCA$ implies that $\angle ABC < \angle DCA$. Thus the exterior angle DCA is bigger than each of $\angle A$ and $\angle B$ of ΔABC. □

Corollary 2. In any triangle ABC, at most one of the angles A, B, C can be obtuse.

Proof. Immediate from Theorem 4. □

Theorem 5. (ASA theorem)

Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other.

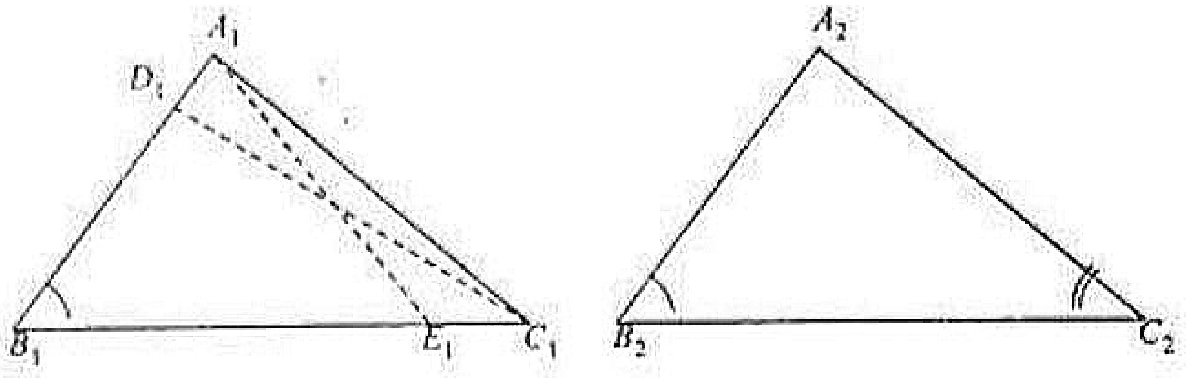


Fig. 3.21

Proof. Case (*i*) Suppose in triangles $A_1B_1C_1$ and $A_2B_2C_2$ we have $\angle B_1 = \angle B_2$, $\angle C_1 = \angle C_2$ and $B_1C_1 = B_2C_2$. Take D_1 on B_1A_1 such that $B_1D_1 = B_2A_2$ (we may assume without loss of generality $A_2B_2 \le A_1B_1$) (Fig. 3.21). Then in the two triangles $B_1C_1D_1$ and $B_2C_2A_2$, we have, the two sides B_1C_1 and B_1D_1 of $\Delta B_1C_1D_1$ equal in length to the two sides B_2C_2 and B_2A_2 respectively of $\Delta B_2C_2A_2$. Further the included angles B_1 and B_2 are equal. Therefore by *SAS* test, the two triangles are congruent. This means that $\angle B_1C_1D_1 = \angle B_2C_2A_2$. But $\angle B_2C_2A_2 = \angle B_1C_1A_1$ by our assumption. Therefore $\angle B_1C_1D_1 = \angle B_1C_1A_1$ and hence by our fundamental principle of measuring angles, the ray C_1D_1 coincides with the ray C_1A_1. This in turn implies D_1 coincides with A_1. Thus $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$. □

Case (*ii*) $\angle B_1 = \angle B_2$, $\angle C_1 = \angle C_2$ and the side A_1B_1 of $\Delta A_1B_1C_1$ = side A_2B_2 of $\Delta A_2B_2C_2$. Take E_1 on B_1C_1 such that $B_1E_1 = B_2C_2$ (as in Fig. 3.21; again without loss of generality we may assume that $B_1C_1 > B_2C_2$). Then the two triangles $B_1E_1A_1$ and $B_2C_2A_2$ are congruent by the *SAS* test. Now, for $\Delta A_1E_1C_1$, the exterior angle $A_1E_1B_1$ = the interior angle $A_1C_1E_1$ which is against the corollary 1 to Theorem 4 unless the point E_1 concides with C_1 in which case $\Delta A_1B_1C_1 \equiv \Delta A_2B_2C_2$. □

Theorem 6. If two sides of a triangle are equal then the angles opposite to these sides are equal.

Proof. In $\angle ABC$ let $AB = AC$. Compare the two triangles ABC and ACB. We have $AB = AC$, $AC = AB$ and $\angle BAC = \angle CAB$. Therefore by the *SAS* test $\Delta ABC \equiv \Delta ACB$ and hence $\angle ABC = \angle ACB$.

Aliter. Suppose AD bisects $\angle BAC$ (Fig. 3.22) meeting BC at D. In the triangles ABD and ACD, we may apply the *SAS* test to get $\Delta ABD \equiv \Delta ACD$. Therefore $\angle ABD = \angle ACD$ or $\angle B = \angle C$ in ΔABC. □

Definition 4. A triangle in which two sides are equal is an *isosceles triangle*.

Theorem 7. If in a triangle two angles are equal, then it is isosceles.

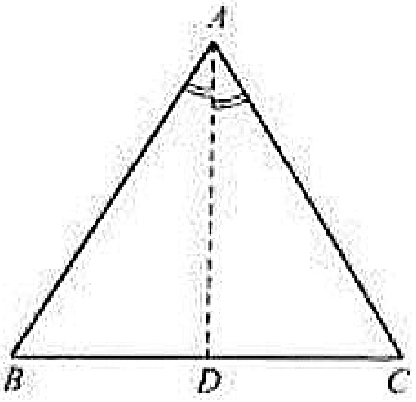


Fig. 3.22

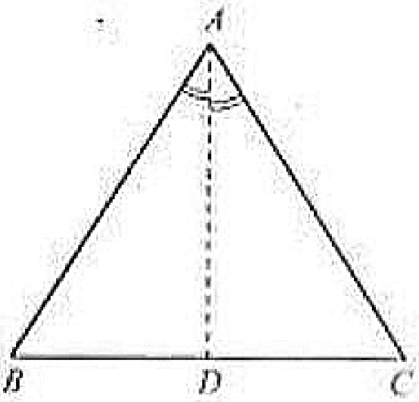


Fig. 3.23

Proof. Let $\angle ACB = \angle ABC$ in ΔABC. Compare the two triangles BCA and CBA. We have $BC = CB$, $\angle BCA = \angle CBA$ (assumption) and $\angle BAC = \angle CAB$ (common angle). Therefore by the *ASA* theorem $\Delta\, BCA \equiv \Delta\, CBA$ and hence $AB = AC$.

Aliter. Let AD bisect $\angle BAC$ meeting BC at D (Fig. 3.23). In the triangles ABD and ACD, $\angle ABD = \angle ACD$ (hypothesis), $\angle BAD = \angle CAD$ (construction), $AD = AD$ (common side). Therefore by the *ASA* theorem, $\Delta ABD \equiv \Delta ACD$ and hence we have $AB = AC$. □

Corollary. In an isosceles triangle, the median to the base bisects the vertical angle and further, is perpendicular to the base.

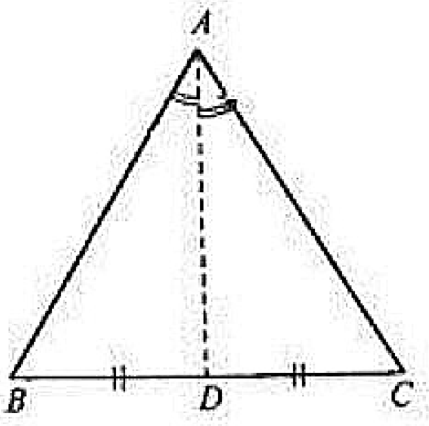


Fig. 3.24

Proof. Let ABC be an isosceles triangle with $AB = AC$ and let AD be the median to the base (the line joining a vertex of a triangle to the midpoint of the opposite side is called a median). In triangles ABD and ACD, we have $AB = AC$, $BD = CD$ and $\angle ABD = \angle ACD$ (in view of Theorem 6). Therefore, by the SAS test $\Delta ABD \equiv \Delta ACD$ and hence $\angle BAD = \angle CAD$ or AD bisects the vertical angle A. Now $\Delta ABD \equiv \Delta ACD$ also implies that $\angle ADB = \angle ADC$ and therefore by Theorem 1, each one of them must be a right angle. In other words the median $AD \perp BC$. ❑

Theorem 8. (SSS Theorem)

If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.

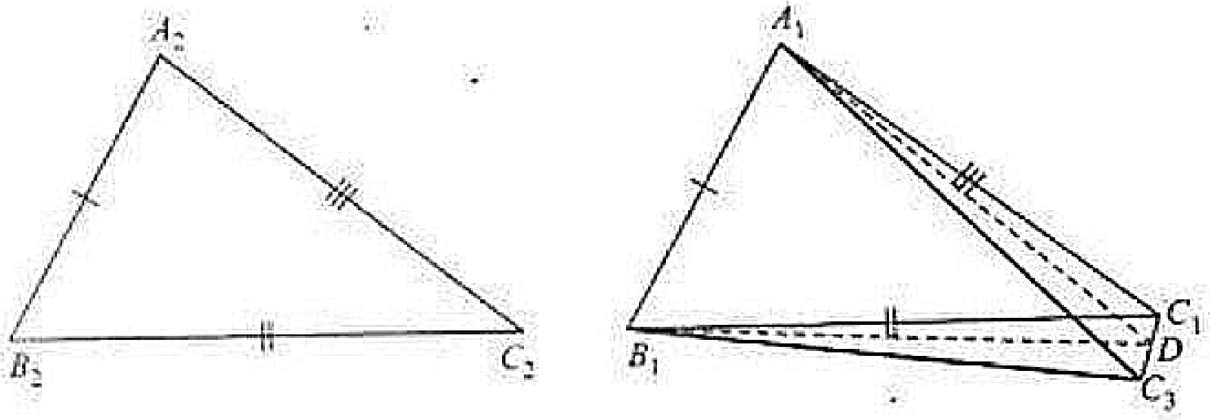


Fig. 3.25

Proof. In triangles $A_1B_1C_1$ and $A_2B_2C_2$ we are given that $A_1B_1 = A_2B_2$, $B_1C_1 = B_2C_2$ and $C_1A_1 = C_2A_2$. Suppose $\angle A_1 = \angle A_2$ or $\angle B_1 = \angle B_2$ then by our fundamental SAS test the two triangles are congruent. Otherwise assume that $\angle A_1 \neq \angle A_2$ and $\angle B_1 \neq \angle B_2$. Draw AC_3 as in Fig. 3.25 such that $\angle B_1A_1C_3 = \angle B_2A_2C_2$ and C_3, C_1 lie on the same side of the side A_1B_1, and further such that $A_1C_3 = A_2C_2$. Now, $A_1C_1 = A_2C_2$ by assumption. Therefore $\Delta A_1C_3C_1$ is isosceles. Again in triangles $A_1B_1C_3$ and $A_2B_2C_2$ we have $A_1B_1 = A_2B_2$, $A_1C_3 = A_2C_2$ and $\angle B_1A_1C_3 = \angle B_2A_2C_2$ (by construction). Applying the SAS test we note that $\Delta A_1B_1C_3 \equiv \Delta A_2B_2C_2$. Therefore $B_1C_3 = B_2C_2$; also $B_1C_1 = B_2C_2$ by hypothesis. This implies that $\Delta B_1C_1C_3$ is isosceles. If D is the mid point of C_1C_3 then A_1D and B_1D are both $\perp C_1C_3$ (Cor. to Theorem 7). Now, the straight lines A_1D and B_1D are distinct since D lies on the segment C_1C_3 and C_1, C_3 are both on the same side of the straight line A_1B_1. This means that we have two distinct perpendiculars DA_1, DB_1 through the point D on C_1C_3 to the sttaight line C_1C_3, which is impossible. Our assumption $\angle A_1 \neq \angle A_2$ and $\angle B_1 \neq \angle B_2$ has led to this impossibility. Therefore $\angle A_1 = \angle A_2$ or $\angle B_1 = \angle B_2$ and in either case we have already observed that the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are congruent. ❑

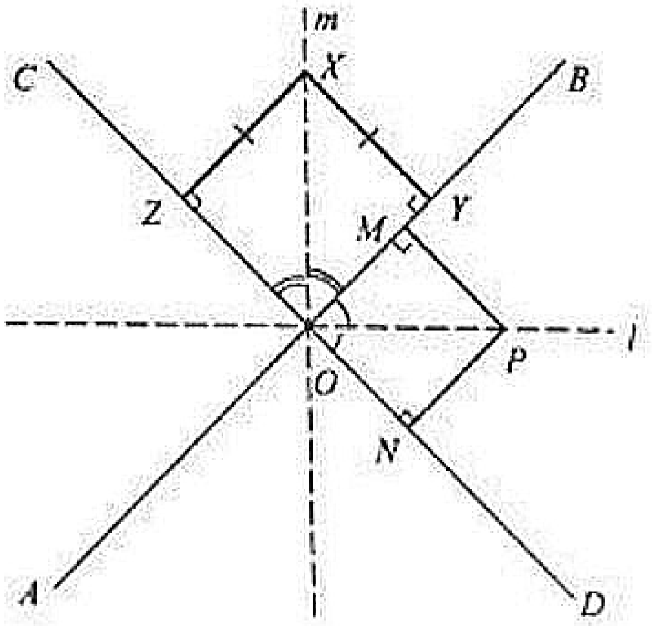


Fig. 3.31

equidistant from AB and CD, then $XY = XZ$ where Y and Z are the feet of the perpendiculars from X on AB, CD respectively. Again, comparing the right angled triangles XOZ and XOY, we note that hypotenuse OX is common and $XZ = XY$; therefore by RHS theorem $\Delta XOZ \equiv \Delta XOY$. Hence $\angle XOY = \angle XOZ$ or X lies on an angular bisector of the angles formed by AB and CD.

Thus the locus of a point equidistant from two intersecting lines is the pair of angular bisectors of the angles formed by the two straight lines. ☐

Remark. The angular bisectors l and m of the angles formed by AB and CD are mutually perpendicular (Fig. 3.32). If $\angle DOB = \alpha$, then $\angle BOC = 180° - \alpha$. Hence the angle between l and m is

$$\angle POB + \angle BOQ = \frac{\alpha}{2} + \frac{180° - \alpha}{2} = 90°$$

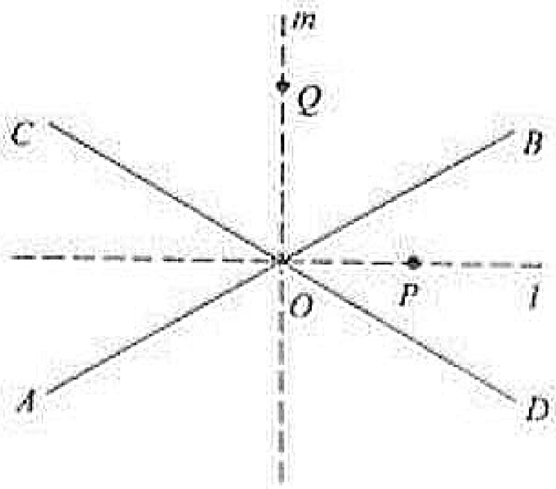


Fig. 3.32

EXERCISE 3.2

1. In a quadrilateral $ABCD$, the diagonals AD and BC meet at O. If it is given that $OA = OC$ and $OB = OD$, prove that $BC = AD$ and that $\angle ACB = \angle CAD$.
2. In ΔABC, D and E are the midpoints of the sides AB and AC respectively. If the perpendiculars at D and E to the sides AB and AC meet at O, prove that $OA = OB = OC$.

3. If two straight line segments AB and CD bisect each other at right angles, show that the sides of the quadrilateral are all equal.
4. Two isosceles triangles whose vertical angles are equal are placed so as to have their vertices coincident. (For an isosceles ΔABC with $AB = AC$, A is called the vertex of the ΔABC.) Prove that two of the lines joining their other angular points are equal.
5. ABC is a triangle and O is any point in it. Prove that
$$\angle BOC > \angle BAC.$$
6. If two isosceles triangles have equal bases and equal vertical angles prove that they are congruent.
7. AB and CD are two straight lines meeting at O and XY is another straight line. Show that in general two points can be found in XY which are equidistant from AB and CD. When is there only one such point?
8. $ABCD$ is a quadrilateral in which diagonals bisect each other. Show that B and D are equidistant from AC.
9. If in a quadrilateral $ABCD$, AC bisects angles A and C, show that AC is perpendicular to BD.
10. ABC and DBC are two triangles on the same base BC and on the same side of it such that $BA = CD$ and $BD = CA$. If AC and BD meet at O, prove that ΔOBA is congruent to ΔOCD.
11. Deduce from theorem 10, that the sum of any two sides is greater than the third side in any triangle.
12. In ΔABC, $AB > AC$. Let D on AB be such that $AD = AC$.
Then prove that
$$\angle ADC = (\angle B + \angle C)/2 \text{ and } \angle BCD = (\angle C - \angle B)/2.$$
13. The bisector of angle A of ΔABC meets BC at U.
Prove that if $AB > AC$ then $\angle AUC$ is acute.
14. With the same notations as in problem 13, prove that $AB > BU$ and $AC > CU$.
15. If the sides AB, BC, CD, DA of a quadrilateral $ABCD$ are in the descending order of magnitude, show that $\angle CDA > \angle CBA$.
16. If AD is the altitude through A of ΔABC, prove that $AB > AC$, $AB = AC$ or $AB < AC$ according as $BD > DC$, $BD = DC$ or $BD < DC$.
17. If X is any point on BC of ΔABC, prove that either AB or AC is greater than AX.
18. If BC is the greatest side of ΔABC, and D, E are points on BC, CA respectively prove that, $BC \geq DE$.
19. If BC is the greatest side of ΔABC, and E and F are points on AB, AC respectively prove that, $BC \geq EF$.
20. Prove that no straight line can be drawn within a triangle which is greater than the greatest side.
21. $ABCD$ is a quadrilateral having $AD = BC$, and $\angle ADC = \angle BCD$. If X is the midpoint of DC prove that $AX = BX$.
22. If the bisector of an angle of a triangle is perpendicular to the opposite side prove that the triangle is isosceles.
23. If two triangles are congruent prove that the straight lines joining the vertices to the midpoints of their bases are equal.
24. If two triangles are congruent prove that the perpendiculars from the vertex to the base of each are equal.
25. If triangles ABC and DEF are congruent and the bisectors of $\angle A$ and $\angle D$ meet BC and EF at X and Y respectively, then prove that $AX = DY$.

26. AU is the bisector of $\angle BAC$ and SUT is drawn perpendicular to AU meeting AB and AC at S and T respectively. Prove that ΔASU is congruent to ΔATU.
27. Through C the midpoint of a straight line segment AB, a straight line is drawn. Perpendiculars AD and BE are dropped upon it from A and B. Prove that $AD = BE$.
28. ABC is an isosceles triangle. The base BC is produced on either side to D and E so that $BD = CE$. Prove that $AD = AE$.
29. If the hypotenuse AC of a right angled ΔABC is of length $2AB$, prove that
$$\angle BAC = 2\angle ACB.$$
30. ABC is an isosceles triangle having $\angle B = \Delta C = 2\Delta A$. If BD bisecting $\angle B$ meets AC in D, prove that $AD = BC$.

3.3 PARALLEL STRAIGHT LINES

Definition 5. Two straight lines on a plane are *parallel* if they have either no common point or two common points; in the latter case the two straight lines coincide.

The fundamental property of parallel straight lines in Euclidean Geometry is the following

Axiom. For each point P and straight line l there is just one straight line through P parallel to it.

We observe that

1. any straight line l is parallel to itself;
2. if a straight line l_1 is parallel to a straight line l_2 then l_2 is parallel to l_1;
3. if a straight line l_1 is parallel to another straight line l_2 and if l_2 is parallel to l_3 then l_1 is parallel to l_3.

We write $l \parallel m$ to mean that l is parallel to m. A straight line drawn to cut two or more given straight lines is called a *transversal*. In the Fig. 3.33, the straight lines l_1 and l_2 are cut by the transversal m.

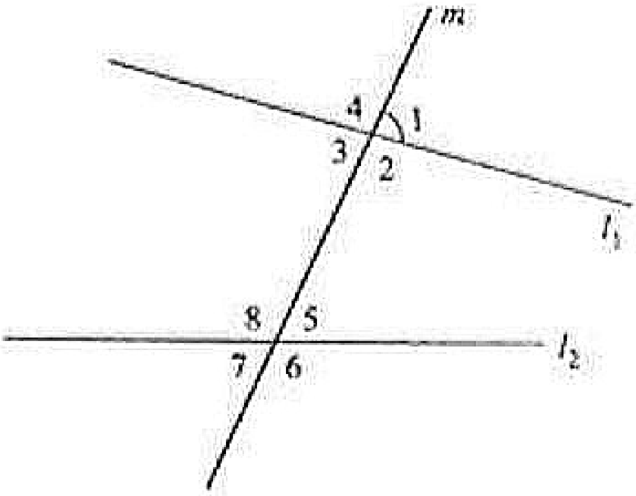


Fig. 3.33

In Fig. 3.33, the angles 2 and 8 are *alternate angles*; so are the angles 3 and 5. The angles 1 and 5 are *corresponding angles*; so are 4, 8; 2, 6; and 3,7.

Theorem 14. Suppose two straight lines are cut by a transversal and any one of the following three conditions hold, namely,

1. a pair of alternate angles are equal,
2. a pair of corresponding angles are equal,
3. a pair of interior angles on the same side of the transversal add upto 180°,

then the two straight lines are parallel.

Proof. The straight lines AB and CD are cut by the transversal EF meeting AB and CD at G, H respectively.

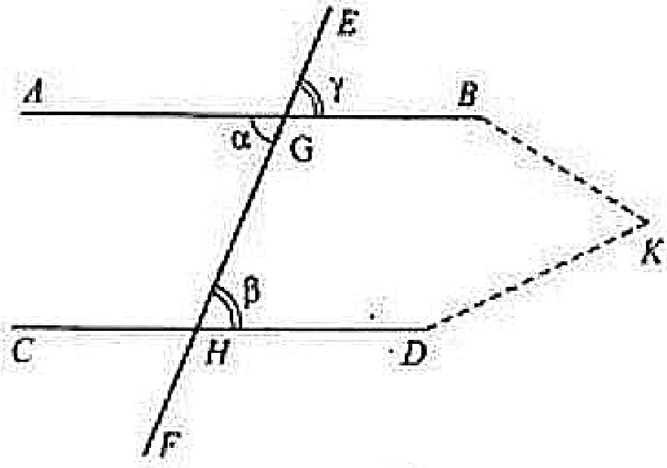


Fig. 3.34

1. Suppose the alternate angles $AGH = \alpha$ and $GHD = \beta$ are equal. If possible let AB and CD meet at a point K. If K is as shown in Fig. 3.34, then α is an exterior angle of ΔGKH for which β is an interior opposite angle. If K is on the other side of GH, then β will be an exterior angle of ΔGKH for which α is an interior opposite angle. Thus either $\alpha > \beta$ or $\beta > \alpha$, both contradicting our hypothesis that $\alpha = \beta$. Hence AB and CD do not meet and therefore are parallel.
2. Assume now that the corresponding angles $EGB = \gamma$ and $GHD = \beta$ are equal. In this case $\angle AGH = \angle EGB$ and hence $\alpha = \angle AGH = \angle EGB = \gamma = \angle GHD = \beta$. Now α and β is a pair of equal alternate angles and hence by (1) $AB \parallel CD$.
3. Suppose the sum of the interior angles BGH and GHD is 180°. Then $\alpha = \angle AGH = 180° - \angle BGH = \angle GHD$ and therefore again by (1) $AB \parallel CD$.

Corollary. If two straight lines l and m are both perpendicular to the same straight line p, then $l \parallel m$.

Proof. l and m are cut by the transversal p and the corresponding angles α and β are equal, each being equal to 90° (Fig. 3.35). Hence $l \parallel m$.

Theorem 15. If two parallel straight lines are cut by a transversal then

1. the alternate angles are equal
2. the corresponding angles are equal
3. the sum of the interior angles on the same side of the transversal is equal to 180°.

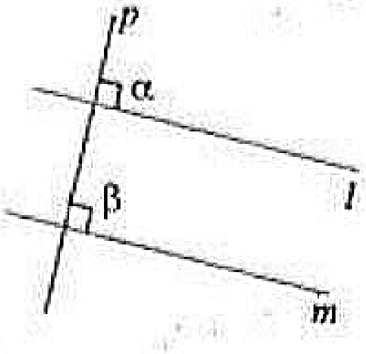


Fig. 3.35

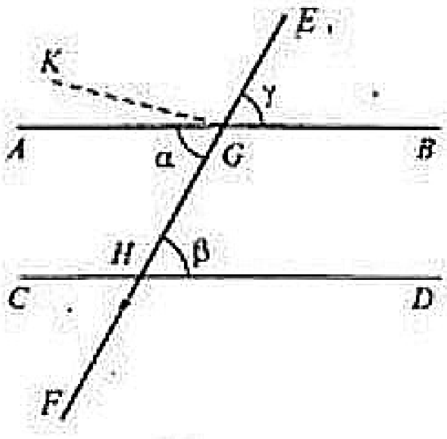


Fig. 3.36

Proof. In Fig. 3.36, $AB \parallel CD$ and the transversal EF cuts AB and CD at G, H respectively.

1. Suppose $\angle AGH \neq \angle GHD$. Draw the ray GK such that $\angle KGH = \angle GHD = \beta$. Then the straight lines KG and CD are cut by the transversal EF; and by construction the alternate angles KGH and GHD are equal. Therefore by Theorem 14 $KG \parallel CD$. Now KG and AB are two straight lines parallel to CD passing through G contradicting our axiom on parallel straight lines. Hence our supposition that $\angle AGH \neq \angle GHD$ is false and we must have the alternate angles AGH and GHD as equal angles.
2. The equality of the alternate angles AGH and GHD implies that the corresponding angles EGB and GHD are equal.
3. Again the equality of the alternate angles AGH and GHD implies that the sum of the interior angles BGH and GHD is 180°. ❑

Corollary. If a straight line is perpendicular to one of two or more parallel straight lines, then it is also perpendicular to the other parallel straight lines.

Proof. Suppose l, m, n are parallel straight lines and p is a straight line perpendicular to l. For the transversal p cutting the parallel lines l and m, the equality of the alternate angles α and β implies that $90° = \alpha = \beta$ (Fig. 3.37). Hence $p \perp m$. The same reasoning tells us that p is perpendicular to all the straight lines parallel to l. ❑

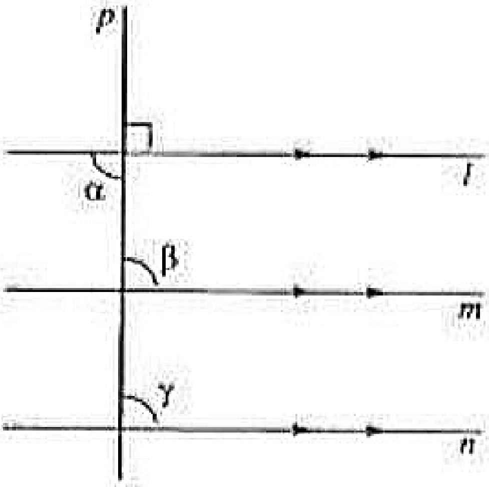


Fig. 3.37

Theorem 16. The sum of the interior angles of a triangle is 180°.

Proof. Let ABC be a triangle. Produce BC to D and draw $CE \parallel BA$ (Fig. 3.38). Then $\angle ABC = \angle ECD$ being corresponding angles for the transversal BD cutting the parallel lines AB and EC. Again the angles BAC and ACE are alternate angles and therefore they are equal. Thus we have

$$\angle A + \angle B + \angle C = \angle ACE + \angle ECD + \angle ACB = 180°.$$ □

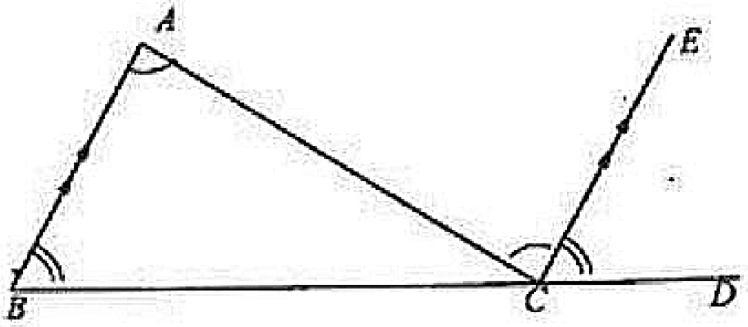


Fig. 3.38

Corollary. An exterior angle of a triangle is equal to the sum of the interior opposite angles.

Proof. For ΔABC, we note that the exterior $\angle DCA = \angle DCE + \angle ECA$ (Fig. 3.38) $= \angle A + \angle B$ (see the proof of Theorem 16). Also $\angle DCA = 180° - \angle BCA$

$$= 180° - \angle C = \angle A + \angle B \text{ (since } \angle A + \angle B + \angle C = 180°).$$ □

Theorem 17. The sum of the interior angles of any convex polygon having n sides is $(2n - 4)$ right angles. [Recall that two right angles = 180° and that a polygon is a non-self intersecting closed polygonal line $A_1A_2A_3 \ldots A_nA_1$ consisting of line segments A_1A_2, A_2A_3,.... $A_{n-1}A_n$ and A_nA_1; a *convex polygon* is a polygon for which the line segment joining any two points in its interior lies entirely within the polygonal region. Note that a polygon is convex iff it entirely lies on one side of any straight line containing a side of the polygon.]

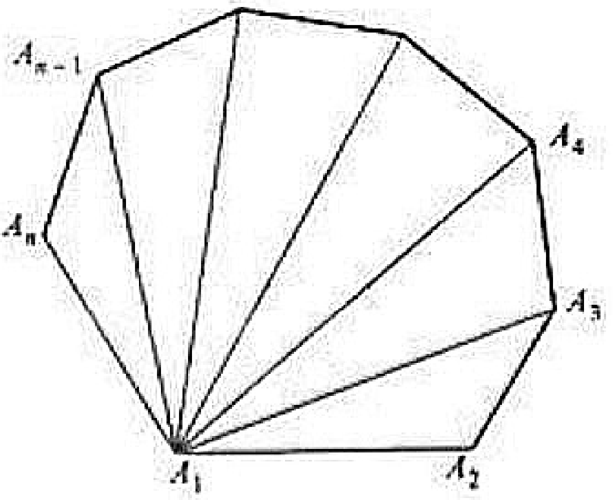


Fig. 3.39

Proof. If $A_1A_2 \ldots A_n A_1$ is any convex polygon, Join A_1 with its other vertices to get $(n - 2)$ triangles namely,

$$\Delta A_1A_2A_3,\ \Delta A_1A_3A_4,\ \Delta A_1A_4A_5,\ \ldots \Delta A_1A_{n-1}A_n.$$

We note that the sum of the interior angles of the polygon, the sum of the angles of all the $(n - 2)$ triangles $= 2(n - 2)$ right angles. □

Corollary. The sum of the exterior angles of a convex polygon is 360° or four right angles.

Proof. The sum of all the exterior angles and the sum of all the interior angles of a convex polygon of n sides is $(180n)°$ since each exterior angle is supplementary to its corresponding interior angle. Therefore the sum of the exterior angles $= n180° - (n - 2)180° = 2(180°) = 360°$. □

Note (*i*) The sum of the interior angles of a convex quadrilateral (convex polygon of four sides) is $2(4) - 4 = 4$ right angles $= 360°$.

(*ii*) The theorem is true for non-convex polygon also (see Exercise 3.3, problem...).

Definition 6. A *parallelogram* is a quadrilateral with both pairs of opposite sides being parallel.

Theorem 18. In a parallelogram, opposite sides are equal, opposite angles are equal, the sum of any pair of adjacent angles is 180° and the diagonals bisect each other.

Proof. Let $ABCD$ be a parallelogram. Then $AB \parallel DC$ and $AD \parallel BC$. The triangles ABC and CDA are congruent since AC is a common side, $\angle CAB = \angle ACD$ (alt angles), $\angle ACB = \angle CAD$ (alt angles) (by the ASA theorem). Therefore $AB = CD$ and $AD = BC$. Thus the opposite sides are equal.

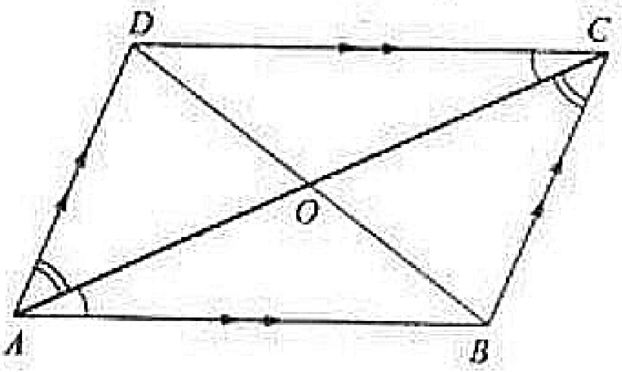


Fig. 3.40

Also $\Delta ABC \equiv \Delta CDA$ implies that $\angle ABC = \angle CDA$, *i.e.*, the opposite angles B and D are equal. Similarly, the congruence of the trianlges BAD and DCB implies that $\angle A = \angle C$.

Comparing the triangles AOD and COB (Fig. 3.40) we see that $\angle AOD = \angle COB$ (vertically opposite angles), $\angle DAO = \angle BCO$ (alternate angles) and $AD = BC$. Therefore, $\Delta AOD \equiv \Delta COB$ which implies that $AO = OC$. Similarly, one can show that $BO = OD$. Thus the diagonals bisect each other, in any parallelogram. ❑

Corollary. If two straight lines are parallel, then any two points on one line are at the same distance from the other.

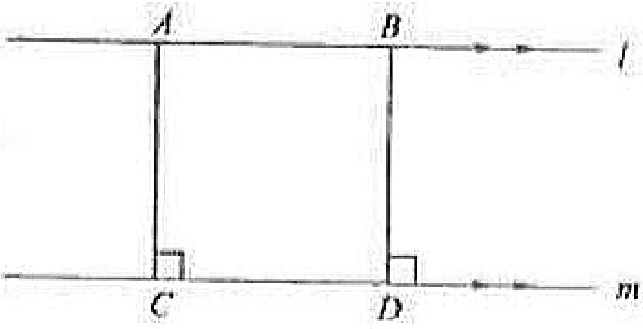


Fig. 3.41

Proof. Let l and m be any two parallel straight lines. If A and B are any two points on l, draw the perpendiculars AC and BD to m from A and B respectively (Fig. 3.41). It is required to prove that $AC = BD$. We note that $ABDC$ is a parallelogram and hence we have $AC = BD$. ❑

Theorem 19. In a convex quadrilateral, the following are equivalent:

1 the quadrilateral is a parallelogram

2. the opposite sides are equal
3. the opposite angles are equal
4. the diagonals bisect each other.

Proof. (1) $\Rightarrow$ (2) follows from Theorem 18.

To prove (2) $\Rightarrow$ (3):

We assume that $AB = CD$ and $AD = BC$. We want to prove that $\angle A = \angle C$ and $\angle B = \angle D$. In ΔABC and ΔCDA we have $AB = CD$, $BC = AD$ and AC is a common side. Therefor by the SSS theorem $\Delta ABC \equiv \Delta CDA$ and so $\angle B = \angle D$. Similarly one can show that $\angle A = \angle C$. Thus (2) $\Rightarrow$ (3).

To prove (3) $\Rightarrow$ (4):

We assume that $\angle A = \angle C$ and $\angle B = \angle D$ and we want to prove that the diagonals AC and BD bisect each other. Suppose $\angle A = \angle C = \alpha$ and $\angle B = \angle D = \beta$. Then we have $2\alpha + 2\beta = 2(\alpha + \beta) = 360^\circ$ and hence $\alpha + \beta = 180^\circ$. The straight lines AB and DC are cut by the transversal AD; the sum of the interior angles $\angle CDA + \angle DAB = \beta + \alpha = 180^\circ$ and hence $AB \parallel DC$. Similarly $AD \parallel BC$. Therefore, the alternate angles OCD and OAB are equal; also $\angle ODC = \angle OBA$. Further $AB \parallel DC$ and $AD \parallel BC$ implies that $ABCD$ is a parallelogram and hence $AB = CD$. This means that $\Delta OAB \equiv \Delta OCD$ and hence $AO = OC$ and $BO = OD$. This proves that (3) $\Rightarrow$ (4).

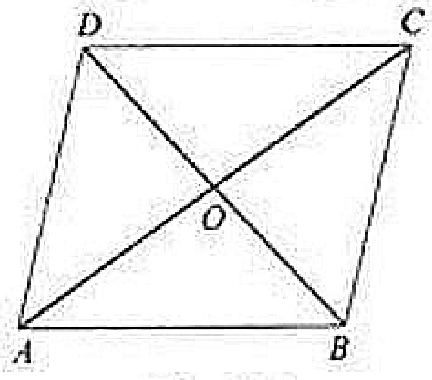


Fig. 3.42

To prove (4) $\Rightarrow$ (1):

We assume that the diagonals AC and BD bisect each other. We want to prove that $ABCD$ is a parallelogram. In triangles AOB and COD we have $\angle AOB = \angle COD$ (vert. opp. angles), $AO = OC$, $BO = OD$. Therefore, by the SAS test $\Delta AOB \equiv \Delta COD$. This implies that the alternate angles OCD and OAB are equal. Therefore, $AB \parallel CD$. Similar argument shows that $AD \parallel BC$ and thus $ABCD$ is a parallelogram. ❑

Theorem 20. If there are three or more parallel straight lines, and the intercepts made by them on a transversal are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

Proof. Suppose the three parallel straight lines l, m, n are cut by the transversals ACE and BDF and $AC = CE$. We want to prove that $BD = DF$ (Fig. 3.43). Draw AG and CH parallel to BF meeting m at G and n at H respectively. Then in triangles CAG and ECH we have $\angle CAG = \angle ECH$ (cor. angles), $\angle ACG = \angle CEH$ (cor. angles) and $AC = CE$. Therefore by the ASA theorem $\Delta CAG \equiv \Delta ECH$; and so $AG = CH$. But by construction $ABDG$ and $CDFH$ are parallelograms. This means that $AG = BD$, $CH = DF$ and hence $BD = DF$.

Theorem 21. In any triangle ABC, the line FE joining the mid points of AB and AC is parallel to BC and $FE = (1/2)\ BC$.

Proof. Draw $AX \parallel BC$ (Fig. 3.44). Then if FY is the straight line parallel to BC, the transversal AB has equal intercepts AF and FB on the parallel lines AX, FY and BC. By Theorem 20, AC also should make equal intercepts on these parallel straight lines and hence E lies on FY. This means that $FE \parallel BC$.

Draw $CY \parallel BA$ meeting FE at Y (Fig. 3.45). Then $BCYF$ is a parallelogram and $FY = BC$. Further ΔAFE and ΔCYE are congruent since $\angle AEF = \angle CEY$ (vert. opp. angles), $\angle FAE = \angle YCE$ (alt. angles) and $AE = EC$.

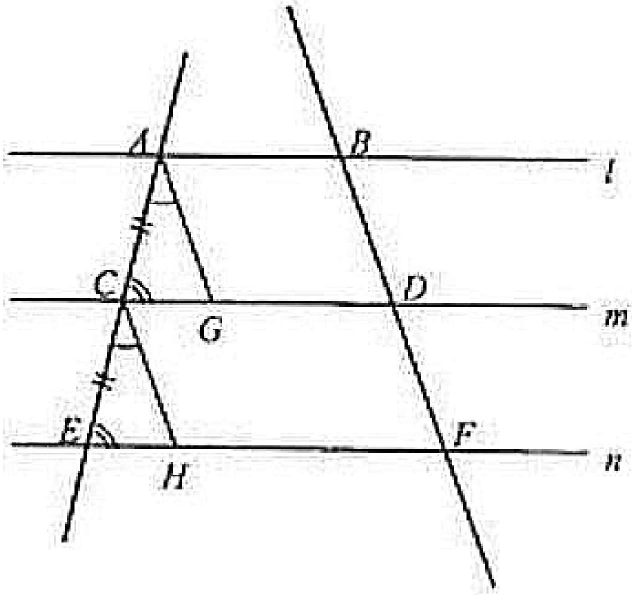


Fig. 3.43

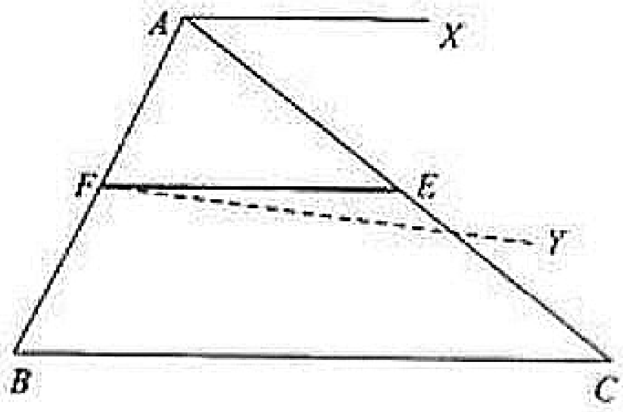


Fig. 3.44

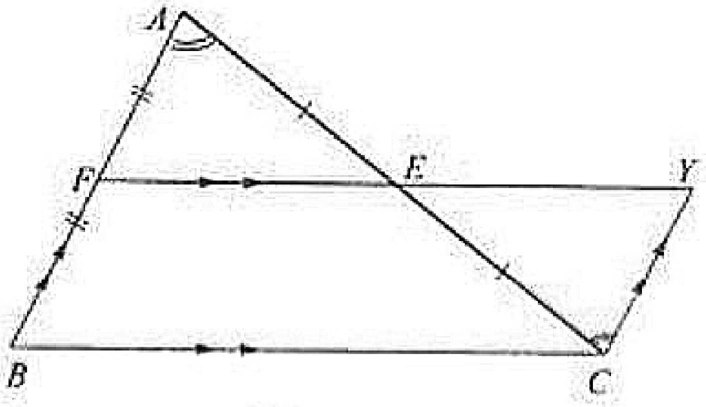


Fig. 3.45

Therefore we have $FE = EY$ and $2FE = FY = BC$. Thus $FE = (1/2)$ BC. ☐

Theorem 22. If in ΔABC, a straight line is drawn parallel to BC through the midpoint F of AB, then it bisects the side AC.

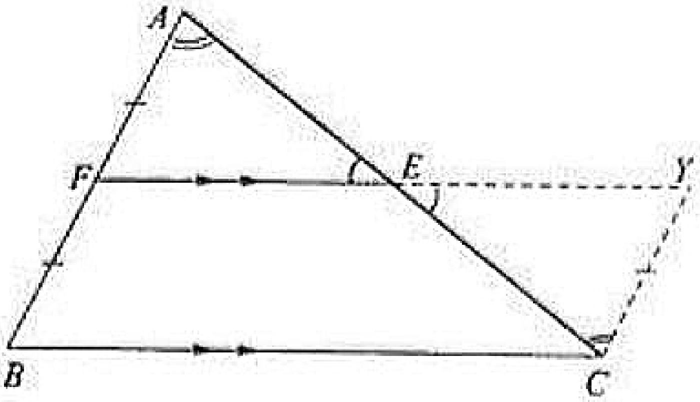


Fig. 3.46

Proof. Let the straight line through F parallel to BC meet AC at E. It is required to prove that $AE = EC$. Draw $CY \parallel BA$ meeting FE produced at Y. We have $\angle AEF = \angle CEY$ (vert. opp. angles), $\angle FAE = \angle YCF$ (alt. angles) and $AF = FB = CY$ (since $BCYF$ is a parallelogram). Therefore

$$\Delta AFE \equiv \Delta CYE \text{ and this gives } AE = EC.$$ □

EXERCISE 3.3

1. In a parallelogram prove that
   (*i*) if one angle is a right angle then all the angles are right angles.
   (*ii*) opposite angles are equal
2. If each angle of a rectilinear figure is $7/8\pi$ radians, find the number of sides.
3. Prove that the bisectors of angles in a parallelogram form a rectangle.
4. In ΔABC the internal bisectors of angles B and C meet at I. Prove that $\angle BIC = 90° + \angle A/2$.
5. In problem 4, if the external bisectors of angles B and C meet at X, then $\angle BXC = 90° - \angle A/2$.
6. In ΔABC perpendiculars AD and BF are drawn to BC and CA respectively to meet at the point H. Prove that $\angle AHE = \angle C$.
7. $ABCD$ is a trapezium with side $BC \parallel AD$. If E is the midpoint of AB and the line through E parallel to DC meets AD and BC at X and Y respectively, prove that $ABCD$ and $XYCD$ have equal areas.
8. If two quadrilaterals $ABCD$ and $PQRS$ have angles A, B, C, D equal to angles P, Q, R, S respectively and $AB = EF$, $DC = HG$ and if AD is not parallel to BC, prove that the quadrilaterals are congruent.
9. Equilateral triangles BAD and CAE are drawn on the sides AB, AC of an equilateral ΔABC externally to the triangle. Show that D, A, E are collinear.
10. If on the sides BC, CA, AB of an equilateral triangle, the equilateral triangles BCX, CAY and ABZ are drawn externally to ΔABC prove that XYZ is also equilateral.
11. Equilateral triangles ABX and ACY are described on sides AB, AC of a ΔABC externally to ΔABC. Prove that $CX = BY$.
12. If in a parallelogram $ABCD$, the diagonal AC bisects $\angle A$, then prove that $ABCD$ is a rhombus.
13. Show how to find points D and E on the side AB, AC of ΔABC such that $DE \parallel BC$ and $DE = BD$.
14. Let $ABCDE$ be a regular pentagon. If the internal angular bisectors of angles A and B meet at O, prove that OC, OD, OE also bisect angles C, D and E.
15. If in ΔABC BC is the greatest side, prove that $\angle A > \angle 60°$.
16. Let D be the midpoint of the side BC of ΔABC. Prove that if $AD > BD$ then $\angle A$ is acute; else if $AD < BD$, then $\angle A$ is obtuse.
17. Let D be the midpoint of the hypotenuse BC of the right angled ΔABC. Prove that $2AD = BC$.
18. If the sum of the distances of any vertex of a quadrilateral from the other three is same for all the four vertices, prove that the quadrilateral is a rectangle.
19. $ABCD$ is a parallelogram and O is any point. The parallelograms $OAEB$, $OBFC$, $OCGD$, $ODHA$ are completed. Show that $EFGH$ is a parallelogram.
20. Any point X is taken on the side BC of ΔABC. Prove that AX is bisected by the straight line joining the midpoints of AB and AC.

21. Suppose the straight line AB of ΔABC is bisected at C and the perpendiculars AX, BY, CZ are drawn to any straight line OP. Prove that
    (*i*) if A, B are on the same side of OP, then $2CZ = AX + BY$.
    (*ii*) if A, B are on the opposite sides of OP, then $2CZ$ = difference of AX and BY.
22. Prove that the straight lines joining the midpoints of the diagonals of a trapezium is parallel to the parallel sides.
23. Prove that in any quadrilateral, the midpoints of the sides form the vertices of a parallelogram.
24. Prove that the lines joining the midpoints of opposite sides of a quadrilateral and the line joining the midpoints of diagonals are concurrent.
25. Let X be the midpoint of the side AB of ΔABC. Let Y be the midpoint of CX. Let BY cut AC at Z. Prove that $AZ = 2ZC$.
26. A is a given point and P is any point on a given straight line. If $AQ = AP$ and AQ makes a constant angle with AP, find the locus of Q.
27. ABC is a equilateral triangle with vertex A fixed and B moving in a given straight line. Find the locus of C.
28. If a trisector of an exterior angle of a triangle is parallel to a bisector of an interior angle, prove that the other trisector of the exterior angle is parallel to a trisector of an interior angle.

3.4 SOME PROPERTIES OF A TRIANGLE

Theorem 23. The perpendicular bisectors of the three sides of a triangle concur at a point.

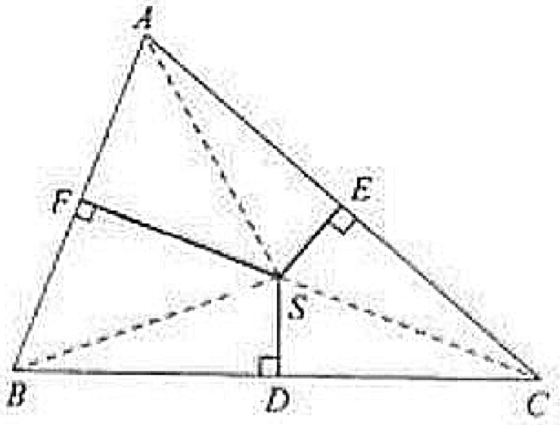


Fig. 3.47

Proof. Let ABC be a triangle; let D, E, F be the midpoints of the sides BC, CA, AB respectively. Suppose the $\perp$ bisectors of BC and CA meet at S. Then it is required to prove that $SF \perp AB$. S being a point on the perpendicular bisector of BC, we have $SB = SC$ (Theorem 12). Again, S lies on the perpendicular bisector of CA implies that $SC = SA$. Thus $SA = SB = SC$. In triangles ASF and BSF we have $AF = BF$ (Hypothesis), $SA = SB$ (proved above) and $SF = SF$ (common side). Therefore, $\Delta ASF \equiv \Delta BSF$. This implies that $\angle AFS = \angle BFS = 90°$ or $SF \perp AB$. ❑

Note. The point of concurrence S, of the perpendicular bisectors of the sides of a triangle ABC is known as the *circumcentre* of ΔABC.

Theorem 24. The bisectors of the three angles of a triangle meet at a point.

Proof. Let ABC be a triangle and let the bisectors of the angles B and C meet at I (Fig. 3.48). It is required to prove that AI bisects $\angle A$. Draw IX, IY, IZ perpendicular to

BC, CA, AB respectively (Fig. 3.48). Since I lies on the bisector BI of $\angle B$ we have $IX = IZ$ (Theorem 13). Similarly, the fact that I lies on the bisector of $\angle C$ implies that $IX = IY$. Thus $IX = IY = IZ$. Now $IY = IZ$ implies, again by Theorem 13, that I lies on the bisector of $\angle A$. Hence the bisectors of the three angles of a triangle concur at a point. ❑

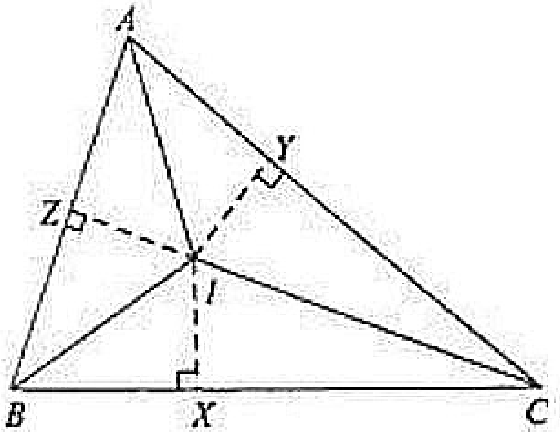


Fig. 3.48

Note. The point of concurrence I, of the internal bisectors of the angles of a triangle is known as the *incentre* of the triangle.

Theorem 25. The three medians of a triangle meet at a point and the point of concurrence trisects each median.

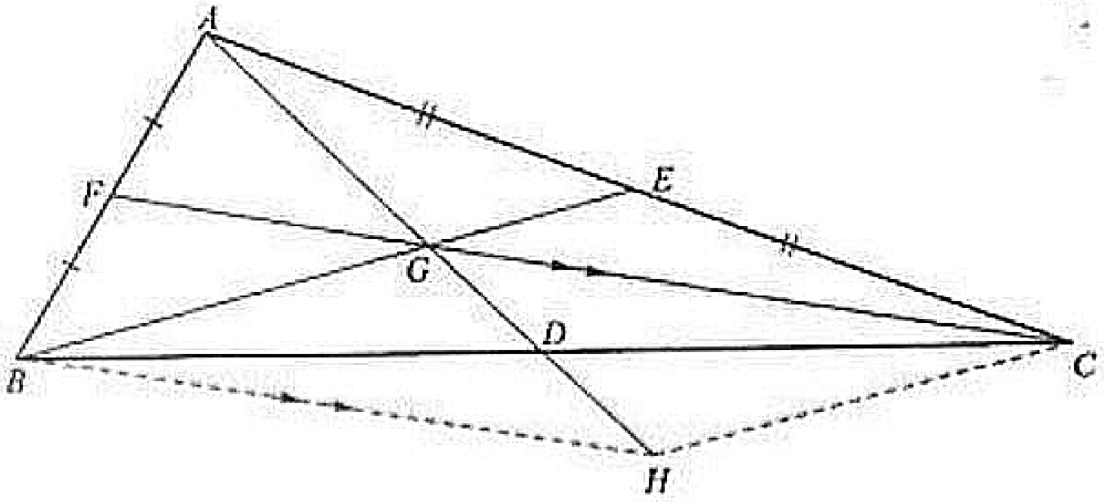


Fig. 3.49

Proof. Let the medians BE and CF of ΔABC meet at G and let AG meet BC at D. It is required to prove that $BD = DC$ and that $AG/GD = BG/GE = CG/GF = 2/1$. Draw $BH \parallel FC$ meeting AD at H. In ΔABH, $FG \parallel BH$ and F is the midpoint of AB. Therefore, by Theorem 22, G must be the midpoint of AH. This means that in ΔAHC, $GE \parallel HC$ (by Theorem 22). This in turn implies that $BHCG$ is a parallelogram and the diagonals BC and GH bisect each other at D. Hence $BD = DC$, proving that the three medians of a triangle meet at a point.

Further, as already observed, $AG = GH$ and $GD = DH = (1/2)\ GH$ implies that $AG/GD = 2/1$. Similarly BE and CF are also trisected by G. ❑

Note. G is known as the *centroid* of ΔABC.

Theorem 26. **The altitudes of a triangle meet at a point.**

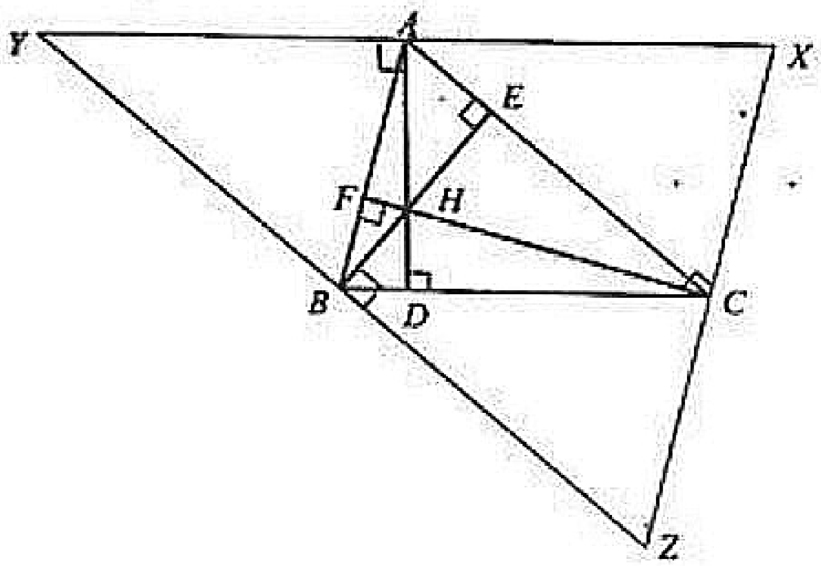


Fig. 3.50

Proof. Let ABC be a triangle. Draw XY, YZ, ZX through A, B, C parallel to BC, CA, AB respectively. Then $BCAY$ and $BCXA$ are parallelograms and hence $YA = BC = AX$ or A is the midpoint of XY. Similarly, B is the midpoint of YZ and C is the midpoint of ZX. If the perpendiculars AD, BE, CF are drawn to the sides XY, YZ, ZX respectively (Fig. 3.50), then by Theorem 23, AD, BE and CF meet at the circumcentre H of ΔXYZ. Now $XY \parallel BC$, $YZ \parallel CA$ and $ZX \parallel AB$ implies that AD, BE, CF are also perpendicular to BC, CA, AB of ΔABC. Hence, they are the altitudes of ΔABC and they meet at the point H.

Note. The point of concurrence of the altitudes of a triangle is known as the *orthocentre* of the triangle.

EXERCISE 3.4

1. C and D are two points on the same side of a straight line AB. Find a point X on AB such that the angles CXA and DXB are equal.
2. C and D are two points on the same side of a straight line AB and P is any point on AB. Show that $PC + PD$ is least when the angles CPA and DPB are equal.
3. In any ΔABC, prove that the bisectors of the interior angle A and the exterior angles at B and C are concurrent.
4. If the medians BE and CF are equal in a ΔABC prove that $AB = AC$.
5. If P is any point on a straight line drawn through the vertex A of an isosceles ΔABC, parallel to the base, prove that $PB + PC > AB + AC$.
6. If S is the circumcentre of ΔABC prove that $\angle BSD = \angle BAC$ where D is the midpoint of BC.
7. If $ABCD$ is a parallelogram, prove that the circumcentres of the triangles ABC and ADC are at the same distance from AC.
8. X is any point in the base BC of an isosceles ΔABC; P and Q are the circumcentres of the triangles ABX and ACX. Prove that $APXQ$ is a parallelogram.
9. Let D, E, F be the feet of the altitude from A, B, C in a ΔABC. Prove that the perpendicular bisector of EF also bisects BC.
10. Suppose the diagonals of a quadrilateral $ABCD$ meet at a point O; then prove that the circumcentres of the four triangles OAB, OBC, OCD and ODA form a parallelogram.

11. If H is the orthocentre of ΔABC, prove that A is the orthocentre of ΔBHC.
12. Prove that the circumcentre S of ΔABC where A', B', C' where A', B', C' are the midpoint of BC, CA, AB respectively.
13. If S is the circumcentre of a ΔABC and D, E, F are the feet of the altitudes of ΔABC then prove that $SB \perp DF$.
14. Let P be any point inside a regular polygon. If d_i is the distance of P from the i^{th} side of the polygon, prove that $d_1 + d_2 \ldots + d_n =$ constant, where n is the number of sides of the polygon.
15. If I is the incentre and S is the circumcentre of ΔABC prove that $\angle IAS$ half the difference in $\angle B$ and $\angle C$.
16. AB and CD are two fixed straight lines and a variable straight line cuts them at X and Y respectively. The angular bisectors of $\angle AXY$ and $\angle CXY$ meet at P. Find the locus of P.
17. I is the incentre of ΔABC. X and Y are the feet of the perpendiculars from A to BI and CI. Prove that XY is parallel to BC.

3.5 SIMILAR TRIANGLES

Theorem 27. Parallelograms on the same base and between the same parallels are equal in area.

Proof. Let $ABCD$ and $ABXY$ be two parallelograms having the same base AB and lying between the same parallels, namely the straight line AB and the straight line YC. (Fig. 3.51) The parallelogram $ABCD =$ Trapezium $ABCY - \Delta ADY$ and the parallelogram $ABXY =$ Trapezium $ABCY - \Delta BCX$ (in area). Therefore to prove that the two parallelograms $ABCD$ and $ABXY$ have equal areas, it is enough to prove that ΔADY and ΔBCX have equal areas. We have $\angle ADY = \angle BCX$, (cor. angles) $\angle AYD = \angle BXC$ (cor. angles) and $AY = BX$ (since $ABXY$ is a parallelogram). Therefore by the ASA theorem the two triangles ADY and BCX are congruent. So they have equal areas. ☐

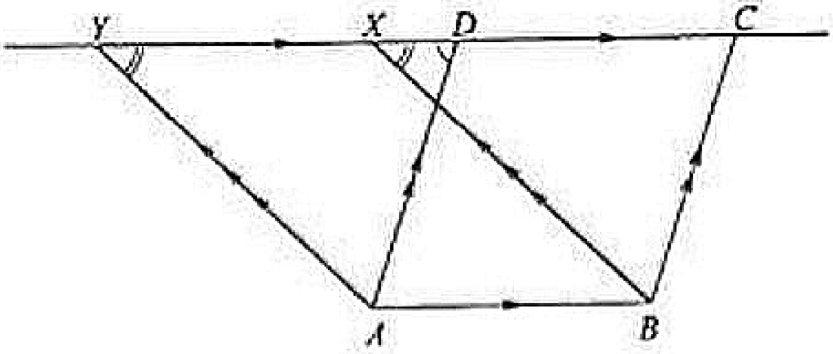


Fig. 3.51

Note. The above proof works even when Y lies between C and D. (Fig. 3.52)

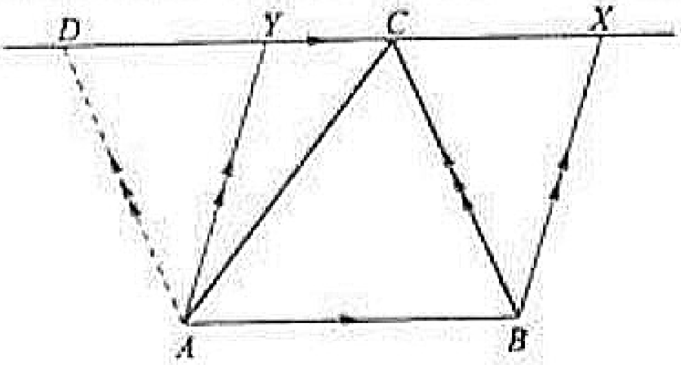


Fig. 3.52

Corollary. The area of a parallelogram is equal to the area of the rectangle whose adjacent sides are equal to the base and the altitude of the parallelogram. In other words, the area of a parallelogram is the product of its base and the altitude.

Proof. Let $ABCD$ be a parallelogram and $ABXY$ be the rectangle on the base AB and with BX equal to the altitude of the parallelogram (Fig. 3.53). Then the parallelograms $ABCD$ and $ABXY$ have the same base and lie between the same parallels. Therefore by Theorem 27, the rectangle $ABXY$ and the parallelogram $ABCD$ have the same area. ❑

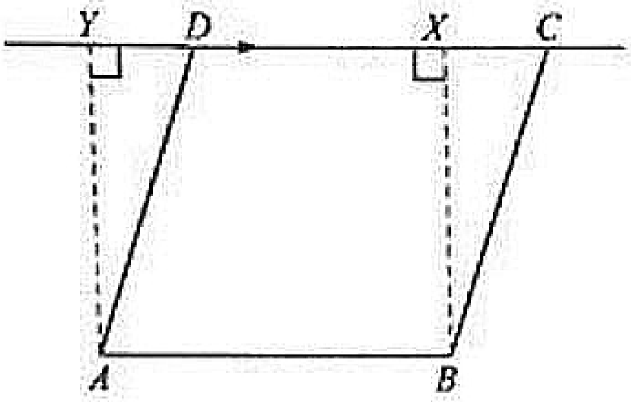


Fig. 3.53

Theorem 28. If a parallelogram and a triangle are on the same base and lie between the same parallels, then the area of the triangle is half that of the parallelogram.

Proof. Consider the ΔABC and the parallelogram $ABXY$ having the same base AB and lying between the same parallels AB and YX. Draw $AD \parallel BC$ (Fig. 3.54) so that $ABCD$ is a parallelogram. Then by Theorem 27, the two parallelograms $ABCD$ and $ABXY$ have equal areas. Now, the triangles ABC and CDA are congruent (Why ?). Hence, area of ΔABC = half the area of parallelogram $ABCD$ = half the area of parallelogram $ABXY$. ❑

Corollary. Triangles on equal bases and between the same parallels are equal in area.

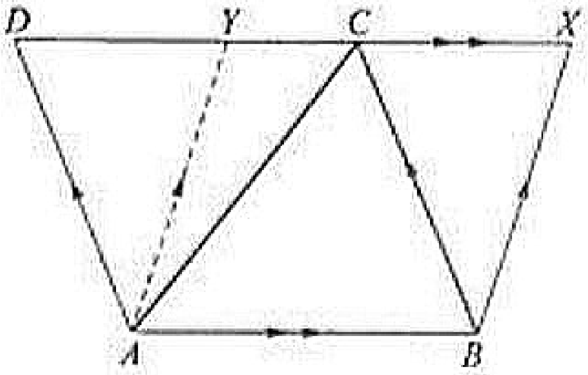


Fig. 3.54

Proof. Exercise. ❑

Note. (1) By the corollary to Theorem 27, the area of a parallelogram $ABCD$

$= AB \times$ altitude through A

$=$ base $\times$ height.

(2) Area of ΔABC

$$= \frac{1}{2} \text{ area of the parallelogram } ABCD$$

$$= \frac{1}{2} (\text{base} \times \text{height}) \text{ (Fig. 3.55).}$$

(3) The ratio of the areas of two triangles of equal altitudes is equal to the ratio of their bases.

(4) The ratio of the areas of two triangles on equal bases is equal to the ratio of their altitudes.

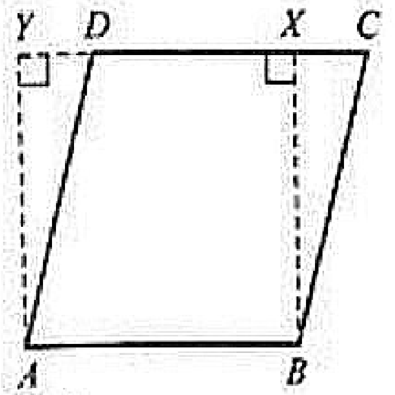


Fig. 3.55

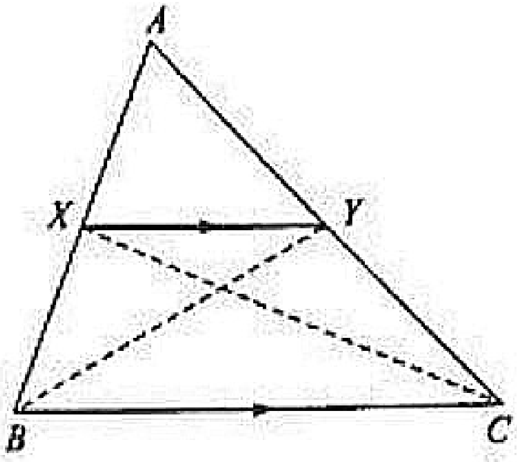


Fig. 3.56

Theorem 29. If a straight line is drawn parallel to one side of a triangle, then it divides the other two sides proportionally. Also, conversely, if a straight line divides two sides of a triangle proportionally, then it is parallel to the third side.

Proof. Let XY be a straight line parallel to BC meeting AB, AC at X, Y. (Fig. 3.56). We want to prove that $AX/XB = AY/YC$. The triangles AXY and BXY have equal altitudes and hence

$$\frac{\Delta AXY}{\Delta BXY} = \frac{AX}{XB}$$

Similarly
$$\frac{\Delta AXY}{\Delta CXY} = \frac{AY}{YC}$$

Now the triangles BXY and CXY have the same base XY and are between the same parallels. Hence $\Delta BXY = \Delta CXY$.

This gives
$$\frac{AX}{XB} = \frac{\Delta AXY}{\Delta BXY} = \frac{\Delta AXY}{\Delta CXY} = \frac{AY}{YC}.$$

Conversely, suppose the straight line XY meets AB, AC at X, Y and

$$\frac{AX}{XB} = \frac{AY}{YC}.$$

We want to prove that $XY \parallel BC$. Again, we have

$$\frac{\Delta AXY}{\Delta BXY} = \frac{AX}{XB} \quad \text{and} \quad \frac{\Delta AXY}{\Delta CXY} = \frac{AY}{YC}$$

By our hypothesis, $\frac{AX}{XB} = \frac{AY}{YC}$ and therefore $\Delta BXY = \Delta CXY$. Now BXY and CXY are two triangles of equal areas on XY and are on the same side of XY. Hence $XY \parallel BC$. □

Note. The points X, Y in the above theorem may lie on AB, AC produced as in Fig. 3.57. Still the same proof works.

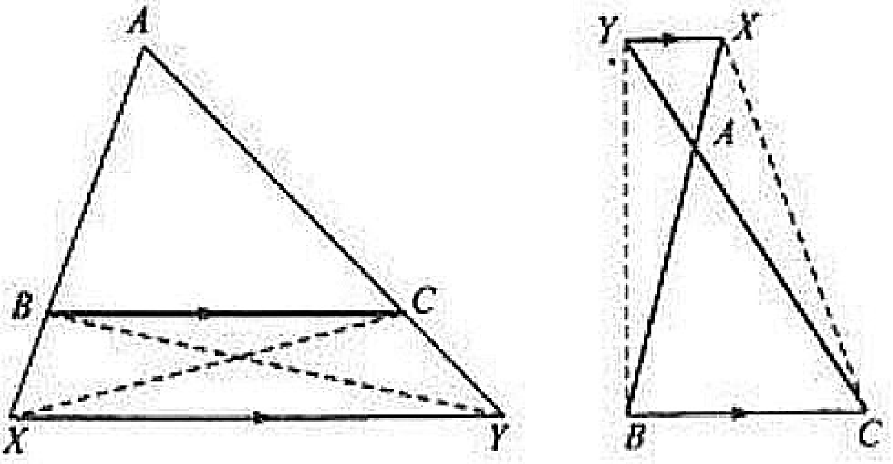
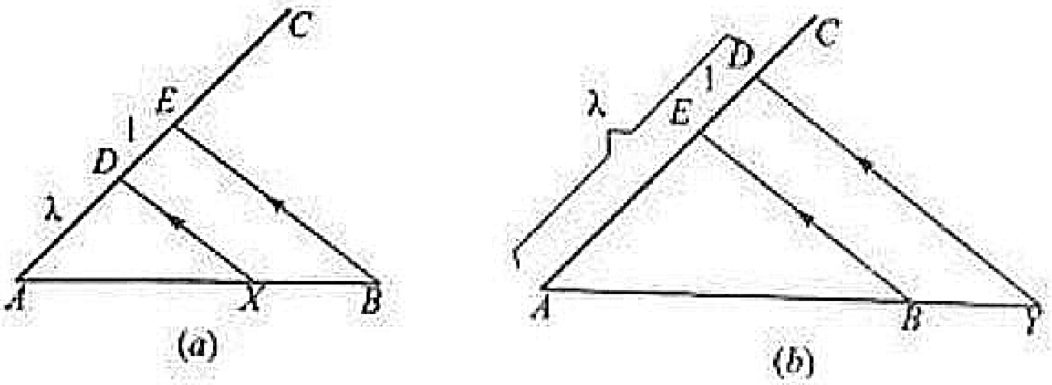


Fig. 3.57

Theorem 30. Given any ratio $\lambda/1 > 0$ there exist exactly two points X, Y on a given straight line AB dividing the line segment AB in the ratio $\lambda : 1$, one point dividing it internally and the other externally. However if $\lambda = 1$, there is no point on the straight line AB which divides the segment externally in the ratio $\lambda : 1$.



(a) (b)

Fig. 3.58

Proof. First, let us prove that there cannot be two points X, X' on AB dividing the segment AB internally in the same ratio $\lambda : 1$. Suppose X, X' are two such points then

$$\frac{AX}{XB} = \frac{AX'}{X'B} = \lambda$$

This implies that
$$\frac{AX + XB}{XB} = \frac{AX' + X'B}{X'B}.$$

This gives
$$\frac{AB}{XB} = \frac{AB}{A'B} \text{ and hence } XB = X'B \text{ or } X = X'.$$

Similarly one observes that there cannot be two points Y, Y' on AB dividing AB externally in the ratio $\lambda : 1$. Now we prove that there exist points X, Y on AB dividing the line segment AB internally and externally in the ratio $\lambda : 1$. Take a convenient ray AC as shown in Fig. 3.58. Cut off points D, E on AC such that $AD = \lambda$ and $DE = 1$, for internal division as shown in Fig. 3.58(*a*); and such that $AD = \lambda$, $ED = 1$ for external division as in Fig. 3.58(*b*). Draw $DX \parallel EB$ meeting AB at X for internal division; and draw $DY \parallel EB$ meeting AB produced at Y for external division. Then by Theorem 29,

we have
$$\frac{AX}{XD} = \frac{AD}{DE} = \frac{\lambda}{1} \quad \text{and} \quad \frac{AY}{YB} = \frac{AD}{DE} = \frac{\lambda}{1}$$

Hence the theorem. If $\lambda = 1$, then there exists no point Y on the straight line AB produced such that $AY/YB = 1$; for, any such point Y satisfies either $AY > YB$ or $AY < YB$ (Fig. 3.59). ❑

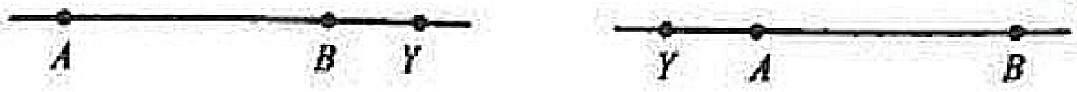


Fig. 3.59

Note. Sometimes it is convenient to attach signs to lengths of directed line segments. Taking the direction A to B to be positive, we note that length $BA = -$ length AB. Therefore if P divides AB internally then $\frac{AP}{PB} > 0$ as AP and PB have the same direction. On the other hand if P divides AB externally, then $\frac{AP}{PB} < 0$ since AP and PB have opposite directions.

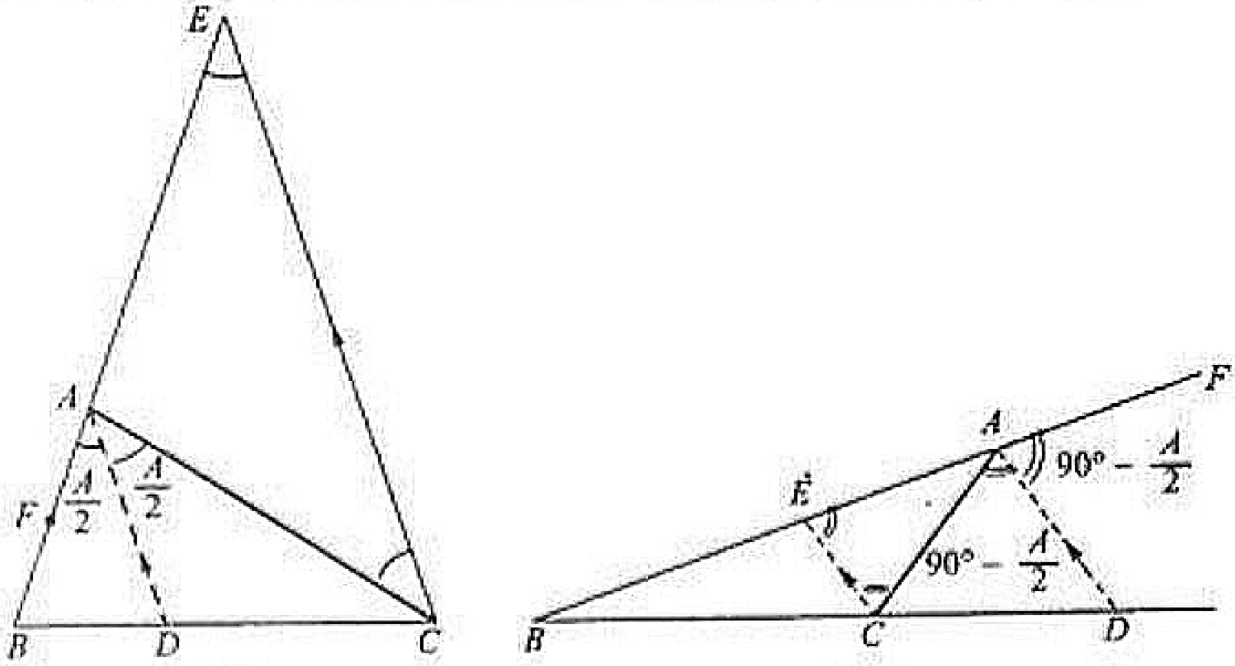
Theorem 31. The internal (or external) bisector of an angle of a triangle divides the opposite side internally (or externally) in the ratio of the sides containing the angle.

Proof. Let AD be the internal (external) bisector of $\angle A$ of ΔABC meeting BC(BC produced) at D. It is required to prove that

$$\frac{BD}{DC} = \frac{AB}{AC}$$

Draw $CE \parallel AD$ meeting BA or BA produced at E.

Let F be a point on BA or BA produced (See Fig 3.60(a) and Fig. 3.60(b)).



(a) (b)

Fig. 3.60

Then $\angle FAD = \angle AEC = \frac{\angle A}{2}$ or $\left(90° - \frac{\angle A}{2}\right)$

Also $\angle DAC = \angle ACE = \frac{\angle A}{2}$ or $\left(90° - \frac{\angle A}{2}\right)$. Thus in $\angle AEC$, we have $\angle AEC = \angle ACE$ and hence $AE = AC$. In ΔBCE, we have $AD \parallel EC$ and therefore by Theorem 29, we get

$$\frac{BD}{DC} = \frac{BA}{AE} = \frac{BA}{AC} \quad \text{(Since } AE = AC \text{ as already observed)}$$

Thus $$\frac{BD}{DC} = \frac{AB}{AC}$$ ☐

Definition 7. If a line segment AB is divided internally and externally in the same ratio at P and Q respectively, then AB is said to be divided harmonically at P and Q. P and Q are called *harmonic conjugates* with respect to AB.

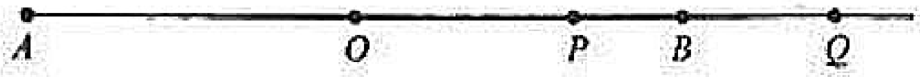


Fig. 3.61

In Fig. 3.61, P and Q are harmonic conjugates with respect to AB.

$$\frac{AP}{PB} = \frac{AQ}{BQ}$$

Note. 1. If P and Q are harmonic conjugates with respect to AB, then A and B are harmonic conjugates with respect to PQ.

For, $$\frac{AP}{PB} = -\frac{AQ}{QB}$$

implies that $$\frac{PA}{AQ} = -\frac{AP}{AQ} = \frac{PB}{QB} = -\frac{PB}{BQ}.$$

2. If P and Q divide AB harmonically, then AP, AB and AQ are in harmonic progression. (Three numbers a, b, c are in arithmetic progression if $2b = a + c$. Three numbers x, y, z are in harmonic progression if $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in arithmetic progression or equivalently

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

(See chapter 15 for more on progressions).

We have $$\frac{AP}{AQ} = \frac{PB}{BQ} = \frac{AB - AP}{AQ - AB}$$ and therefore,

$$AP(AQ - AB) = AQ(AB - AP) \text{ or}$$
$$2AP \cdot AQ = (AQ + AP)AB$$

This gives $$\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}.$$

3. If O is the mid point of AB and P, Q divide AB harmonically then $OB^2 = OP \cdot OQ$.

We have $$\frac{AP}{PB} = \frac{AQ}{BQ} \quad \text{or} \quad \frac{AP + PB}{AP - PB} = \frac{AQ + BQ}{AQ - BQ}$$

Therefore, $$\frac{AB}{(AO + OP) - (OB - OP)} = \frac{(AO + OQ) + (OQ - OB)}{(AO + OQ) - (OQ - OB)}$$

Simplifying, $$\frac{AB}{2OP} = \frac{2OB}{2OP} = \frac{2OQ}{2OB} \quad \text{or} \quad OB^2 = OP \cdot OQ.$$

Definition 8. Two triangles are *similar* if the three angles of the one are equal to the three angles of the other taken in order and the sides about the equal angles are proportional.

In other words $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$ are similar if $\angle A_1 = \angle A_2$, $\angle B_1 = \angle B_2$, $\angle C_1 = \angle C_2$ and

$$\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1A_1}{C_2A_2}.$$

We write $\Delta A_1B_1C_1 \,|||\, \Delta A_2B_2C_2$ to mean that the two triangles are similar. We observe the following.

1. Any triangle is similar to itself.
2. If $\Delta A_1B_1C_1 \,|||\, \Delta A_2B_2C_2$ then $\Delta A_2B_2C_2 \,|||\, \Delta A_1B_1C_1$.
3. If $\Delta A_1B_1C_1 \,|||\, \Delta A_2B_2C_2$ and $\Delta A_2B_2C_2 \,|||\, \Delta A_3B_3C_3$ then $\Delta A_1B_1C_1 \,|||\, \Delta A_3B_3C_3$.
4. If two triangles are congruent, then they are also similar. (How about the converse statement?)

Theorem 32. If a straight line XY parallel to BC meets the sides AB, AC of a triangle ABC at X and Y respectively, then $\Delta AXY \,|||\, \Delta ABC$.

Proof. We have $\angle AXY = \angle ABC$ and $\angle AYX = \angle ACB$ being pairs of corresponding angles formed by the transversals AB, AC cutting the parallel lines XY and BC. Also $\angle XAY = \angle BAC$ (Fig. 3.62) Therefore the two triangles are *equiangular* in the sense that the angles of the one are respectively equal to the angles of the other.

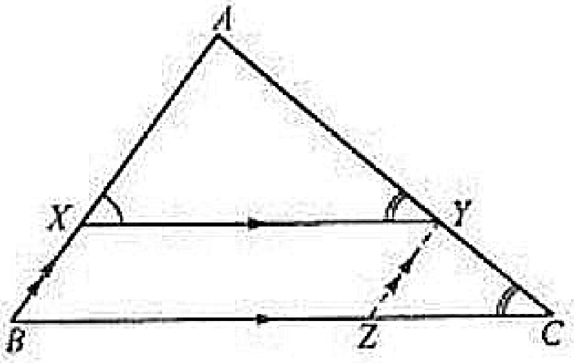


Fig. 3.62

Further $\quad \dfrac{AX}{XB} = \dfrac{AY}{YC}$ since $XY \parallel BC$ (Theorem 29)

Therefore $\quad \dfrac{AX}{AX + XB} = \dfrac{AY}{AY + YC}$ which gives $\dfrac{AX}{AB} = \dfrac{AY}{AC}$.

Through the point Y, draw $YZ \parallel AB$ meeting BC at Z. Then in ΔCAB, $YZ \parallel AB$ gives

$$\frac{AY}{AC} = \frac{BZ}{BC} = \frac{XY}{BC},$$

since $BZ = XY$ in the parallelogram $BZYX$. Thus we have

$$\frac{AX}{AB} = \frac{AY}{AC} = \frac{XY}{BC}.$$

Hence $\Delta AXY \,|||\, \Delta ABC$. □

Theorem 33. Equiangular triangles are similar triangles.

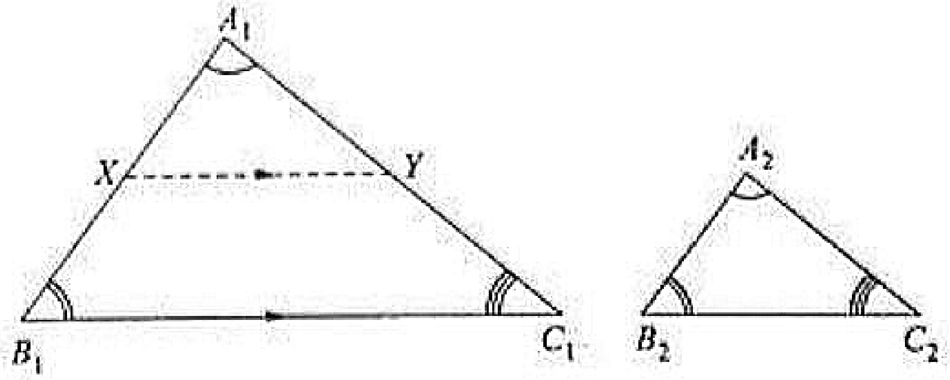


Fig. 3.63

Proof. Let the triangles $A_1B_1C_1$ and $A_2B_2C_2$ be equiangular. Then $\angle A_1 = \angle A_2$, $\angle B_1 = \angle B_2$ and $\angle C_1 = \angle C_2$. It is required to prove that the corresponding sides are proportional, *i.e.*,

$$\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1A_1}{C_2A_2}$$

We may assume that $A_1B_1 \geq A_2B_2$. Let X be a point on A_1B_1 such that $A_1X = A_2B_2$ (Fig. 3.63). Draw $XY \parallel B_1C_1$ meeting A_1C_1 at Y. Now, $\angle A_1XY = \angle A_1B_1C_1$ (cor. angles). Therefore $\angle A_1XY = \angle A_2B_2C_2$. Also $\angle A_1YX = \angle A_1C_1B_1$ (cor. angles) implies that $\angle A_1YX = \angle A_2C_2B_2$. Further $A_1X = A_2B_2$ by our construction. Therefore the triangles A_1XY and $A_2B_2C_2$ are congruent and hence similar. By Theorem 32, $\Delta A_1XY \,|||\, \Delta A_1B_1C_1$ and so we get $\Delta A_1B_1C_1 \,|||\, \Delta A_2B_2C_2$ (See the observation (3) following the definition of similar triangles).

Note. If two angles of one triangle are respectively equal to the two corresponding angles of another triangle, then the two triangles are equiangular (*i.e.*, the third pair of angles also must be equal; this is true because the angles of a triangle add upto 180°) and hence similar.

Theorem 34. If two triangles have three sides of the one proportional to the three sides of the other, taken in order, then the two triangles are similar.

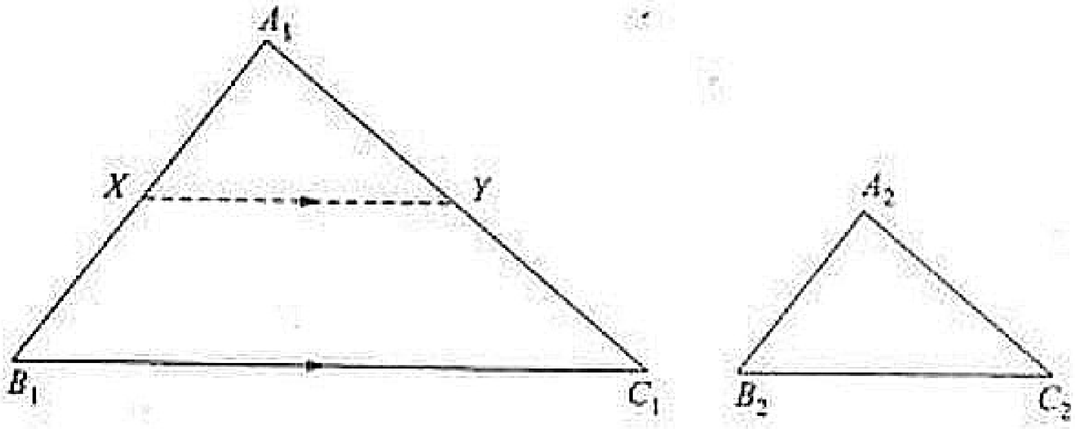


Fig. 3.64

Proof. In the triangles $A_1B_1C_1$ and $A_2B_2C_2$ we are given that

$$\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1A_1}{C_2A_2}$$

It is required to prove that the two triangles are similar; in other words we have to prove that the corresponding angles of the two triangles are equal. We may assume without loss of generality that $A_1B_1 \geq A_2B_2$. Cut off A_1X on A_1B_1 such that $A_1X = A_2B_2$. Also cut off A_1Y on A_1C_1 such that $A_1Y = A_2C_2$.

We have $$\frac{A_1X}{A_1B_1} = \frac{A_2B_2}{A_1B_1} = \frac{A_2C_2}{A_1C_1} = \frac{A_1Y}{A_1C_1},$$

Therefore by Theorem 29, $XY \parallel B_1C_1$ and so we see that

$$\Delta A_1XY \,|||\, \Delta A_1B_1C_1 \text{ (Theorem 32)}$$

This gives $$\frac{A_1X}{A_1B_1} = \frac{A_1Y}{A_1C_1} = \frac{XY}{B_1C_1} \text{ and hence}$$

$$\frac{XY}{B_1C_1} = \frac{A_1X}{A_1B_1} = \frac{A_2B_2}{A_1B_1} = \frac{B_2C_2}{B_1C_1}$$

Therefore $XY = B_2C_2$.

Now by the *SSS* theorem, $\Delta A_1XY \equiv \Delta A_2B_2C_2$.

So, the two triangles A_1XY and $A_2B_2C_2$ are equiangular. Already, we have observed that $\Delta A_1XY \mathrel{|||} \Delta A_1B_1C_1$ and hence they are equiangular. Therefore $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$ are equiangular and hence similar. ☐

Theorem 35. If two sides of one triangle are respectively proportional to two corresponding sides of the other and if the included angles are equal, then the two triangles are similar.

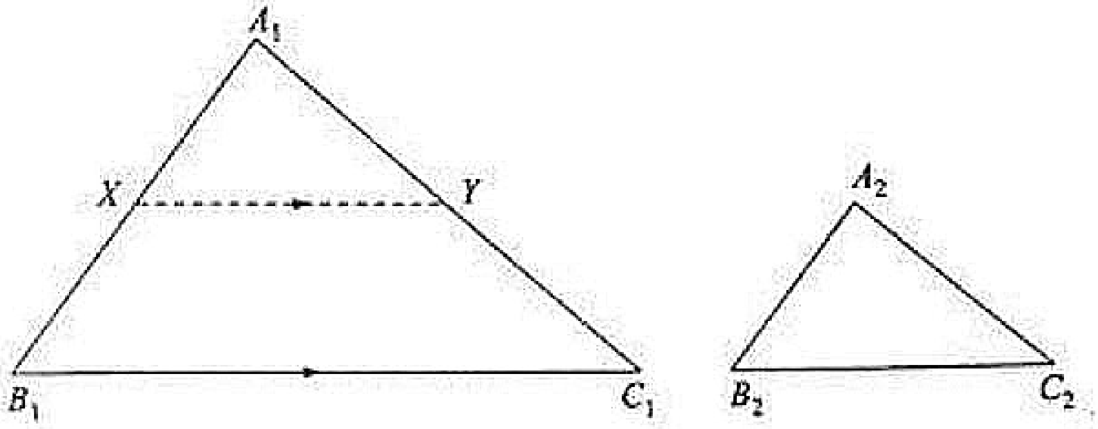


Fig. 3.65

Proof. We are given that $\dfrac{A_1B_1}{A_2B_2} = \dfrac{A_1C_1}{A_2C_2}$ and $\angle A_1 = \angle A_2$.

It is required to prove that the two triangles are similar. We may assume that $A_1B_1 \geq A_2B_2$. Let X be the point such that $A_1X = A_2B_2$. Draw $XY \parallel B_1C_1$ meeting A_1C_1 at Y. As before in Theorem 34, we have $\Delta A_1XY \mathrel{|||} \Delta A_1B_1C_1$ and hence

$$\frac{A_1X}{A_1B_1} = \frac{A_1Y}{A_1C_1} = \frac{XY}{B_1C_1}$$

But $A_1X = A_2B_2$ and therefore $\dfrac{A_2B_2}{A_1B_1} = \dfrac{A_1Y}{A_1C_1}$

Also, the hypothesis $\dfrac{A_1B_1}{A_2B_2} = \dfrac{A_1C_1}{A_2C_2}$ gives $A_1Y = A_2C_2$.

Therefore by the *SAS* test $\Delta A_1XY \equiv \Delta A_2B_2C_2$. Thus we have $\Delta A_1B_1C_1 \mathrel{|||} \Delta A_1XY$ and $\Delta A_1XY \equiv \Delta A_2B_2C_2$. This implies that $\Delta A_1B_1C_1 \mathrel{|||} \Delta A_2B_2C_2$. ☐

Theorem 36. In the two triangles $A_1B_1C_1$ and $A_2B_2C_2$ we have $\angle A_1 = \angle A_2$. Then their areas are proportional to the rectangles contained by the sides containing $\angle A_1$ and $\angle A_2$.

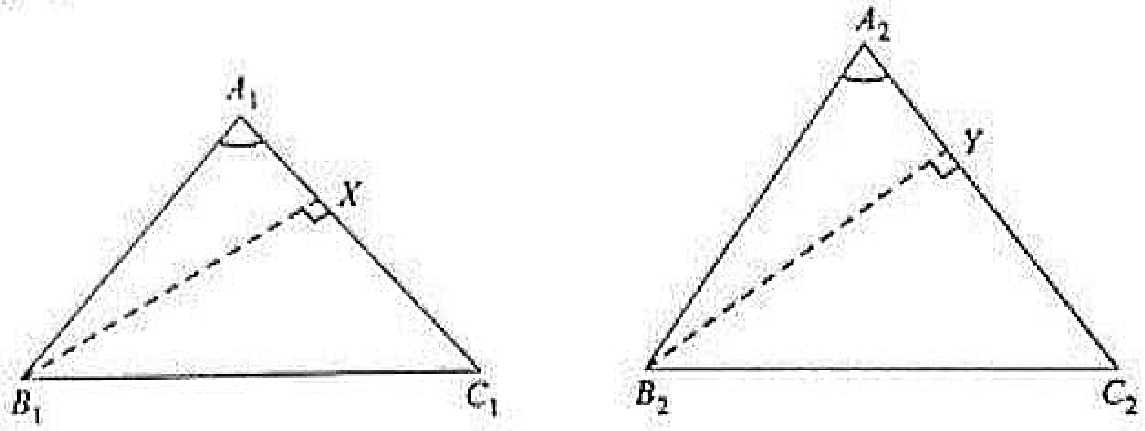


Fig. 3.66

Proof. Draw the altitudes B_1X and B_2Y (Fig. 3.66). Then the triangles A_1B_1X and A_2B_2Y are equiangular and hence similar. Therefore

$$\frac{A_1B_1}{A_2B_2} = \frac{A_1X}{A_2Y} = \frac{B_1X}{B_2Y} \qquad (*)$$

We have area of $\Delta A_1B_1C_1 = (1/2)\, A_1C_1 \,.\, B_1X$ and
area of $\Delta A_2B_2C_2 = (1/2)\, A_2C_2 \,.\, B_2Y$.

Therefore $$\frac{\Delta A_1B_1C_1}{\Delta A_2B_2C_2} = \frac{A_1C_1 \,.\, B_1X}{A_2C_2 \,.\, B_2Y} = \frac{A_1C_1}{A_2C_2} \cdot \frac{A_1B_1}{A_2B_2} \quad \text{from}(*)$$ □

Theorem 37. The areas of two similar triangles are proportional to the squares on corresponding sides.

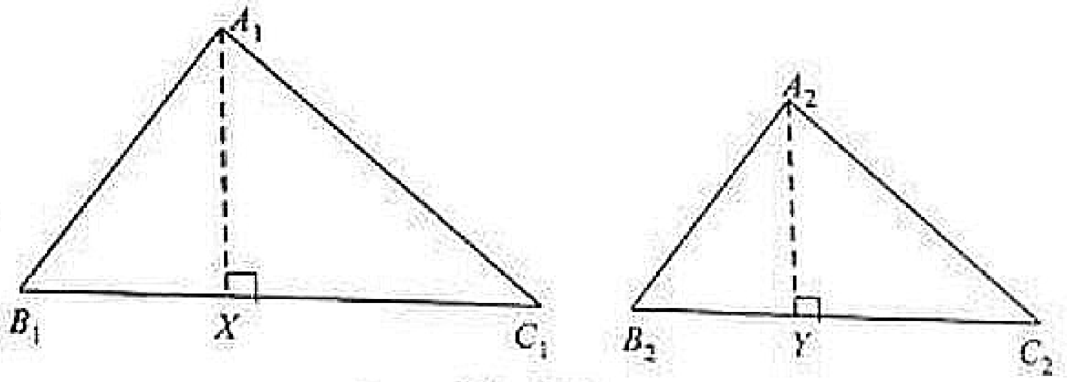


Fig. 3.67

Proof. Let $\Delta A_1B_1C_1$ be similar $\Delta A_2B_2C_2$. It is required to prove that

$$\frac{\Delta A_1B_1C_1}{\Delta A_2B_2C_2} = \frac{B_1C_1^2}{B_2C_2^2}$$

Draw the altitudes A_1X and A_2Y (Fig. 3.67).

Then $$\frac{\Delta A_1B_1C_1}{\Delta A_2B_2C_2} = \frac{B_1C_1}{B_2C_2} \cdot \frac{A_1X}{A_2Y}$$

The triangles A_1B_1X and A_2B_2Y are equiangular and hence similar.

Therefore, $$\frac{A_1X}{A_2Y} = \frac{A_1B_1}{A_2B_2}; \quad \text{and} \quad \frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} \quad \text{(by hypothesis)}$$

This gives $$\frac{\Delta A_1B_1C_1}{\Delta A_2B_2C_2} = \frac{B_1C_1}{B_2C_2} \cdot \frac{A_1X}{A_2Y} = \frac{B_1C_1^2}{B_2C_2^2}$$ □

Theorem 38. (Pythagoras's Theorem)

In a right angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the sides containing the right angle.

Proof. The right triangles ABC, ADB and BDC are all similar to each other (Fig. 3.68). Therefore by Theorem 37

$$\frac{\Delta ABC}{AC^2} = \frac{\Delta ADB}{AB^2} = \frac{\Delta BDC}{BC^2}$$

This means that each of the ratios must be equal to

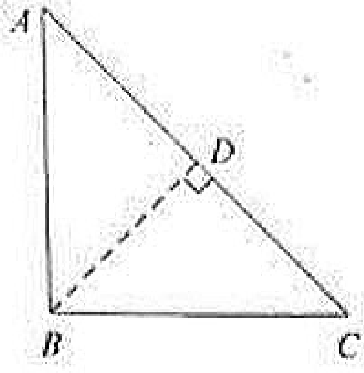


Fig. 3.68

$$\frac{\Delta ADB + \Delta BDC}{AB^2 + BC^2} = \frac{\Delta ABC}{AB^2 + AC^2}$$

Hence $AC^2 = AB^2 + BC^2$.

Note. The converse of Pythagoras's theorem is also true, namely "If in a ΔABC, $AC^2 = AB^2 + BC^2$ then ABC is a right triangle, right angled at B". The proof of this is left as an exercise.

Another proof of Pythagoras's theorem

Construct the squares $BCED$, $CAGF$ and $ABKH$ on the sides of the right triangle ABC, right angled at A. (Fig. 3.69). The triangles KBC and ABD are congruent (Why?).

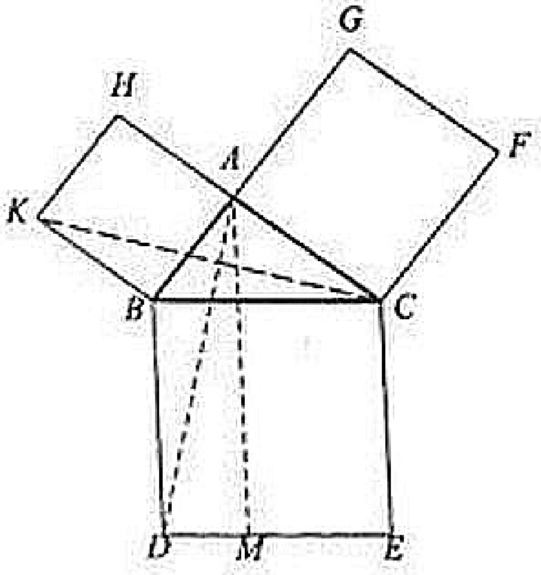


Fig. 3.69

$\therefore$ $\Delta KBC = \Delta ABD$ in area.

Now the square $KBAH$ and ΔKBC are on the same base KB and are between the same parallels.

$\therefore$ Square $KBAH = 2\ \Delta KBC$ in area.

Similarly, Rectangle $BDMX = 2\Delta ABD$ and hence

Rectangle $BDMX = 2\Delta KBC =$ Sq. $KBAH$ (1)

Similarly, Rectangle $CEMX =$ Square ACFG (2)

Adding (1) and (2) we get

Sq. $KBAH$ + Sq. $ACFG$ = Rect. $BDMX$ + Rect. $CEMX$ = Sq. $BCED$.

Hence $BC^2 = AB^2 + AC^2$. □

EXERCISE 3.5

1. $ABCD$ is a parallelogram and E is the midpoint of CD. Prove that area of $\Delta ADE = 1/4$ the area of parallelogram $ABCD$.
2. Consider the family $\mathfrak{R}$ of parallelograms on equal bases whose areas are all equal. Prove that, in $\mathfrak{R}$ that which is a rectangle has the least perimeter.
3. On the line segment AB, parallelograms $ABCD$ and $ABXY$ are drawn on opposite sides and AB or AB produced bisects CX. Then prove that $ABCD$ and $ABXY$ have equal areas.
4. ABC is a fixed triangle. P is any point on the same side of BC as that of A such that ΔPBC and ΔABC have equal areas. Find the locus of P.
5. If the lengths of two sides of a triangle are given, show that its area is greatest when the angle between the sides is a right angle.

6. $ABCD$ is a given parallelogram. What is the locus of the remaining vertices of a parallelogram equal in area to $ABCD$, having AC as a diagonal.

7. Let $ABCD$ be a quadrilateral and X be the midpoint of BD. Prove that the area $AXCB$ is one half the area of $ABCD$.

8. Let $ABCD$ be quadrilateral. Suppose parallels are drawn through A and C to BD and through B and D to AC, prove that the resulting parallelogram has twice the area of $ABCD$.

9. If D is the midpoint of the side BC of ΔABC and X is any point on AD, prove that the triangles AXB and AXC are of equal area.

10. ABC is a triangle and X is any point such that area of $\Delta AXB=$ area of ΔXAC. Find the locus of X.

11. $ABCD$ is a parallelogram and X is any point in the diagonal AC. Show that ΔABX and ΔADX are of equal area.

12. If two triangles have two sides of one equal to two sides of the other and the included angles are supplementary, prove that they are equal in area.

13. If in quadrilateral $ABCD$, AC bisects BD, show that AC also bisects the quadrilateral $ABCD$.

14. If D is a point on the side AB of ΔABC, find a point X on BC such that the triangles XAD and CAX are equal in area.

15. In quadrilateral $ABCD$, the diagonals AC and BD meet at O. Suppose the four triangles AOB, BOC, COD and DOA are equal in area, prove that $ABCD$ is a parallelogram.

16. D, E, F are the midpoints of BC, CA and AB of ΔABC. Prove that ΔBCE and ΔBCF are of equal area; deduce that FE is parallel to BC.

17. In the family $\mathfrak{R}$ of all triangles on the same base whose areas are all equal, prove that the isosceles triangle in $\mathfrak{R}$ has the least perimeter.

18. If the diagonals of a quadrilateral $ABCD$ meet at O, then prove that $\Delta ABC : \Delta ADC = BO : OD$.

19. In ΔABC, the straight lines AD, BE, CF are drawn through a point P to meet BC, CA, AB at D, E, F respectively. Prove that $PD/AD + PE/BE + PF/CF = 1$ and $AP/AD + BP/BE + CP/CF = 2$.

20. Let $ABCD$ be a parallelogram and P be any point on AC. The line $XPY \parallel DA$ meets DC at X and AB at Y. Again, the line $QPR \parallel DC$ meets AD at Q and BC at R. Prove that $PX \,.\, PQ = PY.\, PR$.

21. In problem 20, take $AD = a$, $AB = b$, $XP = x$ and $QP = y$. Show that $x/a + y/b = 1$.

22. $ABCD$ is a trapezium with $AB \parallel CD$. If the diagonals meet at O, prove that $AO : OC = BO : OD$.

23. P is a variable point on a given straight line and A is a fixed point. Q is a point on AP or AP produced such that $AQ : AP =$ constant. Find the locus of Q.

24. ABC is a triangle and XY is variable straight line parallel to AC meeting BC and BA in X, Y respectively. If AX and CY meet at P, find the locus of P.

25. In a quadrilateral $ABCD$, if the bisectors of $\angle A$ and $\angle C$ meet on BD, prove that the bisectors of $\angle B$ and $\angle D$ meet on AC.

26. In an isosceles ΔABC, the bisectors of the base angles B and C meet the opposite sides at E and F respectively. Prove that $FE \parallel BC$.

27. If A' is the midpoint of BC and if the internal bisectors of $\angle AA'B$ and $\angle AA'C$ meet AB and AC at P and Q respectively, prove that $PQ \parallel BC$.

28. The bisector of $\angle A$ in ΔABC meets BC at U. If UX is drawn parallel to AC meeting AB at X, and UY drawn parallel to AB meets AC at Y, prove that $BX/CY = AB^2/AC^2$.

29. ABC is a triangle right angled at A; AP and AQ meet BC or BC produced in P and Q and are equally inclined to AB. Show that $BP : BQ = PC : CQ$.

30. ABC is a triangle with $AB > AC$. The bisector of $\angle A$ meets BC at U and D is the midpoint of BC. Prove that $DU : DB = (AB - AC) : (AB + AC)$.

31. In ΔABC, BE and CF are the angular bisectors of $\angle B$ and $\angle C$ meeting at I. Prove that $AF/FI = AC/CI$.

32. If the bisector of $\angle A$ in ΔABC meets BC at D, prove that $BD = ac/(b + c)$ and $DC = ab/(b + c)$.

33. If the external bisector of $\angle A$ in ΔABC with $AB > AC$ meets BC produced at D' prove that $BD' = ac/(c - b)$ and $CD' = ab/(c - b)$.

34. With notations as in problems 32 and 33, prove that $DD' = 2abc/(c^2 - b^2)$.

35. In ΔABC, we have $AB > AC$. If A' is the midpoint of BC, AD is the altitude through A and if the internal and external bisectors of $\angle A$ meet BC at X and X' respectively, prove that
    (a) $A'X = a(c - b)/2(c + b)$
    (b) $A'X' = a(c + b)/2(c - b)$.
    (c) $A'D = (c^2 - b^2)/2a$.

36. (a) If the bisector of $\angle A$ in ΔABC meets BC at U, prove that $AU^2 = bc(1 - a^2/(b + c)^2)$.
    (b) If the external bisector meets BC at U' then prove that $AU'^2 = bc(a^2/(c - b)^2 - 1)$.

37. $ABCD$ is a parallelogram. The side CD is bisected at P and BP meets AC at X. Prove that $3AX = 2AC$.

38. $ABCD$ is a parallelogram. X divides AB in the ratio $3 : 2$ and Y divides CD in the ratio $4 : 1$. If XY cuts AC at Z, prove that $7AZ = 3AC$.

39. $ABCD$ is a trapezium with $AB \parallel CD$ and $AB = 2CD$. If the diagonals meet at O, then prove that $3AO = 2AC$. If AD and BC meet at F, then prove that $AD = DF$.

40. OA, OB, OC are three given line segments and P is any point on OC. If PM and PN are the perpendiculars from P on OA and OB respectively, prove that $PM : PN$ is a constant.

41. OA_1, OA_2, OA_3 are three given straight lines. Two parallel straight lines AB and CD cut OA_1, OA_2, OA_3 at P, Q, R and P', Q', R' respectively. Prove that $PQ : QR = P'Q' : Q'R'$.

42. $ABCD$ is a trapezium with $AB \parallel CD$ and the diagonals meet at O. If $XOY \parallel AB$ meets AD and BC at X and Y then prove that $XO = OY$.

43. If ABC and DEF are similar triangles and if AX and DY are the altitudes of the triangles through A and D, prove that $AX : DY = BC : EF$.

44. $ABCD$ is a parallelogram, and AXY is a straight line through A meeting BC at X and D at Y. Prove that $BX. DY$ is a constant.

45. $ABCD$ is a parallelogram. A straight line through A meets BD at X, BC at T and DC at Prove that $AX : XZ = AY : AZ$.

46. ABC is a triangle and DAE is a straight line parallel to BC such that $DA = AE$. If C meets AB at X and BE meets AC at Y, prove that $XY \parallel BC$.

47. Given four points A, B, C, D in a straight line, find a point O in the same straight li such that $OA : OB = OC : OD$.

48. $ABCD$ and $AECF$ are two parallelograms and side EF is parallel to AD. Suppose AF a DE meet at X and BF, CE meet at Y, then prove that $XY \parallel AB$.

49. If in triangles ABC and DEF, we have $\angle A = \angle D$ and $AB : DE = BC : EF$. Then prove t $\angle C = \angle F$ or $\angle C + \angle F = 180°$.

50. In a right angled triangle ABC with $\angle A = 90°$, if AD is the altitude, from A on BC, t triangles DBA and DAC are each similar to ΔABC. A also prove DA is a mean proportio to DB and DC.

51. If ΔABC is similar to ΔDEF with X and Y as midpoints of corresponding sides BC and EF, then prove that $AX : DY = BC : EF$.

52. We are given that ΔABC is similar to ΔDEF with X and Y dividing BC and EF in the same ratio. Prove that ΔABX is similar ΔDEY.

53. P is any point within ΔABC. Q is a point outside ΔABC such that $\angle CBQ = \angle ABP$ and $\angle BCQ = \angle BAP$. Show that the triangles PBQ and ABC are similar.

54. In two obtuse angled triangles, an acute angle of the one is equal to an angle of the other; and also the sides about the other acute angles are proportional.

    Prove that the triangles are similar.

55. PM and PN are the perpendiculars from a point to two given straight lines OA and OB. If PM/PN is a constant, prove that the locus of P is a straight line through O.

56. From A, perpendiculars AX, AY are drawn to the bisectors of the exterior angles of B and C of ΔABC. Prove that $XY \parallel BC$.

57. If ABC and XYZ are two triangles such that $AB : BC = XY : YZ$ and the angles B and Y are supplementary prove that $\{\Delta ABC/\Delta XYZ\} = AB^2/XY^2$.

58. A straight line is drawn through the incentre I of ΔABC, perpendicular to AI meeting AB, AC in D and E respectively. Prove that $BD.\ CE = ID^2$.

59. AD bisects $\angle A$ of ΔABC and meets BC at D. If S and S' are the circumcentres of ΔABC, show that $SD/S'D = BD/DC$.

60. If N is a point on the straight line AB and $PN \perp AB$ then prove that $AP^2 - BP^2 = AN^2 - BN^2$.

61. If in quadrilateral $ABCD$, $AC \perp BD$, show that $AB^2 + CD^2 = BC^2 + DA^2$.

62. Let ABC be an equilateral triangle and AD be the altitude through A. Show that $AD^2 = 3BD^2$.

63. In a given straight line AB, find a point P such that the difference in the squares on AP and PB is equal to the difference between two given squares.

64. ABC is a right angled triangle right angled at A. AD is the altitude through A; E is a point on AC such that $AE = CD$ and F is a point on AB such that $AF = BD$. Prove that $BE = CF$.

3.6 CONCURRENCE AND COLLINEARITY

Three points A, B, C are *collinear* if they all lie on a straight line. If A, B, C are as shown in Figure 70, then $AB + BC = AC$. If we use directed segments, we always have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0.$$

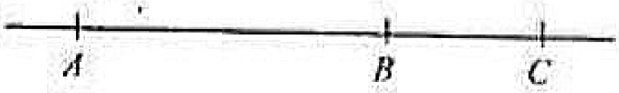


Fig. 3.70

Theorem 39. (Stewart's theorem)

If A, B, C are three collinear points and P any other point, then

$$PA^2\,.\,BC + PB^2\,.\,CA + PC^2\,.\,AB + BC\,.\,CA\,.\,AB = 0.$$

(using directed segments).

Case. (*i*) P does not lie on the straight line ABC.

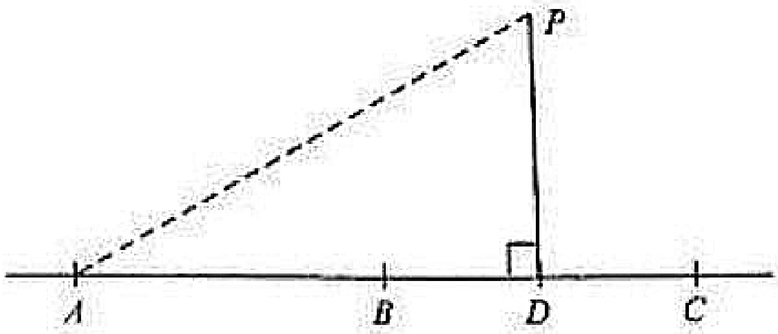


Fig. 3.71

Proof. Let D be the foot of the perpendicular from P on the line ABC. Then we have

$$PA^2 = PD^2 + DA^2 = PD^2 + (DC + CA)^2$$
$$= PD^2 + DC^2 + CA^2 + 2DC \,.\, CA.$$
$$PB^2 = PD^2 + DB^2 = PD^2 + (DC + CB)^2$$
$$= PD^2 + DC^2 + CB^2 + 2DC \,.\, CB.$$

Now $PC^2 = PD^2 + DC^2$ and hence we get

$$PA^2 = PC^2 + CA^2 + 2DC \,.\, CA.$$
$$PB^2 = PC^2 + CB^2 - 2DC \,.\, BC \text{ (since } CB = -BC).$$

This gives

$$PA^2 \,.\, BC + PB^2 \,.\, CA = PC^2 \,.\, BC + CA^2 \,.\, BC + 2DC \,.\, CA \,.\, BC + PC^2 \,.\, CA + CB^2 \,.\, CA - 2DC \,.\, BC \,.\, CA$$
$$= PC^2(BC + CA) + (BC \,.\, CA)(BC + CA).$$
$$= (PC^2 + BC \,.\, CA)\,(BA).$$
$$= (PC^2 + BC \,.\, CA)\,(-AB).$$

Hence $PA^2 \,.\, BC + PB^2\, CA + PC^2 \,.\, AB + BC.\, CA \,.\, AB = 0$.

Case. (*ii*) P lies on the straight line ABC.

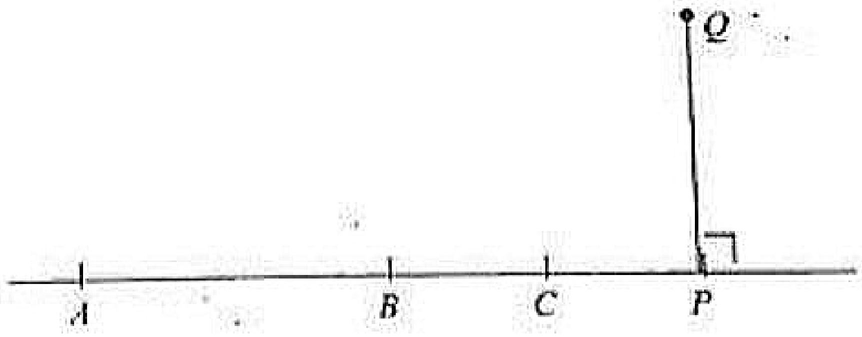


Fig. 3.72

Let Q be any point on the perpendicular through P to the line ABC. Then by case (*i*) we have

$$QA^2 \cdot BC + QB^2 \,.\, CA + QC^2 \,.\, AB + BC \,.\, CA \,.\, AB = 0.$$

We note that $QA^2 = QP^2 + PA^2$, $QB^2 = QP^2 + PB^2$ and $QC^2 = QP^2 + PC^2$.

Substituting in the above equation we get

$$QP^2(BC + CA + AB) + PA^2 \,.\, BC + PB^2 \,.\, CA + PC^2 \,.\, AB + BC \,.\, CA \,.\, AB = 0.$$

Now $BC + CA + AB = 0$ and hence we get our required result. □

Theorem 40. (Menelaus's Theorem) If a transversal cuts the sides BC, CA, AB of a triangle ABC at D, E, F respectively then

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$$

Proof. Draw $CX \parallel BA$ meeting the transversal at X (Fig. 3.73). The triangles FBD and XCD are similar since $BF \parallel CX$. Therefore,

$$\frac{BD}{DC} = \frac{FB}{CX} \text{ (in magnitudes of the segments).}$$

Again the triangles EAF and ECX are similar and so

$$\frac{CE}{EA} = \frac{CX}{AF} \text{ (in magnitudes of the segments)}$$

Hence we get
$$\frac{BD}{DC}\cdot\frac{CE}{EA} = \frac{FB}{CX}\cdot\frac{CX}{AF}$$

Therefore
$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1 \text{ in magnitude.}$$

Now, we shall examine the sign of the product. A transversal cuts two sides internally and the other side externally as in Fig. 3.73 or cuts all the three sides externally as in Fig. 3.74.

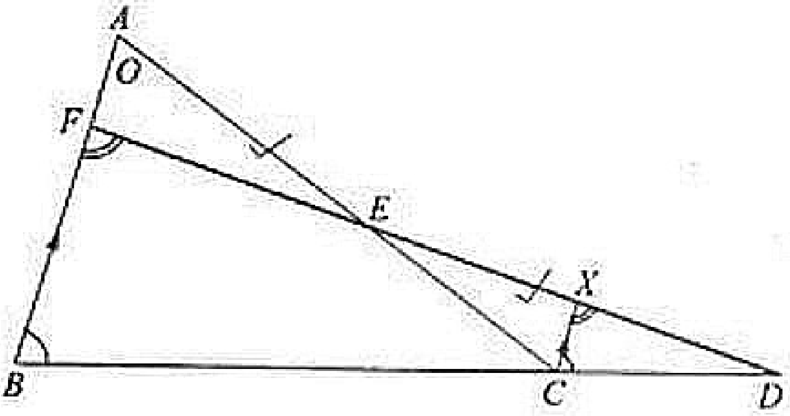


Fig. 3.73

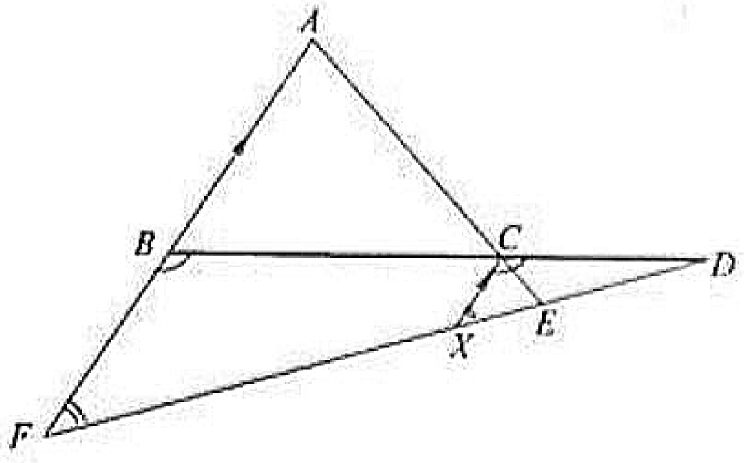


Fig. 3.74

In the former case (as in Fig. 3.73) we have one ratio negative and the other two positive. In our Fig. 3.73,

$$\frac{BD}{DC} < 0;\ \frac{CE}{EA} \text{ and } \frac{AF}{FB} \text{ are positive.}$$

In the latter case (as in Fig. 3.74), all the ratios

$\frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB}$ are negative. Thus the product

$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB}$ is always negative and

we have $$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1$$ □

Theorem 41. If D, E, F are points on the sides BC, CA, AB of a ΔABC such that

$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$ then D, E, F are collinear.

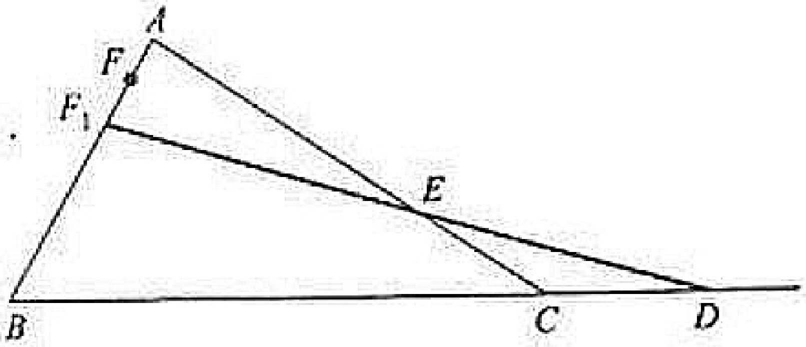


Fig. 3.75

Proof. Suppose DE meets AB at F_1. Then by Menelaus's theorem

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF_1}{F_1B} = -1.$$

This along with our hypothesis

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$$

implies that $$\frac{AF}{FB} = \frac{AF_1}{F_1B}.$$

This means that F and F_1 must coincide with each other (Theorem 30). Therefore D, E and F are collinear. □

Theorem 42. (Ceva's Theorem)

If the lines joining the vertices A, B, C of a triangle ABC to any point S in their plane meet the opposite sides in D, E, F then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1.$$

Proof. If S lies inside ΔABC, then all the three points D, E, F divide the corresponding sides internally. If S is outside ΔABC, then two of the three points D, E, F divide the corresponding sides externally while the remaining point divides the corresponding side internally. (Fig. 3.76(*a*) and Fig. 3.76(*b*)).

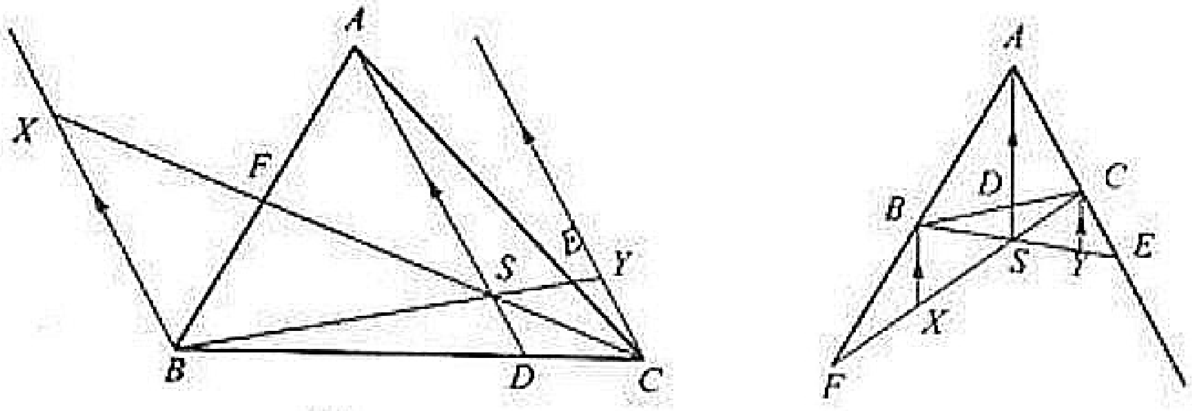
Draw BX and CY parallel to ASD meeting BS and CS at Y and X respectively. The triangles AFS and BFX are similar. Therefore

$$\frac{AF}{FB} = \frac{AS}{BX} \qquad (1)$$

$\Delta XBC \,|||\, \Delta SDC$ gives $\dfrac{BD}{DC} = \dfrac{XB}{SD}$ (2)

$\Delta BDS \,|||\, \Delta BCY$ gives $\dfrac{BD}{BC} = \dfrac{SD}{YC}$ (3)

$\Delta CEY \,|||\, \Delta AES$ gives $\dfrac{CE}{EA} = \dfrac{CY}{AS}$ (4)



(a) (b)

Fig. 3.76

In all these ratios only the lengths of the segments are taken into account. We will examine the signs later. Multiplying the above four equations we get

$$\frac{AF}{FB}\cdot\frac{BC}{DC}\cdot\frac{BD}{BC}\cdot\frac{CE}{EA} = \frac{AS}{BX}\cdot\frac{XB}{SD}\cdot\frac{SD}{YC}\cdot\frac{CY}{AS} = 1$$

This gives $\dfrac{BD}{DC}\cdot\dfrac{CE}{EA}\cdot\dfrac{AF}{FB} = 1$

When S lies inside, all these ratios $\dfrac{BD}{DC}, \dfrac{CE}{EA}, \dfrac{AF}{FB}$ are positive; and when S lies outside ΔABC one ratio is positive and the other two are negative. In Fig. 3.76(*b*), $BD/DC > 0$, $CE/EA < 0$, $AF/FB < 0$.

Thus, $\dfrac{BD}{DC}\cdot\dfrac{CE}{EA}\cdot\dfrac{AF}{FB} = +1$ (both in magnitude and sign).

Second Proof. Consider ΔADC and the transversal BSE. By Menelaus' theorem

$$\frac{DB}{BC}\cdot\frac{CE}{EA}\cdot\frac{AS}{SD} = -1 \qquad (1)$$

Again from ΔADB and the transversal CSX, we get

$$\frac{BC}{CD}\cdot\frac{DS}{SA}\cdot\frac{AF}{FB} = -1 \qquad (2)$$

Multiplying (1) and (2) we get

$$\frac{DB}{CD}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = +1$$

Equivalently $\dfrac{BD}{DC}\cdot\dfrac{CE}{EA}\cdot\dfrac{AF}{FB} = +1$

Third Proof :

We have $$\frac{BD}{DC} = \frac{\Delta ABD}{\Delta ADC} = \frac{\Delta SBD}{\Delta SDC} = \frac{\Delta ABD - \Delta SBD}{\Delta ADC - \Delta SDC} = \frac{\Delta ABS}{\Delta CAS}$$

Similarly, $$\frac{CE}{EA} = \frac{\Delta BCS}{\Delta ABS} \text{ and } \frac{AF}{FB} = \frac{\Delta CAS}{\Delta BCS}$$

Multiplying we get $$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1.$$ □

Theorem 43. (Converse of Ceva's Theorem)

If three Cevians *AD*, *BE*, *CF* satisfy

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

then they are concurrent.

[By a *Cevian* we mean a line segment joining a vertex of a triangle to any given point on the opposite side].

Proof. Suppose the two Cevians *AD* and *BE* meet at *H*. Let the Cevian through C and H be CF_1. Then by Ceva's theorem

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF_1}{F_1B} = 1$$

By assumption $$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Therefore $$\frac{AF}{FB} = \frac{AF_1}{F_1B} \quad \text{and hence } F = F_1.$$

This proves that *AD*, *BE CF are* concurrent. □

Corollary. (1) If *D*, *E*, *F* are the midpoints of *BC*, *CA*, *AB* then the medians *AD*, *BE*, *CF* are concurrent.

(2) If the Cevians *AD*, *BE*, *CF* are the internal bisectors of the angles of the *ABC*, then *AD*, *BE*, *CF* are concurrent.

(3) If the Cevians *AD*, *BE*, *CF* are the altitudes of the *ABC*, then they are concurrent

Proof. (1) If *D*, *E*, *F* are the midpoints of *BC*, *CA*, *AB* then we have

$$\frac{BD}{DC} = 1 = \frac{CE}{EA} = \frac{AF}{FB} \text{ and hence } \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Therefore, *AD*, *BE* and *CF* are concurrent.

(2) If *AD*, *BE*, *CF* are the angular bisectors of the angles of Δ *ABC*,

then $$\frac{BD}{DC} = \frac{AB}{AC}, \frac{CE}{EA} = \frac{BC}{BA} \text{ and } \frac{AF}{FB} = \frac{CA}{CB}$$

Therefore $$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Hence the internal bisectors of the angles of a triangle are concurrent. (Note that, here all the ratios are positive).

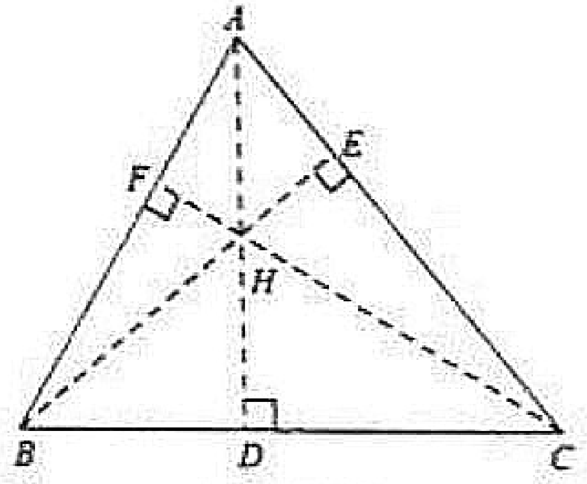


Fig. 3.77

(3) $\Delta ABD \;|||\; \Delta CBF$ gives $\frac{BD}{FB}=\frac{AD}{CF}$ (1)

$\Delta ADC \;|||\; \Delta BEC$ gives $\frac{AD}{DC}=\frac{BE}{EC}$ (2)

$\Delta AFC \;|||\; \Delta AEB$ gives $\frac{AF}{EC}=\frac{AE}{EB}$ (3)

Multiplying the three equations we get $\frac{BD}{FB}\cdot\frac{AD}{DC}\cdot\frac{AF}{FC}=\frac{AD}{CF}\cdot\frac{BE}{EC}\cdot\frac{AE}{EB}$

Therefore $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$ (in magnitude)

Again, all the ratios $\frac{BD}{DC},\frac{CE}{EA},\frac{AF}{FB}$ positive and hence

AD, BE and CF are concurrent. □

EXERCISE 3.6

1. Let ABC be any triangle right angled at A. If D, E trisect the hypotenuse BC then prove that $AD^2 + AE^2 = 5/9BC^2$.
2. If from a point O, OD, OE, OF are drawn perpendicular to the sides BC, CA, AB respectively of ΔABC then prove that
$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0.$$
3. The median AA' of the ΔABC meets the side $B'C'$ of the medial triangle $A'B'C'$ in P and CP meets AB in Q. Show that $AB = 3\,AQ$.
4. If a line through the centroid G of ΔABC meets AB in M and AC in N then prove that $AN.\,MB + AM.\,NC = AM.\,AN$ both in magnitude and sign.
5. Prove that the triangle formed by the points of contact of the sides of a given triangle with the excircles corresponding to these sides is equivalent to the triangle formed by the points of contact of the sides of the triangle with the inscribed circle.
6. (*i*) Prove that the external bisectors of the angles of a triangle meet the opposite sides in three collinear points.

   (*ii*) Prove that the internal bisectors and the external bisector of the third angle meet the opposite sides in three collinear points.
7. Prove that the sides of the orthic triangle meet the sides of the given triangle in three collinear points.

8. B' and C' are the midpoints of AC, AB of ΔABC; Q is the midpoint of $B'C'$ and BQ meets AC at R. Prove that $AR/RC = 1/2$.
9. If H is any point within ΔABC, prove that the external bisectors of the angles AHB, BHC, CHA meet AB, BC, CA respectively at three collinear points.
10. Points X and Y are taken on the sides CA, AB of ΔABC such that $CX/XA = AY/YB = \lambda$. If XY and CB produced meet at D prove that $CD = \lambda^2 BD$.
11. G is the centroid of ΔABC; AG is produced to X such that $GX = AG$. If we draw parallels through X to CA, AB, BC meeting BC, CA, AB at L, M, N respectively, prove that L, M, N are collinear.
12. In ΔABC, XY is drawn parallel to BC cutting AB, AC in X and Y. Prove that BY and CX intersect on the median through A.
13. AD, BE, CF are concurrent lines in a ΔABC. Show that the lines through the midpoints of BC, CA, AB respectively parallel to AD, BE, CF are concurrent.
14. Equilateral triangles DBC, ECA and FAB are constructed externally on the sides BC, CA, AB of ΔABC. Prove that AD, BE and CF are also concurrent.
15. In ΔABC, AD, BE and CF are concurrent lines. P, Q, R are points on EF, FD, DE such that DP, EQ and FR are concurrent. Prove that AP, BQ and CR are also concurrent.
16. AD, BE and CF are three concurrent lines in ΔABC, meeting the opposite sides in D, E, F respectively. Show that the joins of the midpoints of BC, CA, AB to the midpoints of AD, BE and CF are concurrent.
17. AD, BE and CF are three concurrent lines meeting the sides BC, CA, AB of ΔABC in D, E, F respectively. Suppose EF, FD and DE meet BC, CA, AB at X, Y and Z respectively. Prove that B, C divide DX harmonically and that X, Y, Z are collinear.
18. If X and Y are variable points on the sides CA, AB of ΔABC such that $CX/XA + AB/AY = 1$, prove that XY passes through a fixed point.
19. A straight line cuts the sides AB, BC, CD, DA of a quardrilateral in P, Q, R, S. Prove that $AP/PB \,.\, BQ/QC \,.\, CR/RD \,.\, DS/SA = 1$.
20. If two triangles ABC and DEF are such that the perpendiculars from A, B, C to EF, FD, DE are concurrent, prove that the perpendiculars from D, E, F to BC, CA, AB are concurrent.
21. In quadrilateral $ABCD$, let AB and CD meet at E and AD and BC meet at F. Then prove that the midpoints of AC, BD and EF are collinear.
22. Four points P, Q, R, S are taken on the sides AB, BC, CD, DA of a quadrilateral such that $AP/PB \,.\, BQ/QC \,.\, CR/RD \,.\, DS/SA = 1$. Prove that PQ and RS intersect on AC.
23. ABC and $A'B'C'$ are two triangles such that AA', BB', CC' meet at O. Prove that if BC, $B'C'$ meet at L; CA, $C'A'$ meet at M; AB, $A'B'$ meet at N then L, M, N are collinear.

PROBLEMS

1. $ABCD$ is a quadrilateral such that the sum of the angles at A and B is equal to the sum of the angles at C and D. Prove that two sides of the quadrilateral are parallel to one another.
2. Draw four straight lines at random.
   (*a*) In how many points do the lines intersect?
   (*b*) How many triangles are formed?
   (*c*) Into how many regions do the lines divide the plane?
3. A triangle is rotated in its own plane about the point A into a position $A'B'C'$. If AC bisects BB', prove that AB' bisects CC'.

4. If $ABCD$ is a quadrilateral in which $AB + CD = BC + AD$, prove that the bisectors of the angles of the quadrilateral meet in a point which is equidistant from the sides of the quadrilateral.
5. $ABCDEFGH$ is a regular octagon and AF, BE, CH, DG are drawn. Prove that their intersections are the angular points of a square.
6. A figure consists of five equal squares in the form of a cross. Show how to divide it by two straight cuts into four equal figures which will fit together to form a square.
7. $ABCD$ is a quadrilateral and X is a given point in AD. Find a point Y in AB such that the area of the ΔAXY is equal to that of $ABCD$. Hence show how to divide the quadrilateral $ABCD$ into three equal parts by straight lines drawn through X.
8. A square of perimeter 52 is inscribed in a square of perimeter 68. What are the possible distances from a fixed vertex of the inner square to the four vertices of the outer square.
9. Given a hexagon of side $2a$ and 25 points inside it, show that there are at least two points among them whose distance apart is at most a units.
10. Let X be a point inside a rectangle $ABCD$. If $XA = a$, $XB = b$, $XC = c$. Find XD.
11. In ΔABC, find points X, Y, Z on AB, BC, CA such that $AXYZ$ is a rhombus. Show that the area of the rhombus $AXYZ \leq (1/2)\ \Delta ABC$.
12. Equilateral triangles BCX, CAY and ABZ are constructed externally on the sides of ΔABC. I, P, Q, R are the midpoints of BX, BZ and AC prove that ΔPQR is equilateral.
13. Given a parallelogram $ABCD$, a straight line cuts off 1/3 AB, 1/4 and λAC from the segments AB, AD and AC respectively. Find λ.
14. The bisector of each angle of a triangle intersects the opposite side at a point equidistant from the midpoints of the other two sides of the triangle. Find all such triangles.
15. $ABCD$ is a parallelogram in which $AB/BC = \lambda$. If P, Q are points on the line CD with P on CD and M is a point on BC such that AD bisects $\angle PAQ$ and AM bisects $\angle PAB$, $BM = a$, $DQ = b$ prove that $AQ = a/\lambda + b$.
16. In ΔABC a point X is taken on AC and a point Y is taken on BC. If AY and BX meet at O, find the area of ΔCXY if the areas of triangles OXA, OAB and OBX are x, y, z respectively.
17. $ABCD$ is a trapezium with $AD \parallel BC$; $AD = 3BC$ and a transversal XY cuts BC at X and AD at Y. If EF is a line segment contained in XY such that $AE \parallel DF$, $BE \parallel CF$ and $AE/DF = CF/BE = 2$, show that the area of $\Delta EFD = 1/4$ area $ABCD$.
18. In ΔABC, X and Y are points on the sides AC and BC respectively. If Z is on the segment XY such that $\dfrac{AX}{XC} = \dfrac{CY}{YB} = \dfrac{XZ}{ZY}$ prove that the area of ΔABC is given by $\Delta ABC = [(\Delta AXZ)^{1/3} + (\Delta BYZ)^{1/3}]^3$.
19. $ABCD$ is a trapezium with $AD \parallel BC$; X line on AD such that $AX/XD = \lambda$. The straight lines AB and CD meet at E, and the lines BX and AC meet at F. If EF meets AD at Y prove that $AY/YD = \lambda/(\lambda + 1)$.
20. ABC is a triangle and A_1, B_1, C_1 are points on BC, CA, AB such that
$$BA_1/A_1C = CB_1/B_1A = AC_1/C_1B = \lambda.$$
If A_2, B_2, C_2 are points on B_1C_1, C_1A_1, and A_1B_1 such that
$$B_1A_2/A_2C_1 = C_1B_2/B_2A_1 = A_1C_2/C_2B_1 = 1/\lambda.$$
prove that $\Delta\ ABC$ is similar to $\Delta A_2B_2C_2$ and find the ratio of similitude.
21. In ΔABC, D, E, F are points on the sides BC, CA, AB. Also, A, B, C are points on YZ, ZX, XY of ΔXYZ for which $EF \parallel YZ$, $FD \parallel ZX$, $DE \parallel XY$. Prove that area of $\Delta\ ABC = \{\text{area } \Delta\ DEF, \text{ area } \Delta\ XYZ\}^{1/2}$.

22. l and m are two straight lines intersecting at O. If the perpendiculars from A to l, m meet the straight lines l, m at X, Y respectively; and the perpendiculars from B to l, m meet them at P, Q show that the angle between XY and PQ is $\angle AOB$ (assume that $\angle AOB$ is acute).

23. Prove that the locus of the point P which moves such that $AM^2 - MB^2 = \lambda =$ constant is a straight line AB, where A and B are two fixed points.

24. O is a point in the plane of ΔABC with $OA = x$, $OB = y$ and $OC = z$. Prove that there is no $d > o$ and no point P such that $PA = \sqrt{(x^2 + d)}$, $PB = \sqrt{(y^2 + d)}$ and $PC = \sqrt{(z^2 + d)}$.

25. ABC is a triangle P is a point inside ΔABC such that its distances from the sides of ΔABC are x, y, z. If a, b, c, k are given constants, prove that the locus of P such that $ax + by + cz = k$ is either an empty set or a line segment or coincides with the set of all points in ΔABC.

26. Find the locus of points P within a given ΔABC and such that the distances from P to the sides of the given triangle can themselves be the sides of a certain triangle.

27. Let l be a fixed straight line and l_1, l_2, l_3 be three lines perpendicular to l. A, B, C are three fixed points on l; A_1, B_1, C_1 are three points on l_1, l_2, l_3 respectively such that the perpendiculars from A, B, C to B_1C_1, C_1A_1 and A_1B_1 meet at a point. Then prove that for any three points X, Y, Z on l_1, l_2, l_3 respectively, the perpendiculars from A, B, C to YZ, ZX, XY meet at a point.

28. Prove that the feet of the four perpendiculars dropped from a vertex of a triangle upon the four bisectors of the other two angles are collinear.

29. The vertex A of a variable ΔABC is fixed; the side BC of variable length lies on a fixed straight line. Show that the locus of the projection of A upon the bisectors of $\angle B$ and $\angle C$ is a straight line.

30. Show that the sum of the reciprocals of the internal bisectors of a triangle is greater than the sum of the recipocals of the sides of the triangle.

31. The internal bisector of the $\angle B$ of ΔABC meets the sides $B'C'$ and $B'A'$ of the medial triangle in the points A'', C'' respectively. Prove that AA'', CC' are perpendicular to the bisector of $\angle B$ and that $B'A'' = B'C''$. (Similar result holds for external bisector).

32. In any ΔABC if L, M, N are three collinear points on the sides BC, CA, AB and L', M', N' are points on BC, CA, AB respectively such that $A'L = A'L'$, $B'M$, $C'N = C'N'$ where A', B', C' are the midpoints of BC, CA, AB. Prove that L', M', N' are also collinear. (Two such transversals LMN, $L'M'N'$ are called *reciprocal* or *isotomic* transversals).

33. Draw two straight lines of given directions so that they are reciprocal transversals for a given triangle.

34. If D, D'; E, E'; F, F' are isotomic pairs of points on the sides BC, CA, AB of ΔABC then prove that area ΔDEF = area $\Delta D'E'F'$.

35. Two equal segments AE, AF are taken on the sides AB, AC, of a ΔABC. Show that if the median through A of ΔABC meets EF at X then $EX/XF = AC/CB$.

36. D, E, F are any three points on BC, CA, AB of ΔABC such that AD, BE, CF are concurrent at P. If D', E', F' are the isotomics of D, E, F then prove that AD', BE', CF' are also concurrent. (If we call this point as Q then P and Q are called *isotomic conjugates*; see problem 33).

37. If AD, BE, CF are Cevians of ΔABC concurring at a point O, then prove that

$$OD/AD + OE/BE + OF/CF = 1.$$

38. With notations as in 38, prove that

$$AO/OD = AE/EC + AF/FB.$$

39. If two triangles are symmetrical with respect to a poin, show that the reciprocal transversals of the sides of one triangle with respect to the other are concurrent.

40. Show that the lines joining the vertices of a given equilateral triangle to the images of a given point in the respectively opposite sides are concurrent.

41. Prove that the lines joining the midpoints of three concurrent Cevians to the midpoints of ths corresponding sides of the given triangle are concurrent.

42. If O, A, C, D are collinear and $OA^2 = OC \cdot OD$ both in magnitude and sign, then prove that the symmetric of A with respect to O is the harmonic conjugate of A with respect to C, D.

43. If CD divide AB harmonically and O is the midpoint of AB prove that

$$OC^2 + OD^2 = CD^2 + 2OA^2.$$

44. If C, D divide AB harmonically and A', B' are the harmonic conjugates of D with respect to the pairs A, C and B, C prove that C, D also divide $A'\ B'$ harmonically.

45. If C, D divide AB harmonically and O is he midpoint of AB prove that

(*i*) $1/(CA \cdot CB) + 1/(DA\ DB) = 1/(OA \cdot OB)$.

(*ii*) $1/BC = 1/AB + 1/AD + 1/CD$.

(*iii*) $DA \cdot DB = DC\ .\ DO$.

46. Let AB be divided harmonically at C and D . O is a point not on AB and the straight line through B parallel to OA meets OC, OD at P, Q respectively. Prove that $PB = BQ$.

47. If A, B, C, D are collinear points and O is a point not on AB such that the parallel through B to OA meets OC, OD at P, Q with $PB = BQ$, then prove that C, D divide AB harmonically.

48. If AB is divided harmonically by C, D and O is a point not on AB, then prove that any transversal cuts OA, OB, OC, OD in four harmonic points.

49. If C, D divide AB harmonically and C', D' divide $A'\ B'$ harmonically; and the lines AA', BB', CC', meet at a point O, then prove that DD' also passes through O.

50. A straight line through A of the parallelogram $ABCD$ meets BD, BC, CD in E, F, G respectively. Show that $1/AE = 1/AF + 1/AG$.

51. A straight line $PGQR$ through the centroid G of ΔABC meets AB, AC and BC produced at P, Q, R respectively. Prove that $1/GP = 1/GQ + 1/GR$.

52. Consider ΔABC and a point within the triangle, If AP, BP, CP meet the opposite sides in D, E, F respectively, prove that of the numbers AP/PD, BP/PE, CP/PF at least one is less than or equal to 2 and at least one is greater than or equal to 2.

53. AOB is a triangle with $\angle AOB < 90°$. Through a point $M \neq O$, perpendiculars are drawn to OA and OB meeting OA, OB at P, Q respectively. H is the orthocentre of ΔOPQ. What is the locus of H if M is permitted to range over (*i*) the side AB (*ii*) the interior of ΔOAB?

54. P, Q, R are points on the sides BC, CA, AB of ΔABC. Prove that the area of at least one of the triangles AQR, BRP, CPQ is less than or equal to one quarter of the area of ΔABC.

55. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the mid points of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AK, BK, BL, CL, CM, DM, DN, AN$ are twelve vertices of a regular dodecagon.

56. Let A_1A_2, B_1B_2, C_1C_2 be three equal segments on the sides of a equilateral triangle ABC. Prove that in the triangle formed by the lines B_2C_1, C_2A_1, A_2B_1 the segments B_2C_1, C_2B_1 are proportional to the sides in which they are contained.

57. ABC is a triangle and the line YCX is parallel to AB such that AX and BY are the angular bisectors of $\angle A$ and $\angle B$ respectively. If AX meets BC at D and BY meets AC at E and if $YE = XD$ prove that $AC = BC$.

58. Let ABC be an equilateral triangle with side a. M is a point such that $MS = d$ where S is the centre of $\Delta\, ABC$. Prove that the area of the triangle whose sides are of length MA, MB, MC is $[\sqrt{3}/12]\ |a^2 - 3d^2|$.

59. Give two equilateral triangles $A_1B_1C_1$ and $A_2B_2C_2$, find the locus of points M such that the two triangles formed by the line segments MA_1, MB_1, MC_1 and the segments MA_2, MB_2, MC_2 are of the same area.

CHA

4

GEOMETRY–CIRCLES

4.1 CIRCLES-PRELIMINARIES

A circle is a geometric figure in a plane such that all its points are equidistant from a fixed point in the plane. The fixed point is the centre of the circle and the constant distance from the centre is the radius of the circle.

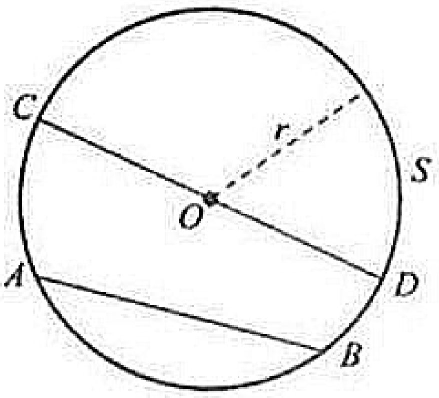


Fig. 4.1

The circle S in Fig. 4.1 has centre O and radius r.

A chord of a circle is a straight line segment joining any two points on the circle. A chord passing through the centre is called a *diameter* of the circle. In Fig. 4.1, AB is a chord of the circle S and CD is a diameter. Circles which have the same centre are called *concentric circles.*

We observe that a point is within, upon or outside a circle according as its distance from the centre is less than, equal to, or greater than the radius. Concentric circles whose radii are unequal, do not intersect with each other. A circle is symmetric about any of its diameters (Fig. 4.2). Also, a circle is symmetric about its centre (Fig. 4.3).

Theorem 1. The perpendicular bisector of any chord of a circle passes through the centre of the circle.

Proof. Let O be the centre of a circle S and AB be any chord; let C be the mid point of AB (See Fig. 4.4). We want to prove that the perpendicular bisector of AB passes through O. In other words, we wish to prove that $OC \perp AB$. In the triangles AOC and BOC we have $OA = OB$ (radius of the circle), $AC = CB$ (by hypothesis) and OC is common. Therefore the two triangles are congruent. Hence $\Delta OCA \equiv \Delta OCB$ which implies that $\angle OCA = \angle OCB = 90°$ or $OC \perp AB$.

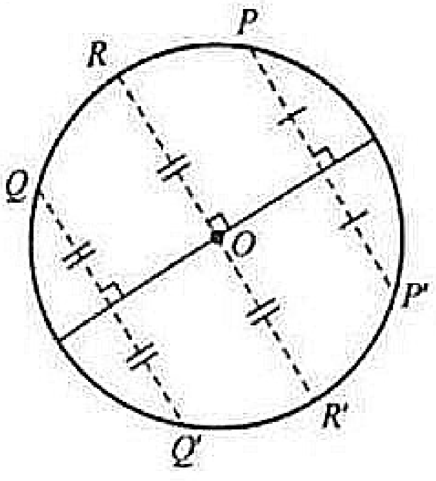


Fig. 4.2

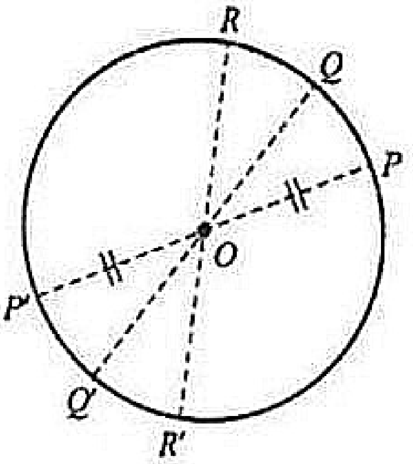


Fig. 4.3

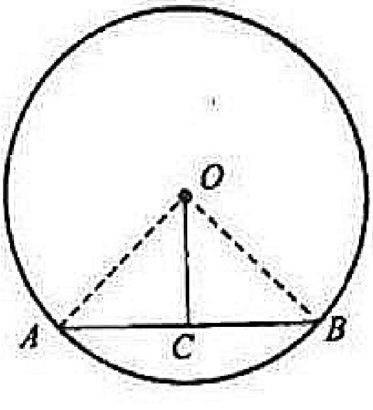


Fig. 4.4

Corollary. A circle is symmetrical about any of its diameters.

Proof. Let AB be any diameter of a circle with centre O (Fig. 4.5). If P is any point on the circle draw $PQ \perp AB$ meeting the circle again at Q. Then by Theorem 1, $PN = NQ$ (Fig. 4.5). Thus the circle is symmetrical about AB. □

Theorem 2. Given any three non-collinear points A, B, C there exists a unique circle passing through A, B and C.

Proof. Let A, B, C be any three non-collinear points. Suppose the perpendicular bisectors of BC and CA meet at S (Fig. 4.6). Then S lies on the perpendicular bisector of BC implies that $SB = SC$ (Theorem 12, Chapter 3). Again, S is also on the perpendicular bisector of CA implies that $SC = SA$. Hence we have $SA = SB = SC$. Further we note that any point equidistant from A, B, and C should lie on the perpendicular bisectors of the BC and CA. Therefore, S is the only point equidistant from A, B, and C and so the circle with centre S and radius SA is the unique circle passing through A, B and C. □

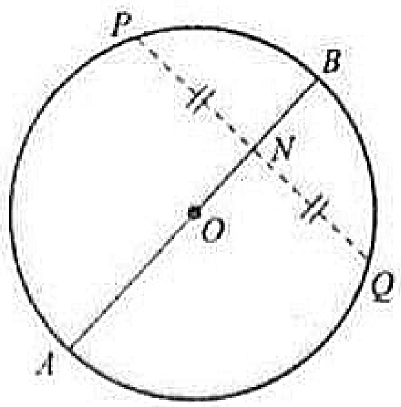


Fig. 4.5

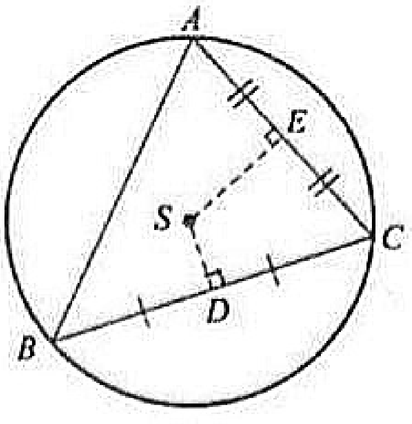


Fig. 4.6

We saw in the previous chapter that the perpendicular bisectors of the sides of a triangle concur at a point. We observe that the point of concurrence is the centre of the unique circle passing through the vertices of the triangle; hence it is called the circumcentre of the triangle and the corresponding circle is called the circumcircle of the triangle.

Corollary 1. If two circles have three points in common, then they must coincide. □

Proof. Immediate from the theorem.

Corollary 2. Two circles cannot intersect in more than two points.

Proof. Immediate from Corollary 1. ❑

Corollary 3. If A, B, C are any three points on a circle and O is a point within the circle such that $OA = OB = OC$ then O is the centre of the circle. ❑

Proof. It is immediate from the theorem.

Corollary 4. Two circles cannot have a common arc unless they coincide.

Proof. Any arc of a circle contains infinitely many points. Hence by Corollary 1, if two circles have a common arc, then they coincide. ❑

Theorem 3. Equal chords of a circle are equidistant from the centre. Conversely, if two chords of a circle are equidistant from the centre, then they are equal.

Proof. Suppose AB and CD are two equal chords of a circle with centre O. Let OX, OY be the perpendiculars from O onto the chords AB, CD respectively (Fig. 4.7). It is required to prove that $OX = OY$. By Theorem 1, OX and OY are the perpendicular bisectors of AB and CD. Hence $AX = CY = 1/2\ AB = 1/2\ CD$. Now, in ΔAOX and ΔCOY we have $\angle AXO = \angle CYO = 90°$, hypotenuse AO = hypotenuse CO (radii) and $AX = CY$. Therefore $\Delta AOX \equiv \Delta COY$ and hence $OX = OY$. Thus, the two equal chords AB and CD are equidistant from the centre O.

Converse. We may use the same figure, Fig. 4.7. But now we assume $OX = OY$ and we wish to prove that $AB = CD$. Again, we compare the right triangles AOX and COY. We have hypotenuse AO = hypotenuse CO and $OX = OY$. Therefore, the two triangles are congruent and hence we get $AX = CY$. But by Theorem 1, $AX = (1/2)\ AB$ and $CY = (1/2)\ CD$. Therefore, $AX = CY$ implies that $AB = CD$. ❑

Theorem 4. Given any two chords of a circle, the one which is nearer to the centre is greater than the one more remote.

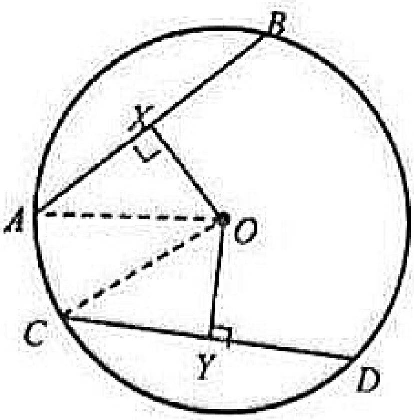


Fig. 4.7

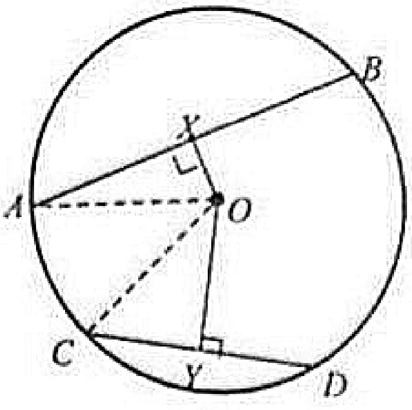
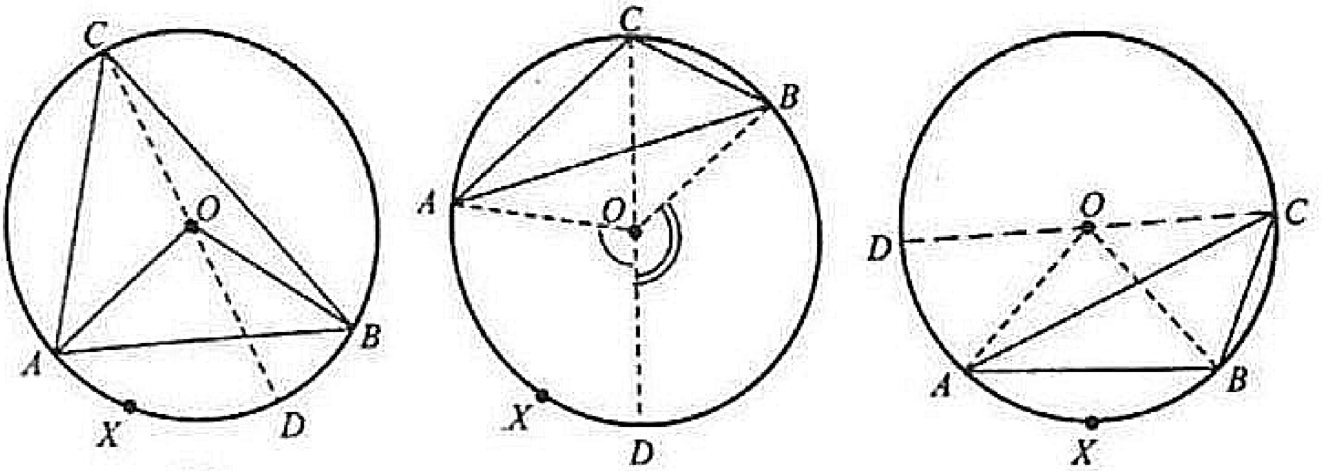


Fig. 4.8

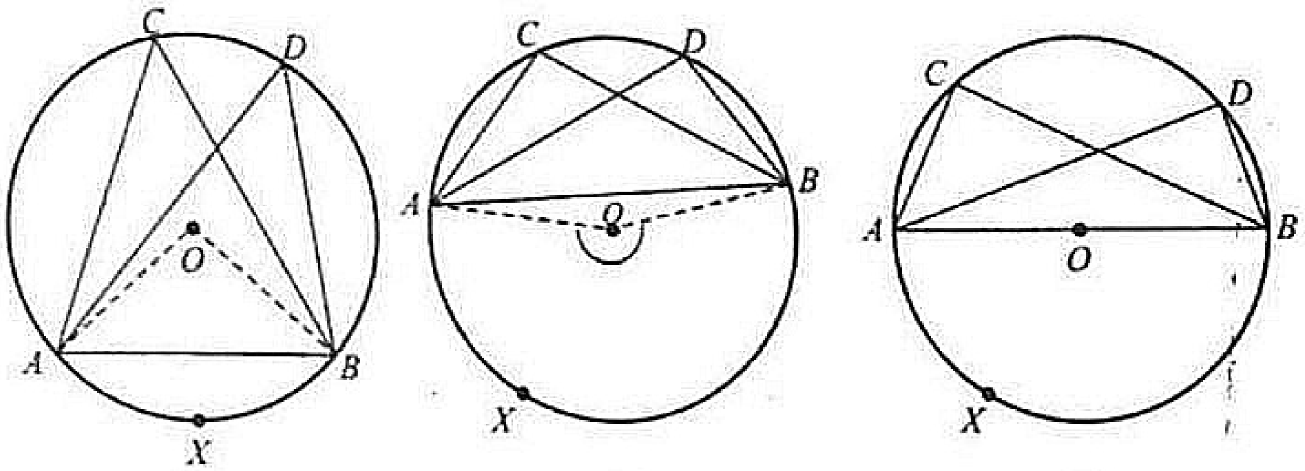
Proof. Let AB, CD be two chords of a circle with centre O. Let OX, OY be the perpendiculars to AB, CD meeting them at X, Y respectively. (Fig. 4.8). Suppose $OX < OY$. Then from the right triangles AOX and COY we have $AO^2 = OX^2 + AX^2$ and $CO^2 = OY^2 + CY^2$. But $AO = CO$ = radius of the circle. Therefore we get $OX^2 + AX^2 = OY^2 + CY^2$. By assumption $OX < OY$ and so $OX^2 + AX^2 = OY^2 + CY^2$ can hold good if and only if $AX > CY$. But by Theorem 1, $AX = (1/2)\ AB$ and $CY = (1/2)\ CD$. ❑

Thus $OX < OY$ implies that $AB > CD$.



(a) (b) (c)

Fig. 4.9



(a) (b) (c)

Fig. 4.10

Theorem 5. The angle subtended at the centre is double the angle subtended at any point on the remaining part of the circumference, for any arc of a circle.

Proof. Let AXB be an arc of a circle with centre O and C be any point on the remaining part of the circumference. It is required to prove that $\angle AOB = 2\ \angle ACB$. Let CO meet the circle again at D. Suppose O lies within the angle ACB as in Fig. 4.9(a) or Fig. 4.9(b). The triangles AOC and BOC are isosceles (Why?) and so we have $\angle OAC = \angle OCA$ and $\angle OBC = \angle OCB$. This gives, $\angle AOD = 2\ \angle ACO$ and $\angle BOD = 2\ \angle BCO$. Adding, we get $\angle AOD + \angle BOD = \angle AOB = 2(\angle ACO + \angle BCO) = 2\ \angle ACB$. If O is not within $\angle ACB$ as in Fig. 4.9(c), then we have $\angle BOD - \angle AOD = 2(\angle BCO - \angle ACO)$ which in turn gives $\angle AOB = 2\angle ACB$. ☐

Theorem 6. Angles in the same segment of a circle are equal.

Proof. Let ACB, ADB be two angles in the same segment $ACDB$ of a circle with centre O.

Then by Theorem 5, we see that $\angle ACB, = (1/2)\ \angle AOB = \angle ADB$. Hence, angles in the same segment are equal. (See Fig. 4.10(a) and 4.10(b)). ☐

If the segment happens to be a semicircle as in Fig. 4.10(c), then $\angle AOB = 2$ right angles $= 180°$ sines AOB is a straight angle in this case. This observation leads to the following corollary.

Corollary. The angle in a semicircle is always 90° ☐

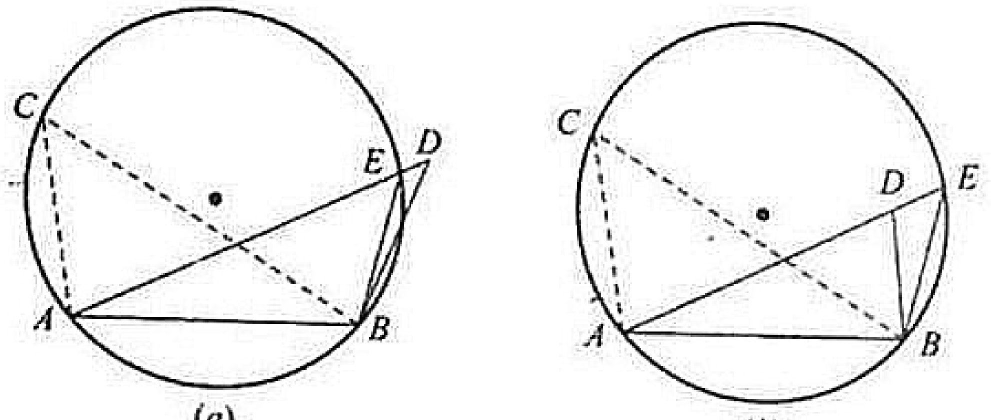
Theorem 7. If a straight line segment joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic.

[We say that four points are *concyclic* if there is a circle passing through all the four points].

Proof. Let the line segment AB make equal angles at C and D, *i.e.*, $\angle ACB = \angle ADB$ as in Fig. 4.11. Then we wish to prove that A, B, C, D are concyclic. Draw the circle ABC passing through A, B and C. Suppose this circle does not pass through D; let it cut AD (or AD produced as in Fig. 11) at E.

Now, by construction, $\angle ACB = \angle AEB$ (angles in the same segment). But by our hypothesis $\angle ACB = \angle ADB$ and therefore $\angle ADB = \angle AEB$. This means that the exterior angle AEB of ΔEDB is equal to an interior opposite angle of the same triangle, which is impossible. Hence the circle ABC passes through D or in other words A, B, C, D are concylic.

□



(*a*) (*b*)

Fig. 4.11

EXERCISE 4.1

1. OA, OB are two equal line segments; the circle with centre C and radius r passes through A, B. Prove that $\angle AOC = \angle BOC$.
2. If A, B are any two points on the circle S, prove that the chord AB which joins them lies entirely within the circle.
3. When two circles (C_1, r_1) and (C_2, r_2) cut, prove that $r_1 - r_2 < d < r_1 + r_2$ where $d = C_1C_2$.
4. When two circles cut each other then prove that the line joining their centres bisects their common chord at right angles.
5. A chord PQ of a circle cuts a concentric circle at P', Q'. Prove that $PP' = QQ'$.
6. Prove that two chords AB, CD of a circle bisect each other if and only if both of them are diameters.
7. Two circles cut each other at A, B. If PAQ and RBS are parallel straight lines meeting the circles again at P, Q, R, S, then prove that $PQ = RS$.
8. Show that the locus of the midpoints of a family of parallel chords of a circle is a diameter which is perpendicular to the given family of chords.
9. If $PQRS$ is a parallelogram whose vertices lie on a circle then PR and QS are diameters of the circle (see problem 6).
10. Prove that every circle passing through a fixed point and having its centre on a fixed straight line must pass though another fixed point.
11. Two straight lines OAB and OCD are drawn from an external point O to cut a given circle at A, B, C, D. Prove that the intersection of AD and BC cannot be the centre of the circle.

12. $ABCD$ is an isosceles trapezium. Prove that a circle can be drawn passing through A, B, C and D.
13. C is the midpoint of an arc ACB of a circle. Prove that C is equidistant from the radius through A and B.
14. AB and CD are two diameters of a circle and CE is a chord parallel to AB. Prove that B is the midpoint of the arc DBE.
15. $ABCD$ is a quadrilateral inscribed in a circle such that $AB = CD$. Prove that $AC = BD$.
16. A, B, C are three points on a circle and D, E are the midpoints of the minor arcs cut off by AB, AC. Prove that DE is equally inclined to AB and AC.
17. O is the centre of the circumcircle of an acute angled ΔABC. Show that $\angle OBC$ is the complement of $\angle BAC$.
18. ABC and $A'B'C'$ are two triangles such that $\angle A = \angle A'$ and $BC = B'C'$. Prove that the circumcircle of ΔABC is equal to the circumcircle of triangle $A'B'C'$.
19. $ABCD$ is a quadrilateral inscribed in a circle. If the diagonals AC and BD are at right angles, show that AB and CD subtend supplementary angles at the centre.
20. Find locus of middle points of chords of a circle which pass through a fixed point.
21. Given the base and the vertical angle of a triangle show that its area is greatest when it is isoscel.
22. ABC is a triangle and A', B', C' are the midpoints of the sides BC, CA, AB respectively. If AD is the altitude though A, prove that $\angle BDC = \angle BCA$. Hence show that the circumcircle of $A'B'C$ also passes through the feet D, E, F of the altitudes of triangle ABC.
23. Two circles intersect at A and B; PAQ is a straight line through A meeting die circles again at P, Q. Find the locus of the midpoint of PQ.
24. The circumferences of three unequal circles whose centres are A, B and C pass through a common point, O, from which lines are drawn through A, B and C meeting the circumferences at A', B' and C'. Show that the sides of $\Delta A'B'C'$ pass through the other points of intersection of the circles and are respectively parallel to the sides of the ΔABC.
25. A chord of constant length slides round a fixed circle. Show that the locus of any point fixed in the chord is a concentric circle.
26. A, B are the midpoints of two equal chords in a circle and the straight line joining A, B meets the circle at P, Q. Prove that $PA = QB$.
27. Prove that of all chords of a circle which are bisected by a fixed chord, the greater is that which meets the fixed chord at a point nearer its midpoint.
28. The internal bisector of $\angle A$ of ΔABC meets the circumcircle in D. If DE, DF are the perpendiculars to AB, AC respectively from D, then prove that $AE = (AB + AC)/2$.
29. If I is the incentre of ΔABC and AI meets the circumcircle at D, then prove that $DB = DC = DI$.
30. If H is the orthocentre of ΔABC and AH meets BC at D and the circumcircle at E, then prove that $HD = DE$.

4.2 TANGENTS

In general, we have seen that a straight line cuts a circle at, utmost two points. If a straight line has just one common point with a circle, we say that the straight line touches the circle. In that case, the straight line is called a *tangent* to the circle and the point at which a tangent touches the circle is known as the point of contact of the tangent. Two circles touch one another when they have only one point in common and have a common tangent at this point. Circles may touch externally in which case they are on opposite sides of the common tangent; or they may touch internally, in which case they are on the same side of the common tangent (Fig. 4.12).

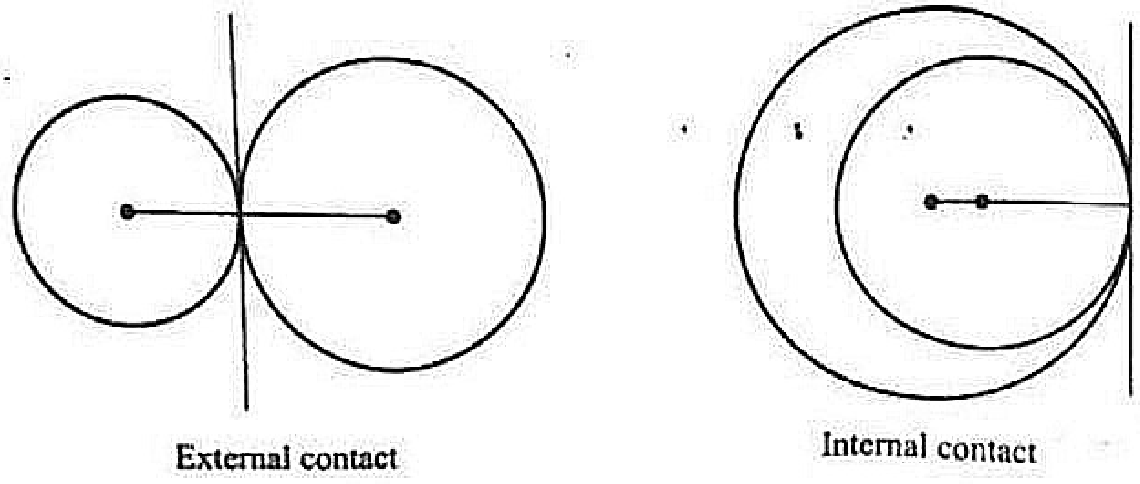


Fig. 4.12

Theorem 8. One and only one tangent can be drawn to a circle at any point on its circumference and this tangent is perpendicular to the radius through the point of contact.

Proof. Let P be any point on a circle with centre O.

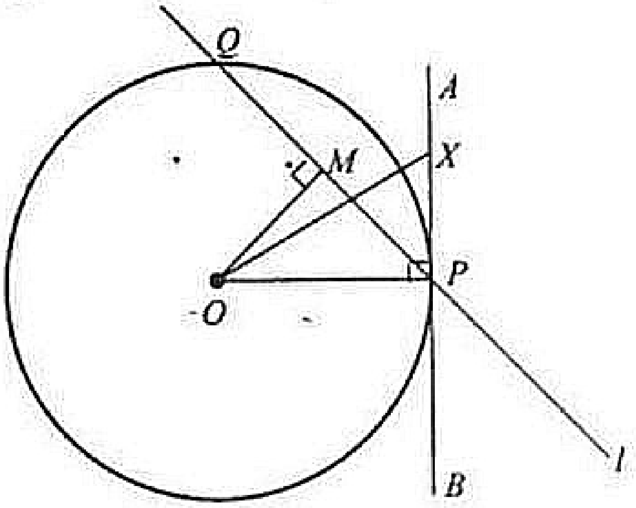


Fig. 4.13

Draw $APB \perp OP$ as in Fig. 4.13. If X is any point on the straight line APB different from P, then $\angle OPX$ is a right triangle with OX as its hypotenuse. Therefore $OX > OP$ = the radius of the circle. This means that X lies outside the given circle. This is true for every point X on the straight line APB except P. Hence the straight line AB touches the circle at P or in other words, the straight line through P perpendicular to the radius OP is a tangent to the circle at P. If any other straight line l through P is considered, let M be the foot of the perpendicular from O on l. Then as l is not perpendicular to OP, we see that $M \neq P$. On this straight line cut off MQ equal to PM (Fig. 4.13). Then by construction, OM is the perpendicular bisector of PQ; and therefore $OP = OQ$. This means that the point Q also lies on the given circle; and the straight line l cuts the circle at two distinct points P and Q. This says that l is not a tangent to the circle. Hence the theorem. ❑

Theorem 9. If two tangents are drawn to a circle from an exterior point then (*i*) the lengths of the tangents are equal (*ii*) they subtend equal angles at the centre (*iii*) the angle between them is bisected by the straight line joining the point and the centre.

Proof. See Fig. 4.14. Let A be an exterior point to the circle with centre O and AP, AQ be two tangents from A to the circle touching the circle at P and Q respectively. Then it is required to prove that (*i*) $AP = AQ$ (*ii*) $\angle AOP = \angle AOQ$ (*iii*) $\angle PAO = \angle QAO$. As AP and AQ are the tangents to the circle at P and Q. We have $\angle APO = \angle AQO = 90°$. We note that the right angled triangles OAP and OAQ are congruent (RHS theorem). Therefore $AP = AQ$, $\angle AOP = \angle AOQ$ and $\angle PAO = \angle QAO$. ☐

Given a circle and a point A exterior to it, how many tangents to the circle can be drawn through A? The following theorem answers this question.

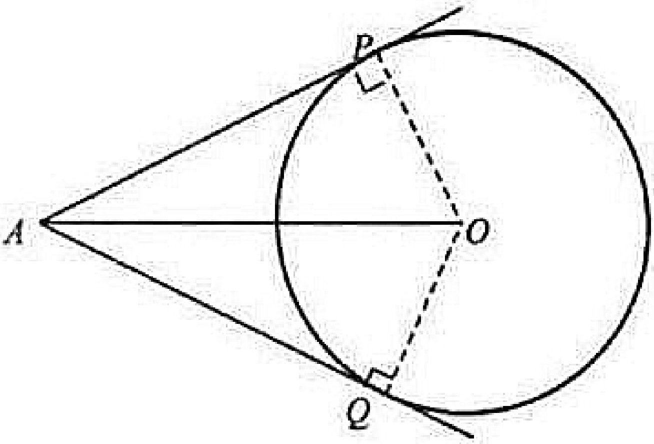


Fig. 4.14

Theorem 10. There are exactly two tangents from an exterior point to a given circle.

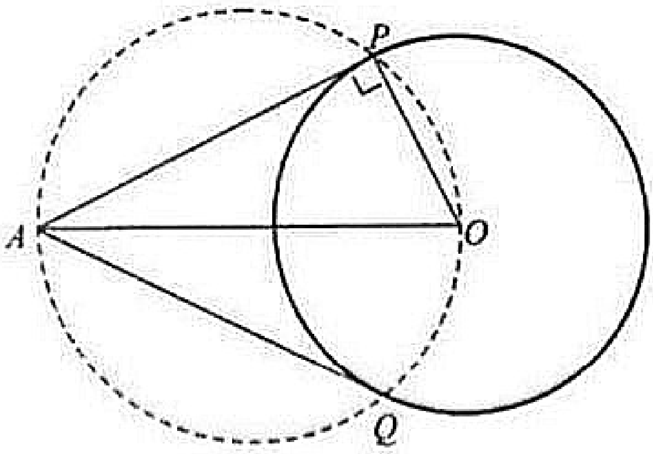


Fig. 4.15

Proof. Suppose P is the point of contact of a tangent to the circle from A. Then as $\angle APO = 90°$, P must lie on the circle AO as diameter (Theorem 7). Now, the circle on AO as diameter and the given circle cut exactly at two distinct points since A lies outside the given circle. (Fig. 4.15). Therefore, the points of contact of the tangents from A to the given circle must be the two points of intersection of the circle on AO as diameter and the given circle. Thus there are exactly two tangents from an exterior point to a given circle. ☐

We have already seen that if A lies on the circle, there is a unique tangent to the circle through A. If A lies inside the circle and AP is a tangent to the circle with P as its point of contact, then ΔAPO must be a right angled triangle, right angled at P. Therefore $AO^2 = AP^2 + OP^2$ and $AP^2 = AO^2 - OP^2$. But A lies inside the circle implies that $AO^2 - OP^2 < 0$. This gives $AP^2 < 0$ which is impossible, since the square of any real number is always non-negative. Hence there is no tangent to the circle through an interior point of the circle.

Theorem 11. If two circles touch one another, then the point of contact lies on the straight line joining the centres.

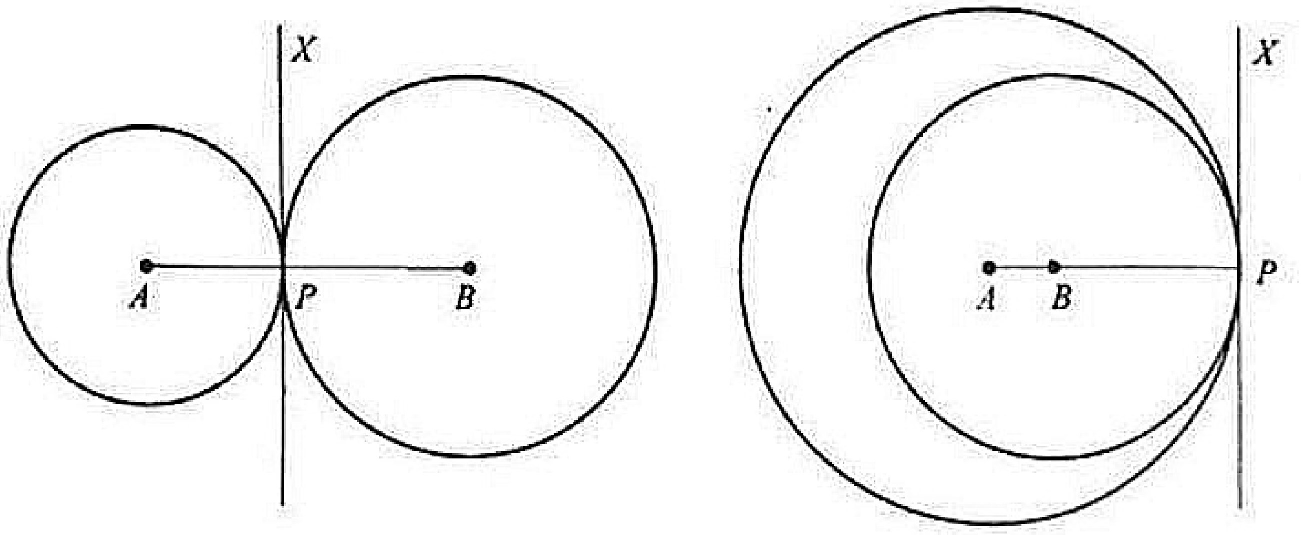


Fig. 4.16

Proof. Let two circles with centres A and B touch each other at P. It is required to prove that A, P, B are collinear. Since the circles touch each other at P (Fig. 4.16), they have a common tangent PX at P. Hence PA and PB are both perpendicular to the common tangent XP at P. This is possible only if A, P and B are collinear and the straight line AB is perpendicular to the common tangent. ❑

Corollary. If two circles touch each other, then the distance between their centres is equal to the sum or difference of their radii.

Proof. If A and B are the centres of two touching circles with radii r_1 and r_2 and if P is their point of contact, then by Theorem 11. A, P and B are collinear. When the circles touch externally P lies in the line segment $\overline{AB}$ and we have $AP + PB = AB$ or $r_1 + r_2 = AB$. When they touch internally, P lies outside the segment $\overline{AB}$ and we have $AB = AP - BP$ or $BP - AP$ depending upon $AP \geq BP$ or $AP \leq BP$. Thus $AB = |AP \pm BP| = |r_1 \pm r_2| =$ sum or difference of their radii. ❑

Remark. The perfect symmetry of a circle about its centre and about any of its diameters tells us that equal arcs subtend equal angles at the centre; and conversely if two arcs subtend equal angles at the centre then they are equal in length.

Theorem 12. In equal circles (or in the same circle) if two chords are equal, then they cut off equal arcs on the circles.

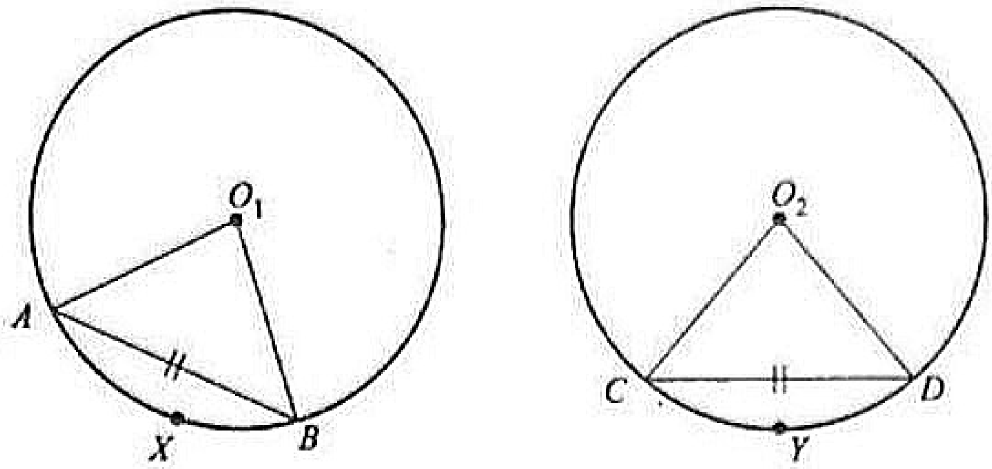


Fig. 4.17

Proof. Let AB and CD be two equal chords of two equal circles with centres O_1 and O_2 respectively. It is required to prove that arc AXB = arc CYD in length. By the *SSS* theorem $\Delta AO_1B = \Delta CO_2D$ and so $\angle AO_1B = \Delta CO_2D$. Therefore by our remark preceding the theorem arc AXB = arc CXD in length. ☐

The converse of the above theorem, namely, "In equal circles (or in the same circle) if two arcs are equal then they cut off equal chords on the circles" is also true. The proof is left as an exercise

Theorem 13. In any circle, the angle between a tangent and a chord through the point of contact of the tangent is equal to the angle in the alternate segment.

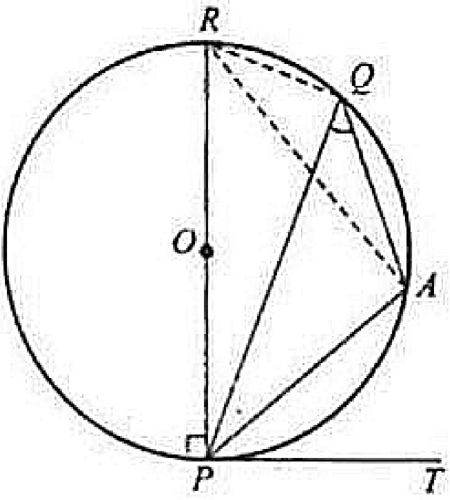


Fig. 4.18

Proof. Let PT be a tangent to a circle with centre O, the point of contact being P. Let AP be any chord through P. It is required to prove that $\angle APT = \angle AQP$ where Q is any point on the other segment determined by AP. Let PR be the diameter through P. Then $\angle PAR = 90°$ (angle in a semi-circle) and $\angle RPT = 90°$ (Fig. 4.18). Therefore $\angle ARP = 90° - \angle APR$ (from the right triangle RAP) $= \angle TPR - \angle APR = \angle APT$. But $\angle ARP = \angle AQP$ and hence $\angle APT = \angle AQP$ = angle in the alternate segment. ☐

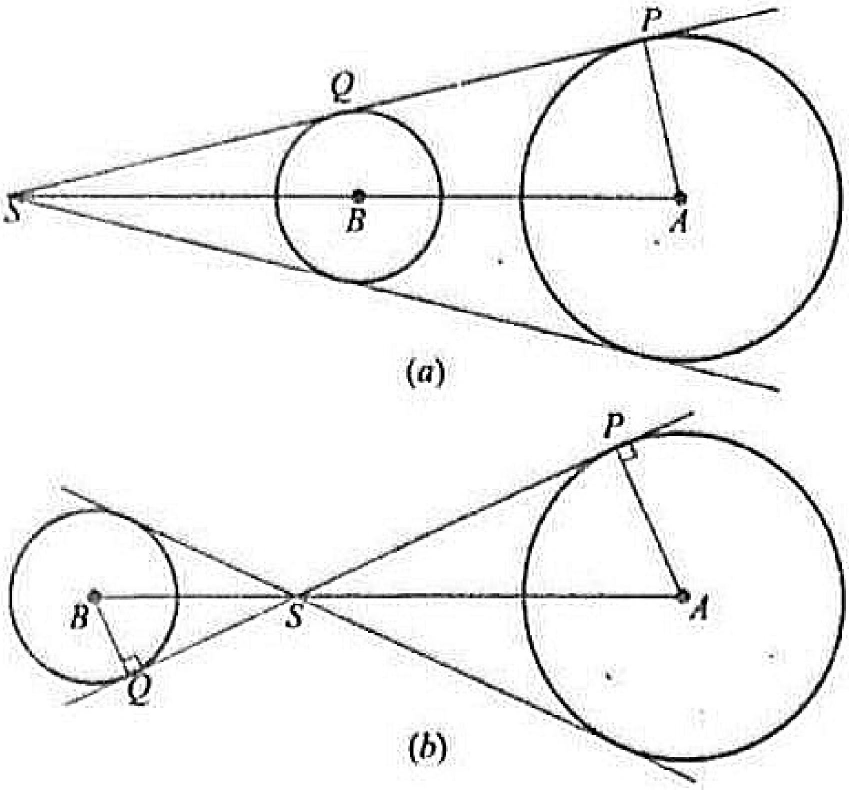


Fig. 4.19

Theorem 14. A common tangent to two circles divides the straight line segment joining their centres, externally or internally in the ratio of their radii.

Proof. Suppose PQ is a common tangent to the two circles with centres A, B and radii r_1, r_2 respectively such that P and Q are the points of contact with the corresponding circles. Let PQ meet the line of centres joining A and B at S (Fig. 4.19(a)) or S' (Fig. 4.19(b)). We have $\angle APS = \angle BQS = 90°$. Therefore $AP \parallel BQ$ and the triangles BQS and APS are equiangular and hence similar.

So, $\frac{AS}{SB} = \frac{AP}{BQ} = \frac{r_1}{r_2}$ and hence S divides AB externally in the ratio $r_1 : r_2$.

When PQ meets AB at S' as in Fig. 4.19(b), S' divides AB internally and again the similarity of the triangles APS' and BQS' gives

$$\frac{AS'}{S'B} = \frac{r_1}{r_2}$$

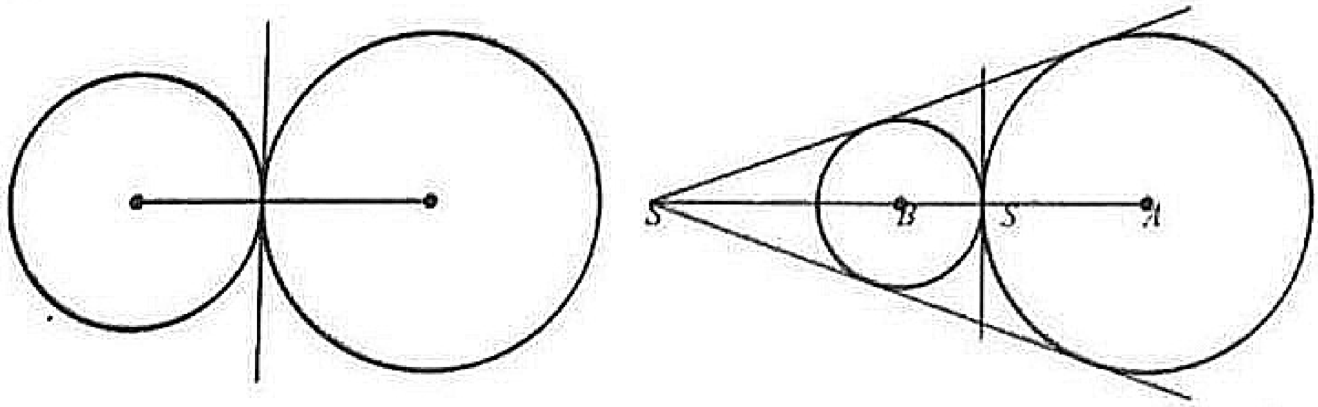


Fig. 4.20

Thus any common tangent to two circles divides the straight line segment joining their centres either internally or externally in the ratio of the radii. ☐

Definition 1. The points S and S' dividing the line segment joining the centres of two circles internally and externally in the ratio of their radii are known as the *centres of similitude* of the two circles.

When S and S' are both exterior to the circles as in Fig. 4.19(a), Fig. 4.19(b), there are two common tangents from S and two common tangents from S'. The two common tangents from the external centre of similitude are the direct common tangents and the two common tangents from the internal centre of similitude are the transverse common tangents. Thus in general, there are four common tangents to two circles. When two circles touch externally (Fig. 4.20) there is only one transverse common tangent and there are two direct common tangents. When two circles touch internally (Fig. 4.21) there is only one direct common tangent and no transverse common tangents, as S' lies inside both the circles. Also when two circles cut each other, there are two direct common tangents and no transverse common tangents (Fig. 4.22). When one circle lies entirely within the other there are no common tangents.

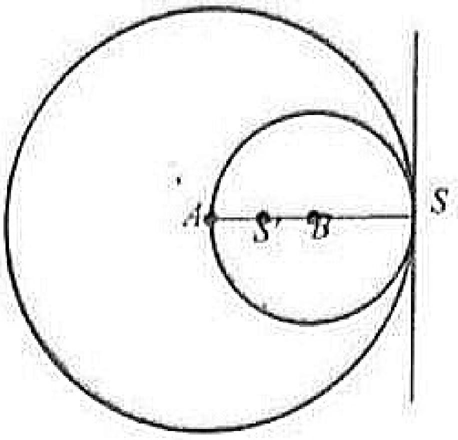


Fig. 4.21

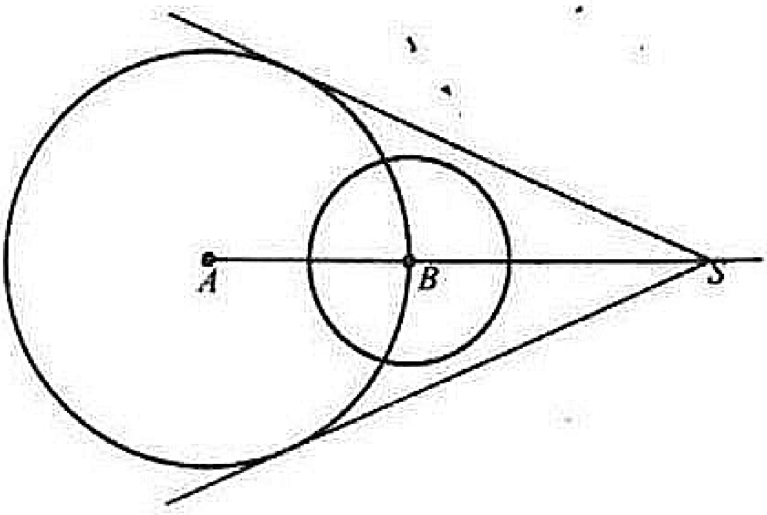


Fig. 4.22

EXERCISE 4.2

1. If PA, PB are tangents to a circle whose centre is O, then prove that $\angle APB + \angle AOB = 180°$.
2. If P, Q and R, S are the points of contact of the two direct common tangents to two circles prove that $PQ = RS$.
3. If P, Q and R, S are the points of contact of the two transverse common tangents to two circles prove that $PQ = RS$.
4. If two circles intersect at A, B prove that the angle between the tangents at A is the same as the angle between the tangents at B.
5. If a circle can be inscribed in a quadrilateral, prove that the sum of one pair of opposite sides is equal to the sum of the other pair.
6. If a circle can be inscribed in a parallelogram, prove that the parallelogram is a rhombus.
7. If a straight line cuts a circle at A, B prove that it cuts the circle at the same angle at each of these points.
8. What is the locus of the centre of a circle which touches two given parallel straight lines?
9. S_1 and S_2 are two concentric circles and AB is a chord of outer circle S_1 touching S_2 at C. Prove that $AC = CB$.
10. What is the locus of the centres of circles which touch a given circle at a given point?
11. What is the locus of centres of circles of given radius which touch a given circle?
12. Two circles with centres A, B touch at P. If XPY is drawn to cut the circles again at X, Y, prove that $AX \parallel BY$.
13. Two circles intersect in A and a straight line XAY is drawn to cut the circles again at X and Y. Tangents at X and Y to the respective circles cut at Z. Prove that $\angle XZY$ is equal to the angle between the tangents at A.
14. A straight line cuts two concentric circles in A_1, A_2 and B_1, B_2. Prove that the four intersections of a tangent at an 'A' and at a 'B' lie on another concentric circle.
15. A straight line of given length subtends given equal angles at two fixed points. Prove that the straight line a always touches a fixed circle.
16. Three equal circles pass through a given point H and meet one another two by two at A, B, C. Prove that H is the orthocentre of ΔABC.
17. A tangent to a circle at a point P on it, is parallel to the chord AB. Prove that P bisects the arc cut off by AB.
18. AB is a chord of a circle and AT is the tangent at A. Prove that the bisector $\angle BAT$ bisects the arc AB.

19. AB, AC are tangents from A to a circle touching the circle at B, C. If D is the midpoint of minor arc BC, prove that D is the incentre of ΔABC.

20. The diagonals of the parallelogram $ABCD$ meet at O. Prove that the circles AOB and COD touch each other.

21. AB is a chord of a circle and PAQ is the tangent at A; C and D are points on the circle such that CA and DA bisect the angles BAP and BAQ. Show that CD is a diameter perpendicular to AB.

22. Two circles touch internally at X and a straight line cuts them at A, B, C, D in order. Prove that AB, CD subtend equal angles at X.

23. Suppose the internal and external bisector of $\angle A$ meet the side BC and BC (produced) at E and F respectively. If the tangent at A to the circle ABC meets BC produced at D, prove that D bisects EF.

24. A triangle ABC circumscribes a circle, with points of contact being X, Y, Z. If the feet of the altitudes of ΔXYZ are D, E, F prove that the sides of ΔDEF are parallel to the sides of ΔABC.

If in a quadrilateral, the sum of one pair of opposite sides is equal to the sum of the other pair, prove that a circle can be inscribed in it. (See problem 4- Problems Ch. 3)

26. If a circle can be inscribed in a quadrilateral prove that the bisectors of the angles of the quadrilateral are concurrent.

27. Four circles of different radii are such that each circle touches two and only two of the others. Show that a circle can be inscribed in the quadrilateral $ABCD$ where A, B, C, D are the centres of the given four circles.

28. If the circumference of a circle is divided into n equal parts, prove that
    (*i*) the points of division are the vertices of a regular polygon.
    (*ii*) the tangents at the points of division are the sides of a regular polygon.

29. Two circles cut at right angles, (*i.e*, the angle between the tangents at their points of intersection is 90°). Show that the area common to the two circles together with the square on the radius is equal to the area of either circle.

30. Two given circles intersect at A and B. A straight line through B meets the circles again at C and D. (*i*) Show that CD is greatest when it is parallel to the line joining the centres. (*ii*) When is the area of ΔACD the greatest possible?

31. AB is a diameter of a circle and BM is the tangent at B. If the tangent at a point C on the circle meets BM at X and if AC produced meets BM at Y, prove that $BX = XY$.

32. Prove that if a chord and a tangent are drawn from a point on a circle, the midpoint of the subtended arc is equidistant from them.

33. A and B are points on two concentric circles. Prove that the angle between the tangents at A and B is the same as the angle subtended by AB at the centre.

34. Two circles are said to cut each other orthogonally if the angle between their tangents at a common point is a right angle. Show that the locus of the centres of circles cutting a given circle orthogonally at a given point is a straight line.

35. Find the locus of centres of circles of given radius cutting a given circle orthogonally.

36. If AB is a common tangent to two circles, prove that the circle on AB as diameter cuts each of the circles orthogonally.

37. If two circles of radii r_1, r_2 cut orthogonally at A, B prove that $AB \cdot d = 2r_1r_2$ where d is the distance between their centres.

38. If H is the orthocentre of ΔABC, show that the circles on AH and BC as diameters cut orthogonally.

39. O is a fixed point; P is a variable point on a fixed circle S. If P' is on the line OP such that $OP'/OP = \lambda$ = a constant, find the locus of P'.

40. O is a fixed point; P is a variable point on a fixed circle with centre C. The line bisecting $\angle OCP$ meets OP at P'. Find the locus of P'.

41. A straight line OPQ is drawn through a centre of similitude O of two circles to cut them at P and Q. Prove that the tangents at P, Q are parallel.

42. Two circles cut at A and B and their common tangents meet at O. If AP, AQ are the tangents at A to the two circles, prove that OA bisects $\angle PAQ$.

43. Prove that if three circles are tangent to one another, the tangents at the points of contact are concurrent.

4.3 CYCLIC QUADRILATERALS

A quadrilateral $ABCD$ is *cyclic if* there is a circle passing through all the four vertices of the quadrilateral.

Theorem 15. The opposite angles of a cyclic quadrilateral are supplementary.

Proof. Let $ABCD$ be a cyclic quadrilateral inscribed in a circle with centre O. It is required to prove that $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$. Now, by Theorem 5, $\alpha = 2\angle A$ and $\beta = 2\angle C$ (Fig. 4.23). Therefore, $360° = \alpha + \beta = 2(\angle A + \angle C)$ or $\angle A + \angle C = 180°$. Similarly $\angle B + \angle D = 180°$. ☐

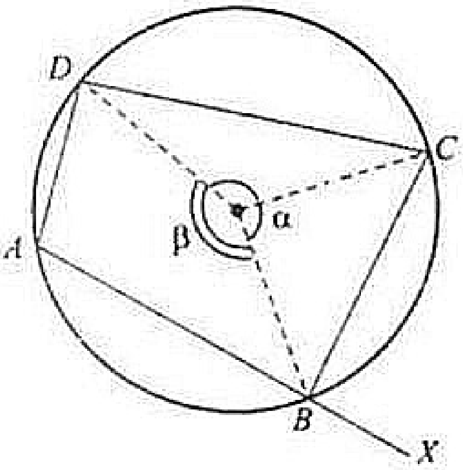


Fig. 4.23

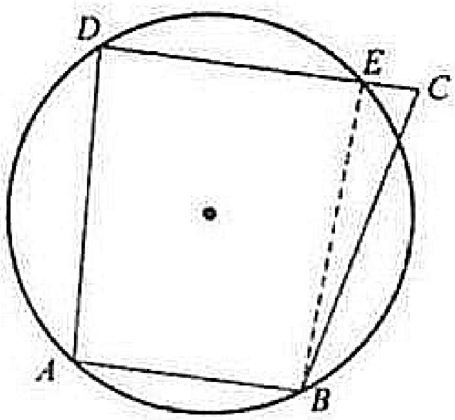


Fig. 4.24

Corollary. If $ABCD$ is a cyclic quadrilateral then any exterior angle of $ABCD$ is equal to the interior opposite angle.

Proof. We wish to prove that the exterior angle XBC is equal to the interior opposite angle ADC (Fig. 4.23); this is immediate since $\angle XBC = 180° - \angle B = \angle D$. ☐

Theorem 16. If two opposite angles of a quadrilateral are supplementary then it is cyclic.

Proof. Let $ABCD$ be a quadrilateral such that $\angle A + \angle C = \angle B + \angle D = 180°$. (See Fig. 4.24). Suppose the circle through A, B and D cuts the straight line DC at E. Then $ABED$ is a cyclic quadrilateral. Therefore $\angle BED = 180° - \angle BAD$. But $\angle BAD + \angle BCD = 180°$ by our hypothesis. Hence $\angle BED = \angle BCD$. If $C \neq E$, this says that an exterior angle of ΔBCE is equal to an interior opposite angle, which is impossible. Therefore $C = E$ or the quadrilateral $ABCD$ is cyclic. ☐

Note. The above proof also works when E lies on DC produced, in which case BCD is an exterior angle of ΔBCA.

Proof. Let AD cut the circumcircle of ΔABC again at X. Then $\angle ABD = \angle AXC$ being angles in the same segment determined by the chord AC. Also, $\angle BAD = \angle CAD$ as AD bisects $\angle A$. Therefore $\Delta ABD \;|||\; \Delta AXC$ (Fig. 4.29) and

we get $$\frac{AB}{AX} = \frac{AD}{AC}.$$

This gives $AB \cdot AC = AD \cdot AX$. Also, $AD^2 = AD\,(AX - DX)$ implies that $AD \cdot AX = AD^2 \cdot + AD \cdot . DX$. Therefore we get $AB \cdot AC = AD \cdot AX = AD^2 + AD \cdot DX$. By the secant property of a circle we have $AD \cdot DX = BD \cdot DC$. Hence $AB \cdot AC = BD \cdot DC + AD^2$. $\square$

Theorem 21. If AD is the altitude through A of ΔABC and if R is the circumradius of ΔABC then $AB \cdot AC = 2R \cdot AD$.

Proof. Let AE be the diameter through A of the circle ABC. (Fig. 4.30). We have $\angle ADC = \angle ABE = 90°$ and $\angle ACD = \angle AEB$ (angles in the same segment).

Therefore $\Delta ADC \;|||\; \Delta ABE$ and so $\dfrac{AB}{AD} = \dfrac{AE}{AC}$.

This gives $AB \cdot AC = AE \cdot AD = 2R \cdot AD$. $\square$

Corollary. $\Delta = \text{Area of } ABC = \dfrac{abc}{4R}$ (usual notations)

Proof. We have $$\Delta = \frac{1}{2} a \cdot AD = \frac{abc}{2.2R} = \frac{abc}{4R}$$ $\square$

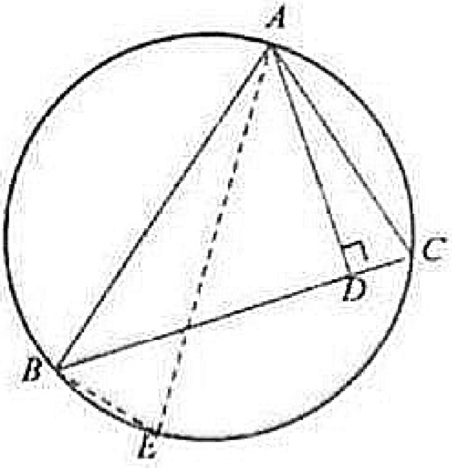


Fig. 4.30

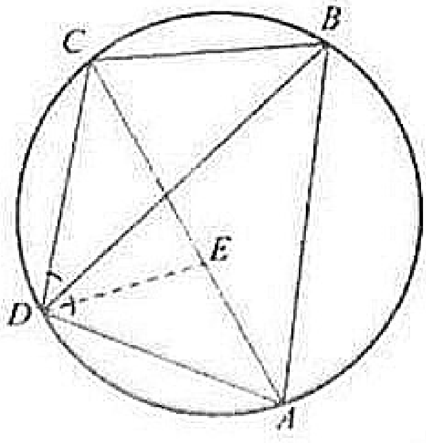


Fig. 4.31

Theorem 22. (Ptolemy's Theorem)

The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by pairs of opposite sides.

Proof. Let $ABCD$ be a cyclic quadrilateral. It is required to prove that $AC \cdot BD = AB \cdot CD + BC \cdot AD$. We may assume without loss of generality that $\angle ADB > \angle BDC$. Draw DE meeting AC at E such that $\angle ADE = \angle BDC$. (Fig. 4.31). Then the triangles ADE and BDC are equiangular and hence similar. Therefore

$$\frac{AD}{BD} = \frac{AE}{BC} \quad \text{or} \quad AD \cdot BC = BD \cdot AE \tag{1}$$

Again, $\angle ADB = \angle ADE + \angle EDB = \angle BDC + \angle EDB = \angle EDC$

and $\angle DBA = \angle DCE$ (angles in the same segment).

Therefore $\Delta ADB ||| \Delta EDC$ and so

$$\frac{AB}{EC}=\frac{BD}{CD} \text{ or } AB\cdot CD = BD\cdot EC \qquad (2)$$

Adding (1) and (2) we get $AB\cdot CD + BC\cdot AD = BD\,(AE+EC) = BD\cdot AC$.

Theorem 23. If $ABCD$ is a quadrilateral which is not cyclic then $AB\cdot CD + BC\cdot AD > AC\cdot BD$.

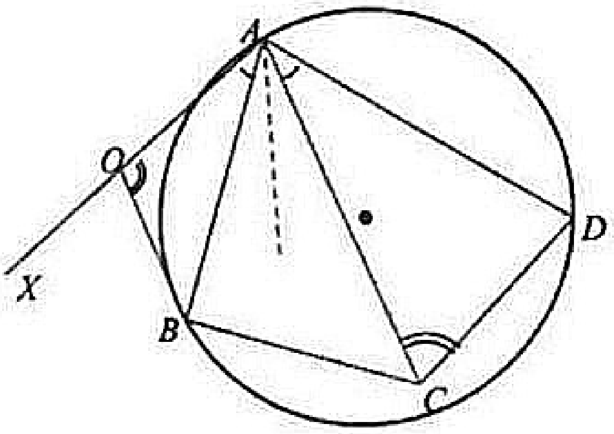


Fig. 4.32

Proof. Suppose $ABCD$ is not a cyclic quadrilateral, draw the circle ABD. Let AX be a straight line symmetric to AD about the bisector of $\angle BAC$. In other words, AX is a straight line such that $\angle XAB = \angle CAD$. There is a unique point O on AX such that $\angle AOB = \angle ACD$. Therefore, comparing the triangles AOB and ACD, the third pair of angles ABO and ADC must also be equal. This means that O does not lie on BC (otherwise $ABCD$ becomes a cyclic quadrilateral).

$\Delta AOB ||| \Delta ACD$ gives $$\frac{AO}{AC}=\frac{AB}{AD}=\frac{OB}{CD} \qquad (1)$$

Also, $\Delta OAC ||| \Delta BAD$ (Why?) gives $= \dfrac{OC}{BD}=\dfrac{AC}{AD}$. (2)

Hence $AC\cdot BD = OC\cdot AD < (OB+BC)\cdot AD = OB\cdot AD + BC\cdot AD$

$= AB\cdot CD + BC\cdot AD$. from (1) □

Note. With notations as in the proof of Theorem 23, we note that if $ABCD$ is cyclic, we must have O lying on BC and $OC = OB + BC$.

$AC\cdot BD = OC\cdot AD = (OB+BC)\cdot AD = OB\cdot AD + BC\cdot AD = AB\cdot CD + BC\cdot AD$ which is Ptolemy's theorem.

Corollary. Quadrilateral $ABCD$ is cyclic iff $AC\cdot BD = AB\cdot CD + AD\cdot BC$. □

Proof. Immediate from the theorem.

EXERCISE 4.3

1. $ABCD$ is a cyclic quadrilateral and AB, CD are produced to meet at X. Prove that ΔXAD and ΔXCB are similar.
2. If I is the incentre of ΔABC and I_a is the excentre opposite to A, prove that $BICI_a$ is a cyclic quadrilateral.

3. In ΔABC, AD and BE are the altitudes through A and B. If H is the orthocentre, prove that the quadrilateral $DHEC$ is cyclic. Hence show that $\angle AHB = \pi - \angle C$ and that the circles AHB and ACB are equal circles.

4. O is the centre of a circle and AB is a diameter of the circle. If OD is perpendicular to AB and meets a chord AC (or AC produced) at D, then prove that the circle AOD is equal to the circle through O, D, B, C.

5. Let P be any point on the circumcircle of triangle ABC and let L, M, N be the feet of the perpendiculars from P on the sides BC, CA, AB respectively. Prove that L, M, N are collinear.

6. In ΔABC let A', B', C', be the midpoints of BC, CA, AB and let H be the orthocentre. If P is the midpoint of AH, prove that the circle $A'\ B'\ C$ passes through P. Hence prove that the midpoints of the sides of triangle ABC and the midpoints of the line segments joining the orthocentre to the vertices all lie on a circle.

7. Using problem 6, prove that the circle through the midpoints of the sides and the midpoints of the lines joining the orthocentre to the vertices also passes through the feet of the altitudes of the triangle.

8. In a quadrilateral $ABCD$, the bisectors of the angles A, B meet at E and those of B, C; C, D; D, A meet in F, G, H respectively. Prove that $EFGH$ is cyclic.

9. If the exterior angles of a quadrilateral are bisected by four straight lines, prove that these four straight lines form a cyclic quadrilateral.

10. X, Y, Z are any three points on the sides BC, CA, AB of triangle ABC. Prove that the three circles AYZ, BZY and CXY meet at a point.

11. A, B, C are three collinear points and P is a point not in the line AB; AFE, BFD and CED are perpendiculars to PA, PB, PC respectively. Prove that P, D, E, F are concyclic.

12. $ABCD$ is a cyclic quadrilateral; the sides AB and DC are produced to meet at X and AD, BC produced meet at Y. Show that the bisectors of the angles BXC, CYD are at right angles.

13. If X is any point on the internal bisector of $\angle A$, prove that
$$\Delta BAX / \Delta CAX = BA/AC.$$

14. Find a point X inside ΔABC such that
$$\Delta AXB : \Delta BXC : \Delta CXA = k : l : m$$
where k, l, m are given constants.

15. The diagonals AC, BD of a cyclic quadrilateral $ABCD$ meet at O. Prove that $AB \cdot BC / AD \cdot DC = BO/OD$.

16. S_1 and S_2 are two circles touching internally at O, with S_2 being the inner circle. A straight line cuts S_1 at A, D and S_2 at B, C. Prove that
$$AB : CD = (OA \cdot OB) : (OC \cdot OD).$$

17. X is any point on the circle through the four vertices of a cyclic quadrilateral $ABCD$. If x, y, z, w, u, t are the perpendicular distances of X from AB, BC, CD, DA, AC, BD respectively, prove that $xz = yw = ut$.

18. AB is a chord of a circle and the tangents at A, B meet at C. If P is any point on the circle and PL, PM, PN are the perpendiculars from P to AB, BC, CA prove that $PL^2 = PM \cdot PN$.

19. X is any point on AB and the median AD of triangle ABC meets XC at Y. Prove that $XY/YC = AX/XB$.

20. AB is a diameter of a circle and PQ is a chord perpendicular to AB meeting AB at X. If the tangent at P meets AB at Y, prove that $YQ/QX = YP/PX$.

21. A, B are fixed points; AP and BQ are parallel chords of a variable circle such that AP/BQ is a constant. Prove that the locus of P is a circle.

22. Two given circles subtend equal angles at a point P. Find the locus of P.

23. X is any point on the minor arc BC of the circum circle of an equilateral triangle ABC. Prove that $XA = XB + XC$.

24. ABC is an isosceles triangle with $AB = AC$. The altitude AD meets the circumcircle at P. Prove that $AP \cdot BC = 2AB \cdot BP$.

25. $ABCDE$ is a regular pentagon; P is any point on the minor arc AB of the circum circle of $ABCDE$. Prove that $PA + PB + PD = PC + PE$.

26. P is any point inside a parallelogram $ABCD$ such that $\angle APB + \angle CPD = 180°$. Prove that $AP \cdot CP + BP \cdot DP = AB \cdot BC$.

4.4 TRIANGLES REVISITED

Note. Here and in the rest of the book, the angles of a ΔABC may be denoted by A, B, C instead of $\angle A, \angle B, \angle C$ wherever the context is clear like for example, $A + B + C = 180°$.

Theorem 24. Let ABC be a triangle, AD the altitude through A and AK the circumdiameter through A. Then $\angle DAK = \angle B - \angle C$. Further the angular bisector AX of $\angle A$ bisects $\angle DAK$.

Proof. We have $\angle ABC = \angle AKC$ (angles in the same segment).

$$\angle BAD = 90° - \angle ABC = 90° - \angle AKC = \angle KAC \text{ (Fig. 33)}$$

$$\angle DAK = \angle BAC - 2\,\angle BAD = \angle A - 2\,(90° - \angle B)$$

$$= \angle A + 2\,\angle B - 180° = \angle B - \angle C \text{ since } A + B + C = 180°.$$

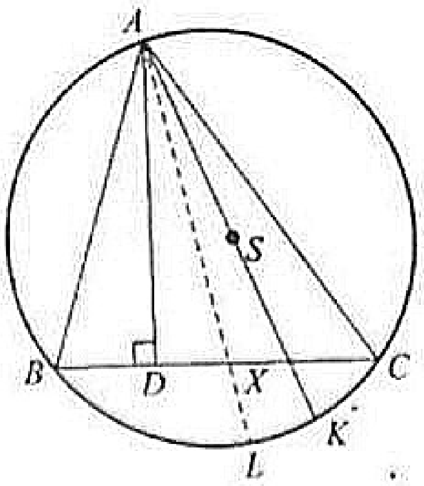


Fig. 4.33

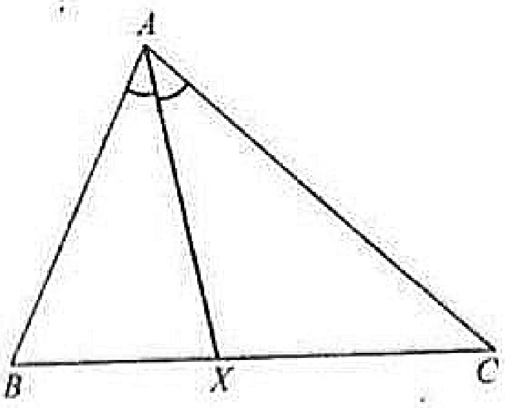


Fig. 4.34

This proves the first part of the theorem. We have taken B, C both acute in Fig. 4.33. The same proof works when one of B and C is obtuse. Let AXL be the angular bisector of $\angle A$ (Fig. 4.33). We have

$$DAX = \frac{\angle A}{2} - \angle BAD = \frac{\angle A}{2} - \angle CAK = \angle XAK. \qquad \square$$

Thus AX also bisects DAK.

Theorem 25. If the internal bisector of $\angle A$ of ΔABC meets BC at X then the difference between $\angle AXB$ and $\angle AXC$ is the same as the difference between $\angle B$ and $\angle C$.

Proof. We have $\angle AXC = \angle B + \dfrac{\angle A}{2}$ and $\angle AXB = \angle C + \dfrac{\angle A}{2}$.

Therefore $\angle AXB - \angle AXC = \angle B - \angle C$. $\square$

Theorem 26. In ΔABC, if the internal and external bisectors of $\angle A$ meet the circumcircle at X and Y, then XY is a circumdiameter perpendicular to BC.

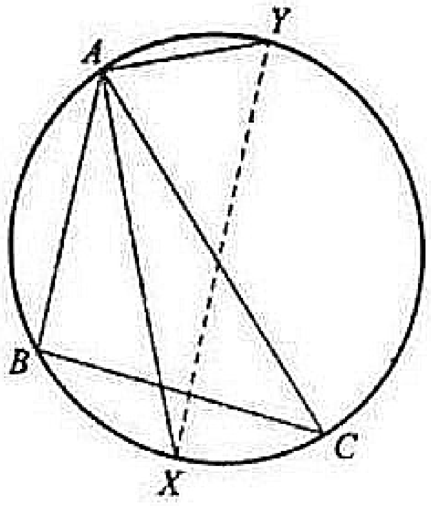


Fig. 4.35

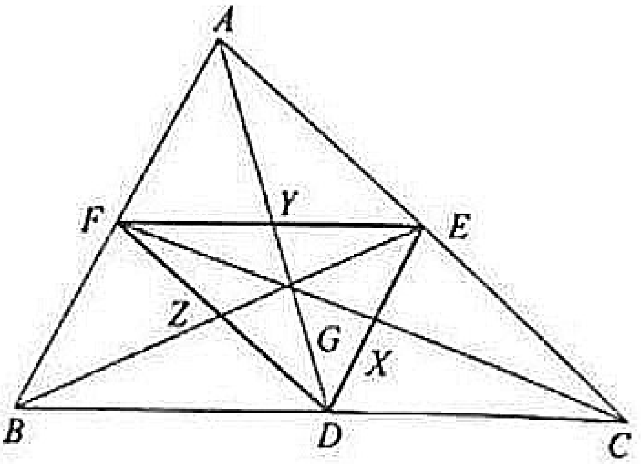


Fig. 4.36

Proof. Since equal arcs subtend equal angles at the circumference, $\angle BAX = \angle XAC$ gives arc BX = arc XC (Fig. 4.35). Therefore the diameter XY' of the circum circle should be the perpendicular bisector of the chord BC. Now, XY' is a diameter implies that $AY' \perp AX$. This means that AY' must be the external bisector of $\angle A$. Hence Y' coincides with Y and $XY \perp BC$. ☐

If ABC is a triangle with D, E, F as the midpoints of the sides BC, CA, AB respectively, then ΔDEF is called the *medial triangle* of ABC.

Theorem 27. A triangle and its medial triangle have the same centroid.

Proof. The median AD of ΔABC also bisects the side EF of ΔDEF (Why?). Therefore DY is a median of ΔDEF (Fig. 4.36). Similarly EZ and FX are also medians of ΔDEF. Thus G is the centroid of ΔDEF as well. ☐

Theorem 28. If m_a, m_b, m_c are the lengths of the medians of ΔABC, through A, B, C respectively then

$$2m_a^2 = b^2 + c^2 - \frac{a^2}{2},$$

$$2m_b^2 = c^2 + a^2 - \frac{b^2}{2} \quad \text{and} \quad 2m_c^2 = a^2 + b^2 - \frac{c^2}{2},$$

where a, b, c are the lengths of the sides BC, CA, AB of ΔABC.

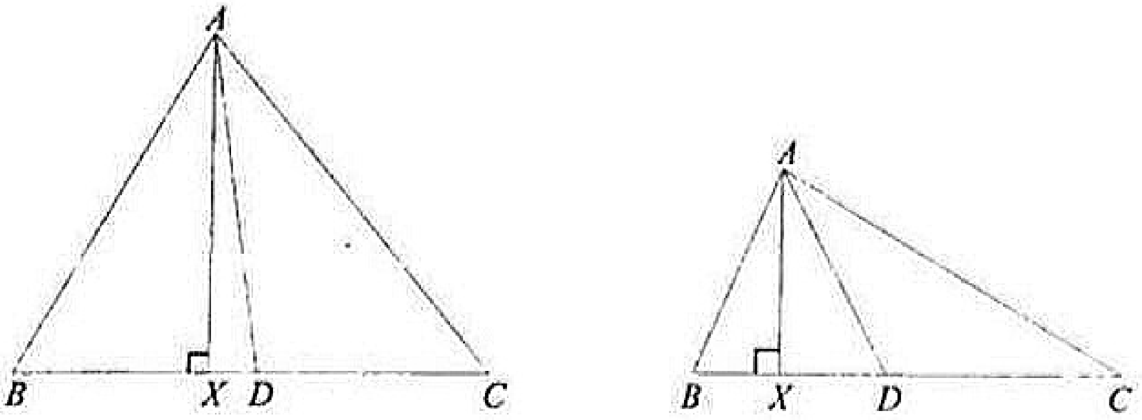


Fig. 4.37

Proof. Let AD be the median through A and AX the altitude through A. We use Pythagoras's theorem repeatedly.

We have $AB^2 = AX^2 + XB^2 \quad = (AD^2 - DX^2) + XB^2$ (Fig. 4.37)

$$= AD^2 + (DB - DX)^2 - DX^2$$

$$= AD^2 + DB^2 - 2DB \cdot DX.$$

Similarly, $AC^2 \quad = AD^2 + DC^2 + 2DC \cdot DX$

Adding we get $AB^2 + AC^2 \quad = 2AD^2 + 2DB^2$ (since $DB = DC$)

or
$$2AD^2 = AB^2 + AC^2 - \frac{1}{2}(BC^2) \text{ as } DB = \frac{BC}{2}$$

i.e.,
$$2m_a^2 = b^2 + c^2 - \frac{a^2}{2}.$$

Similarly
$$2m_b^2 = c^2 + a^2 - \frac{b^2}{2} \quad \text{and} \quad 2m_c^2 = a^2 + b^2 - \frac{c^2}{2}. \qquad □$$

Corollary 1. 1. $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(b^2 + c^2 + a^2)$

2. $GA^2 + GB^2 + GC^2 = \frac{1}{3}(b^2 + c^2 + a^2)$

where G is the centroid of ΔABC.

Proof. 1. Follows immediately from Theorem 28.

2. We have $GA = (2/3)\, m_a$, $GB = (2/3)\, m_b$ and $GC = (2/3)\, m_c$.

$$GA^2 + GB^2 + GC^2 = (4/9)\ (m_a^2 + m_b^2 + m_c^2) = (1/3)\ (a^2 + b^2 + c^2). \qquad □$$

Corollary 2. If P is any point in the plane of ΔABC then

$PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3PG^2$ where G is the centroid of ΔABC.

Proof. Let X be the midpoint of AG. (Fig. 4.38). The median PD of ΔPBC is given by

$$2PD^2 = PB^2 + PC^2 - \frac{BC^2}{2} \qquad (1)$$

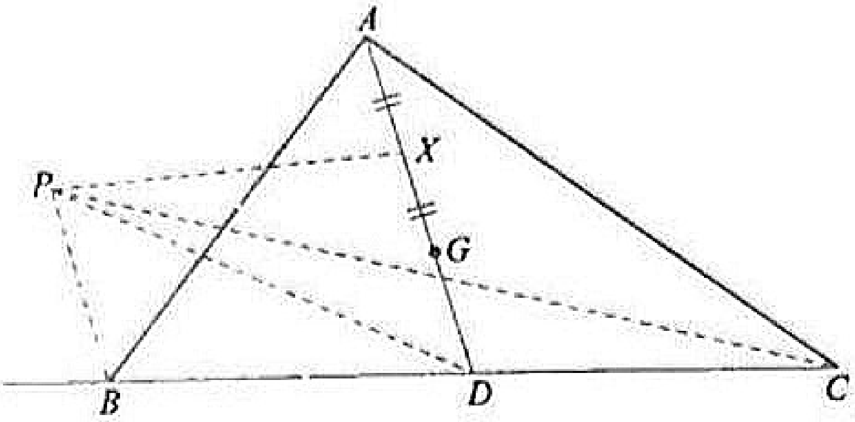


Fig. 4.38

The median PG of ΔPDX is given by

$$2PG^2 = PD^2 + PX^2 - XD^2/2 \qquad (2)$$

The median PX of ΔPAG is given by

$$2PX^2 = PA^2 + PG^2 - AG^2/2 \qquad (3)$$

Therefore (1) + ((2) × (2)) + (3) gives

$$2PD^2 + 4PG^2 + 2PX^2 = PB^2 + PC^2 - BC^2/2 + 2PD^2 + 2PX^2 - XD^2 + PA^2 + PG^2 - AG^2/2$$

Therefore, $PA^2 + PB^2 + PC^2 - 3PG^2 = BC^2/2 + XD^2 + AG^2/2$

$$= BC^2/2 + AG^2 + AG^2/2 \text{ (since } XD = AG)$$

$$= BC^2/2 + (3/2)\,AG^2\,.$$

Similarly, if we consider the other medians BE and CF,

we get $\quad PA^2 + PB^2 + PC^2 - 3PG^2 = CA^2/2 + (3/2)BG^2$ and

$$PA^2 + PB^2 + PC^2 - 3PG^2 = AB^2/2 + (3/2)CG^2.$$

Adding we get,

$$3(PA^2 + PB^2 + PC^2 - 3PG^2)$$

$$= \frac{1}{2}(BC^2 + CA^2 + AB^2) + 3/2(BG^2 + CG^2 + AG^2)$$

$$= \frac{1}{2}(3(GA^2 + GB^2 + GC^2)) + 3/2\,(GA^2 + GB^2 + GC^2) \text{ (from Corollary 1)}$$

$$= 3(GA^2 + GB^2 + GC^2).$$

Hence $\quad PA^2 + PB^2 + PC^2 - 3PG^2 = GA^2 + GB^2 + GC^2.$ □

Corollary 3. If R is the circumradius and S is the circumcentre of ΔABC then

$$SG^2 = R^2 - (1/9)\,(a^2 + b^2 + c^2).$$

Proof. Take $P = S$ in Cor. 2 to get

$$SA^2 + SB^2 + SC^2 = 3R^2 = GA^2 + GB^2 + GC^2 + 3SG^2$$

$$= (1/3)(a^2 + b^2 + c^2) + 3SG^2$$

Therefore $\quad SG^2 = R^2 - (1/9)(a^2 + b^2 + c^2).$ □

Theorem 29. In any ΔABC, if $\angle B > \angle C$ then the internal bisector BE of $\angle B$ is shorter than the internal bisector CF of $\angle C$.

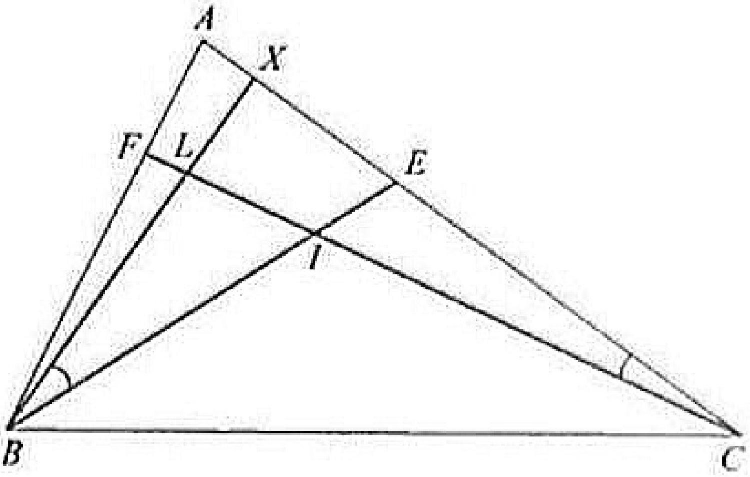


Fig. 4.39

Proof. Since $\angle B > \angle C$, $\angle ABE = \angle B/2 > \angle C/2 = \angle ACF$. Let X be the point on the segment AE such that $\angle XBE = \angle ACF$. Now, BE and CF meet at I the incentre of ΔABC. Let BX meet CF at L. By construction, ΔXBE and ΔXCL are equiangular and hence similar. Therefore $\dfrac{BE}{CL} = \dfrac{BX}{CX}$.

In ΔXBC. we have $\angle XBC = \frac{\angle B}{2} + \frac{\angle C}{2} > \frac{\angle C}{2} + \frac{\angle C}{2} = \angle XCB$ and hence $XC > BX$.

Therefore $1 < \frac{BX}{XC} = \frac{BF}{CL}$ or $BE < CL$.

Hence $BE < CL < CF$. ❑

Another proof of Theorem 29 We use again the same figure Fig. 4.39. By construction $\angle LBE = \angle LCE$ and hence the four points L, B, C, E are concylic. We have,

$$\angle C = \angle BCE < \frac{1}{2}(\angle B + \angle C) = \angle CBL < \frac{1}{2}(\angle A + \angle B + \angle C) = 90°, \text{(Fig. 4.39)}$$

Therefore $\angle BCE < \angle CBL < 90°$ and the chords BE and CL of the circle BCE subtend different acute angles on the circumference of the circle. This implies that $BE \neq CL$; also the shorter chord being farther from the centre, subtends a smaller acute angle at the circumference. Hence $BE < CL$. But $CL < CF$ and therefore $BE < CF$. ❑

Corollary. If two internal bisectors of a triangle are equal, then the triangle is isosceles.

Proof. Immediate from the theorem. ❑

Theorem 30. The external bisectors of any two angles of a triangle are concurrent with the internal bisector of the third angle.

Proof. Let the external bisectors of $\angle B$ and $\angle C$ of ΔABC meet at I_a (Fig. 4.40). Then the distances of I_a from BC and AB are equal as I_a lies on the external bisector of $\angle B$. Also I_a lies on the external bisector of $\angle C$ implies that the distance of I_a from BC and CA are equal. Thus I_a is at the same distance, say r_a, from the three sides of the ΔABC. Hence I_a must lie on the internal bisector AI of $\angle A$. ❑

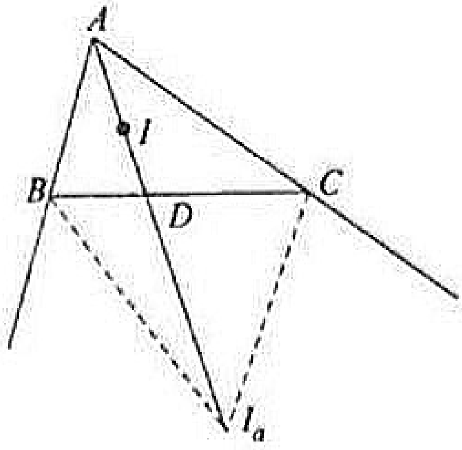


Fig. 4.40

The point I_a is called the *excentre* opposite to A. Similarly, the external bisectors of $\angle C$ and $\angle A$ meet the internal bisector of $\angle B$ at a point I_b called the excentre opposite to B; and the external bisectors of $\angle A$ and $\angle B$ meet the internal bisector of $\angle C$ at I_c, the excentre opposite to C.

We note that the incentre I is equidistant from the three sides BC, CA, AB of ΔABC. If r is the distance of I from the sides of ΔABC then the circle with centre I and radius r touches all the three sides of the triangle and is inscribed in the triangle. It is called the *incircle* of ΔABC. The circle with centre I_a and radius r_a touches the sides BC, CA, AB of ΔABC. It touches BC at a point on the line segment BC, whereas it touches the other two sides CA, AB at points on CA, AB produced. The circles (I_a, r_a), (I_b, r_b), (I_c, r_c) are the three escribed circles known as the *excircles* opposite to A, B and C respectively. (See Fig. 4.41).

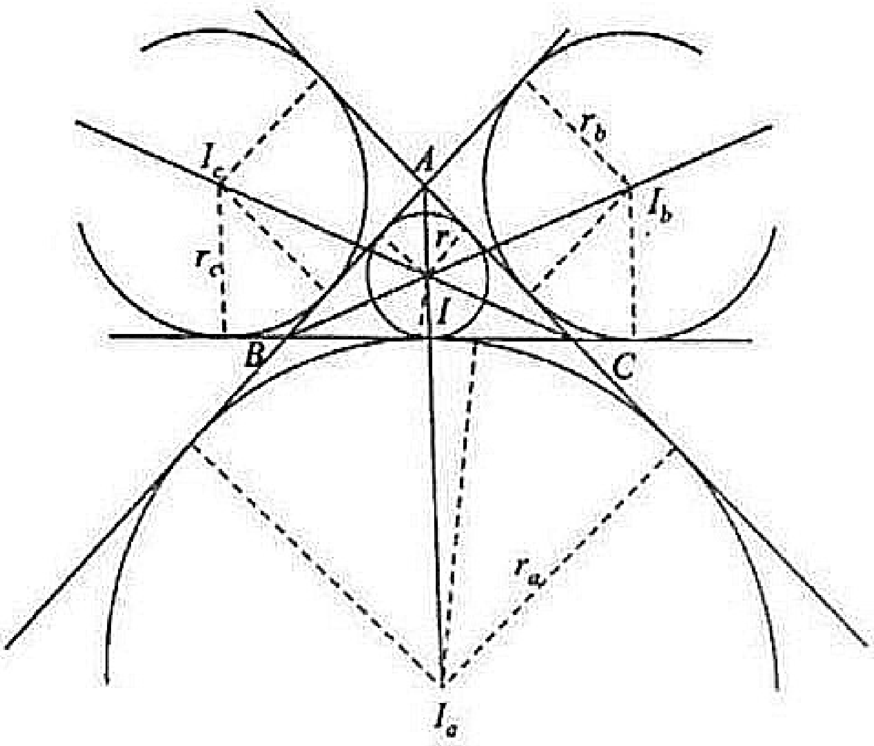


Fig. 4.41

Theorem 31. The incentre I and the excentre I_a opposite to A divide the bisector AU harmonically, where U is the point of intersection of the internal bisector of $\angle A$ and BC.

Proof. It is required to prove that $\frac{AI}{IU}=\frac{AI_a}{UI_a}$. Fig. 4.42.

Consider ΔBAU. By Theorem 31 of Chapter 3 we have

$\frac{AI}{IU}=\frac{BA}{BU}=\frac{AI}{UI_a}$ since BI and BI_a are the bisectors of $\angle ABU$ of ΔABC.

In fact, we have $\frac{BU}{UC}=\frac{c}{b}$ and $\frac{BV}{CV}=\frac{c}{b}$ (Fig. 4.42). □

$$\frac{BU}{BU+UC}=\frac{c}{c+b} \quad \text{or} \quad BU=BC\cdot\frac{c}{c+b}=\frac{ac}{b+c}.$$

$$\frac{BV}{CV-BV}=\frac{c}{b-c} \text{ or } BV=BC\cdot\frac{c}{b-c}=\frac{ac}{b-c}.$$

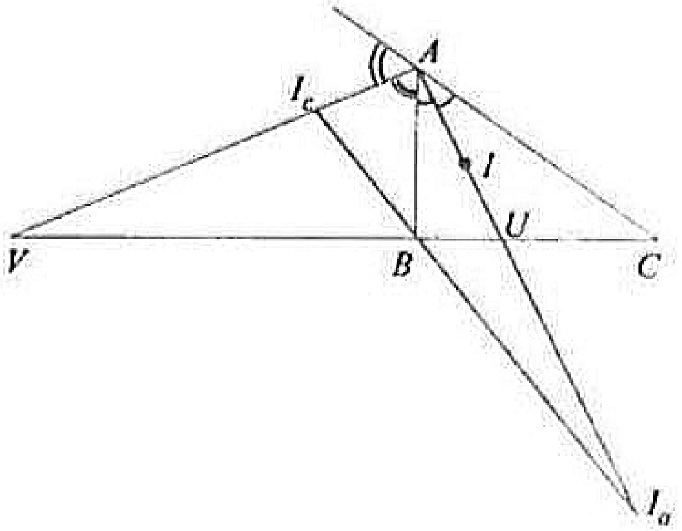


Fig. 4.42

$$\frac{AI}{IU} = \frac{BA}{BU} = \frac{c}{ac/(b+c)} = \frac{b+c}{a} \quad \text{and}$$

$$\frac{AI_c}{I_cV} = \frac{BA}{BV} = \frac{c}{ac/(b-c)} = \frac{b-c}{a}.$$

Theorem 32. If I is the incentre of ΔABC and I_a is the excentre opposite to A then $AI \cdot AI_a = AB \cdot AC$.

Proof. As $\angle IBI_a = \angle ICI_a = 90°$, the circle on II_a as diameter passes through B and C. If AB cuts this circle again at B_1 (Fig. 4.43) then $AB_1 = AC$ (Why?).

Therefore $\quad AI \cdot AI_a = AB \cdot AB_1 = AB \cdot AC.$ □

Theorem 33. If the incircle of ΔABC touches the sides BC, CA, AB of the triangle at X, Y, Z then $BX = s - b$, $CY = s - c$ and $AZ = s - a$ where $2s = a + b + c$, the perimeter of ΔABC.

Proof. We have $\quad AZ = AY$, $BZ = BX$ and $CX = CY$. (Fig. 4.44).

Therefore
$$BZ + BX = BA + BC - AZ - CX$$
$$= BA + BC - AY - CY$$
$$= c + a - b.$$

Therefore $\quad 2BZ = 2s - 2b$ or $BZ = s - b = BX$

Similarly we get $\quad CY = s - c$ and $AZ = s - a$. □

Theorem 34. If the escribed circle opposite to A touches the sides BC, CA, AB of ΔABC at X_a, Y_a, Z_a respectively, then $AZ_a = AY_a = s = BX_b = BZ_b = CY_c = CX_c$ (with obvious meanings for X_b, Y_b, Z_c, X_c, Y_c, and Z_c).

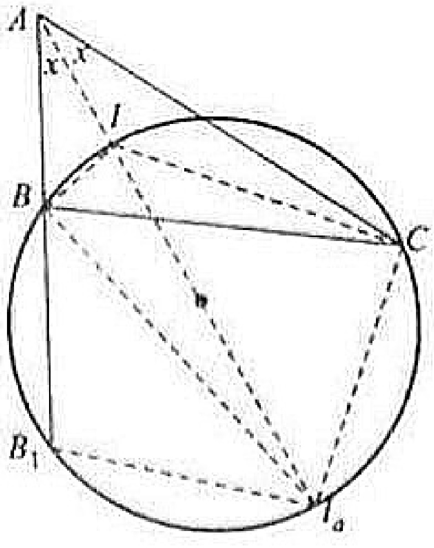


Fig. 4.43

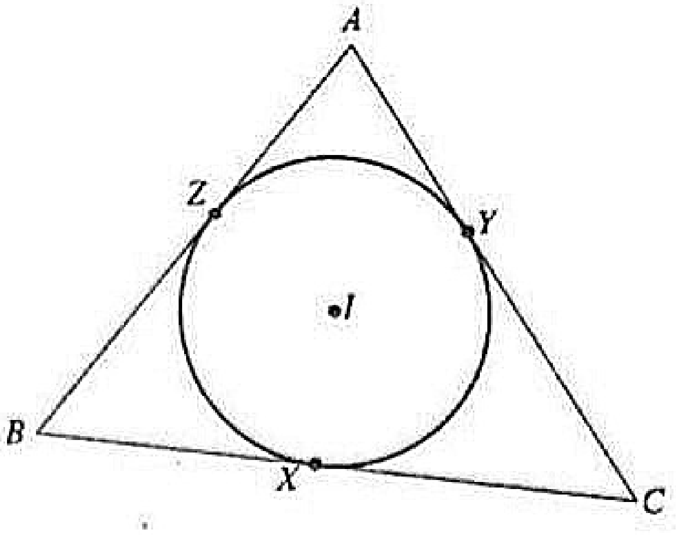


Fig. 4.44

Proof. We have $AY_a = AZ_a$, $BX_a = BZ_a$ and $CY_a = CX_a$, being the tangents from A, B, C to the escribed circle (I_a, r_a).

Therefore
$$2AZ_a = AZ_a + AY_c$$
$$= AB + AC + BZ_a + CY_a = AB + AC + BX_a + CX_a$$
$$= AB + AC + BC = 2s.$$ □

Therefore $\quad AZ_a = AY_a = s$. Similarly we get $BX_b = BZ_b = s = CY_c = CX_c$. □

Corollary. $\quad BX_a = BZ_a = s - c$; $CX_a = CY_a = s - b$

Proof. We have $\quad BX_a = BZ_a = AZ_a - AB = s - c$ and

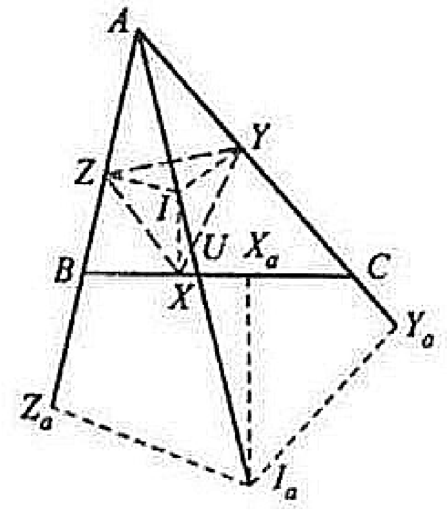


Fig. 4.45

$$CX_a = CY_a = AY_a - AC = s - b.$$ □

Definition 3. Two points on a side of a triangle are *isotomic points* if they are equidistant from the midpoint of this side.

Theorem 35. The points of contact of a side of a triangle with the incircle and excircle corresponding to this side are two isotomic points.

Proof. In our usual notations, we wish to prove that

$BX = CX_a$, $AY = CY_b$ and $AZ = BZ_c$. (Fig. 4.41).

We have $BX = s - b$ and $CX_a = s - b$ (Theorem 33 and Corollary to Theorem 34). This means that X and X_a are isotomic points. Similarly, Y and Y_b are isotomic points on CA and so are Z and Z_c on AB. □

Corollary.

(*i*) $XX_a = b - c$, $YY_b = a - c$, $ZZ_c = a - b$

(*ii*) $ZZ_a = YY_a = a$; $ZZ_b = XX_b = b$; $YY_c = XX_c = c$.

Proof. (*i*) $XX_a = BC - BX - CX_a = a - 2(s - b) = b - c$.

Similarly, $YY_b = a - c$ and $ZZ_c = a - b$. (Fig. 4.41).

(*ii*) $ZZ_a - ZB + BZ_a = (s - b) + (s - c) = a$. Similarly we get the other equations. □

Theorem 36.

(1) X_c and X are isotomic points on BC; Y_a and Y_c are isotomic on CA and Z_a and Z_b are isotomic points on AB. Further $X_bX_c = b + c$, $Y_cY_a = c + a$ and $Z_aZ_b = a + b$.

(2) $Y_bY_c = Z_bZ_c = a$.

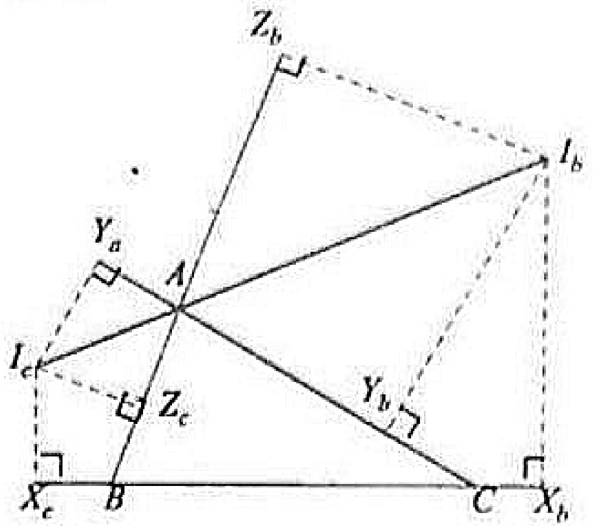


Fig. 4.46

Proof. 1. We have $BX_c = CX_c - a = s - a$ and $CX_b = BX_b - a = s - a$. Therefore X_b and X_c are isotomic points on BC. Furthermore, $X_bX_c = X_bC + CB + BX_c = (s-a) + a + (s-a) = 2s - a = b + c$. Similarly we get the corresponding relations for the other sides.

2. $Y_bY_c = CY_c - CY_b = s - CY_b = s - CX_b = s - (s-a) = a$. Similarly, $Z_bZ_c = a$. ❑

Theorem 37.

(1) For any triangle the area is equal to the product of the inradius and the semiperimeter. *i.e.*, for any ΔABC,

area of $\Delta ABC = \Delta = rs$

(2) $\Delta = r_a(s-a) = r_b(s-b) = r_c(s-c)$.

Proof. (1) If I is the incentre of ΔABC, we have

$$\text{Area of } \Delta ABC = \text{area of } \Delta BIC + \text{area } \Delta CIA + \text{area } \Delta AIB. \text{ (Fig. 4.41)}$$
$$= (1/2)\,ar + (1/2)\,br + (1/2)\,cr = rs$$

Thus $\Delta = rs$.

(2) We see from Fig. 4.41 that

$$\text{Area } \Delta ABC = \text{area } \Delta ABI_a + \text{area } \Delta ACI_a - \text{area } \Delta BCI_a$$
$$= (1/2)cr_a + (1/2)br_a - (1/2)ar_a = (1/2)(b + c - a)r_a$$
$$= (s-a)r_a$$

Thus $\Delta = (s-a)r_a$. Similarly we obtain

$\Delta = (s-b)r_b$,

$\Delta = (s-c)r_c$. ❑

Theorem 38. $rr_a = (s-b)(s-c)$

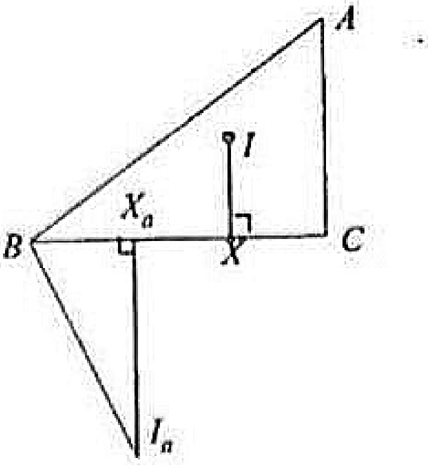


Fig. 4.47

Proof. We have
$$\angle IBX = \angle IBI_a - \angle I_aBX_a$$
$$= 90^\circ - \angle I_aBX_a = \angle BI_aX_a$$

Therefore the right triangles IBX and BI_aX_a are similar,

and
$$\frac{IX}{BX} = \frac{BX_a}{I_aX_s} \text{ or } \frac{r}{s-b} = \frac{s-c}{r_a}$$

i.e.,
$$rr_a = (s-b)(s-c)$$ ❑

Theorem 39. Area of $\Delta ABC = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (known as Hero's formula)

Proof. We have seen that $r = \Delta/s$ and $r_a = \Delta/s - a$

Therefore $$rr_a = \frac{\Delta^2}{s(s-a)}.$$ But by Theorem 38 $rr_a = (s-b)(s-c)$.

This gives $$\frac{\Delta^2}{s(s-a)} = (s-b)(s-c) \text{ or } \Delta^2 = s(s-a)(s-b)(s-c).$$ ❑

Corollary. (1) $rr_ar_br_c = \Delta^2$

(2) $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$.

Proof.

(1) $$rr_ar_br_c = \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \frac{\Delta}{s-c}$$

$$= \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = \Delta^2 \text{ from the theorem.}$$

(2) $$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{(s-c)}{\Delta} + \frac{(s-b)}{\Delta} + \frac{(s-c)}{\Delta}$$

$$= \frac{3s-(a+b+c)}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$ ❑

Theorem 40. For any triangle ABC, if the incircle touches the sides BC, CA, AB at X, Y, Z respectively, then

$$\frac{\text{area of } \Delta ABC}{\text{area of } \Delta XYZ} = \frac{2R}{r}$$

where R is the circumradius and r is the inradius of ΔABC.

Proof. The quadrilateral $AZIY$ is cyclic (Why?) See Fig. 4.48.

Therefore $\angle BAC + \angle YIZ = 180°$

So, by Theorem 36 of Chpater 3, we get

$$\frac{\Delta IYZ}{\Delta ABC} = \frac{IY\,.\,IZ}{AB\,.\,AC} = \frac{r^2}{bc} = \frac{cr^2}{abc}.$$

Similarly $$\frac{\Delta IZX}{\Delta ABC} = \frac{br^2}{abc} \text{ and } \frac{\Delta IXY}{\Delta ABC} = \frac{ar^2}{abc}.$$

Adding we get $$\frac{\Delta XYZ}{\Delta ABC} = \frac{r^2(a+b+c)}{abc} = \frac{2sr^2}{abc} = \frac{2\Delta r}{4R\Delta} = \frac{r}{2R}.$$

(From Theorem 37 and Cor. to Thm. 21 of Ch. 4)

Thus $$\frac{\Delta ABC}{\Delta XYZ} = \frac{2R}{r}.$$ ❑

Theorem 41. $$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$$

where r is the inradius and h_a, h_b, h_c are the altitudes through A, B, C of ΔABC.

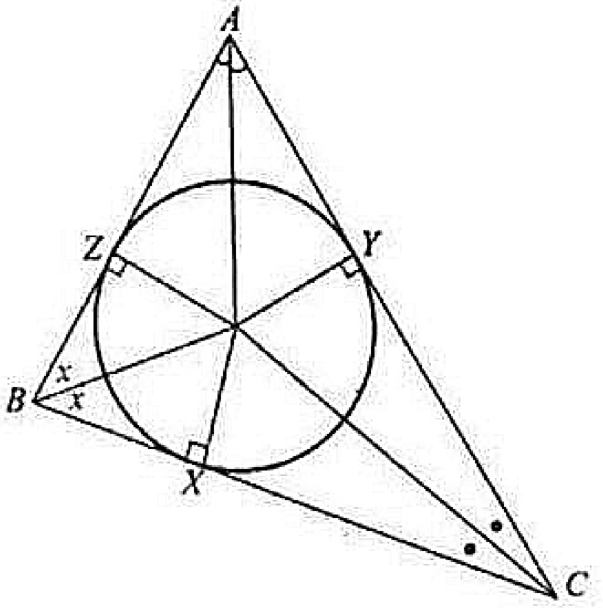


Fig. 4.48

Proof. We have $\quad \Delta = (1/2)ah_a = (1/2)bh_b = (1/2)ch_c$.

Therefore $\quad 2\Delta = ah_a = bh_b = ch_c$; which gives

$$\frac{2s}{(1/r)} = \frac{a}{(1/h_a)} = \frac{b}{(1/h_b)} = \frac{c}{(1/h_c)} = \frac{a+b+c}{(1/h_a + 1/h_b + 1/h_c)}$$ ☐

Hence $$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

Corollary. $$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

Proof. Follows immediately from the Corollary to Theorem 39 and Theorem 41. ☐

Theorem 42. $\quad r_a + r_b + r_c = r + 4R$.

Proof. Let the circumcircle of ΔABC meet the line $AI I_a$ at P. Then the centre of the circle IBI_aC on II_a as diameter lies on II_a as well as the perpendicular bisector of BC. The arc BP and the arc PC make equal angles at A implies that P is the midpoint of arc BC. Hence P lies on the perpendicular bisector of BC. This means that P must be the midpoint of II_a. Let PQ be the circumdiameter of ΔABC through P. Let IK be parallel to BC meeting I_aX_a at K (Fig. 4.49) and meeting SD at L. Then we have,

$$PD + DL = PL = (1/2)I_aK \text{ (since } PL \parallel I_aK) = (1/2)(I_aX_a + X_aK)$$

Also $\quad DL = X_aK = r$. (Fig. 49).

Therefore $\quad PD = (1/2)(I_aX_a + r) - r = (1/2)(r_a - r)$ (since $I_aX_a = r_a$).

Just as we observed that P is the midpoint of II_a, we can prove that Q is the midpoint of I_bI_c. Also,

$$DQ = (1/2)(I_bX_b + I_cX_c) = (1/2)(r_b + r_c).$$

$\therefore \quad 2R = PQ = PD + DQ = (1/2)(r_a + r_b + r_c - r)$ ☐

and hence $\quad r_a + r_b + r_c = 4R + r$.

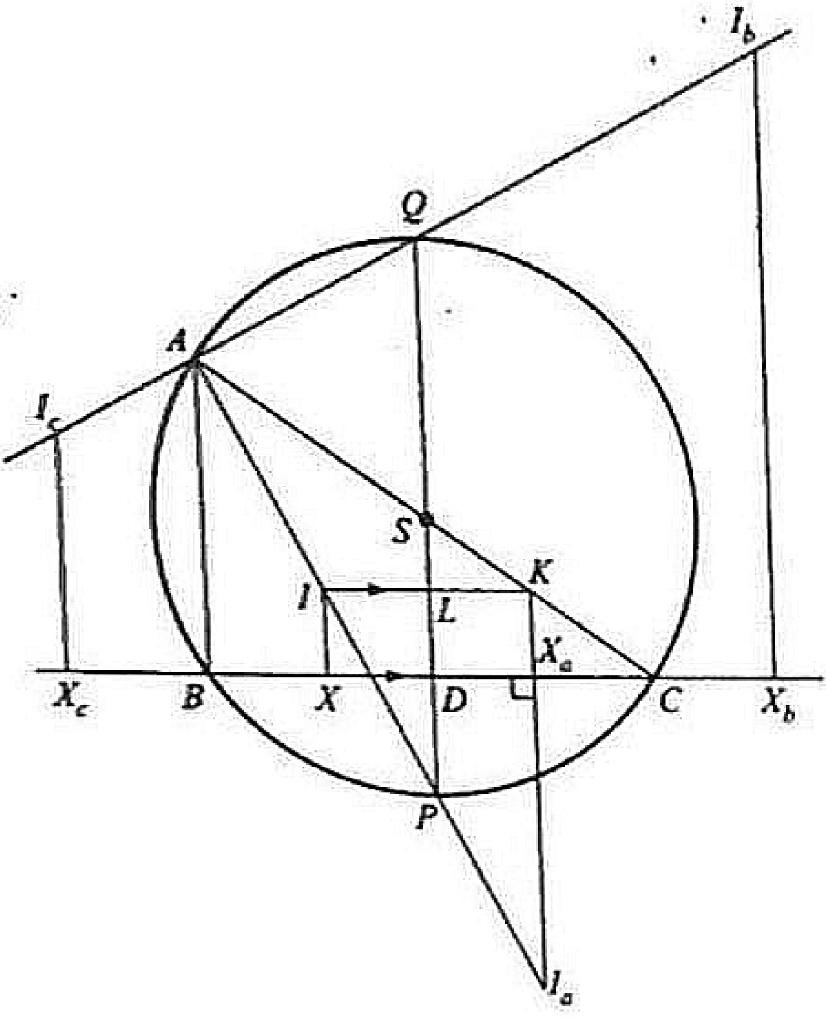


Fig. 4.49

Corollary. $SD + SE + SF = R + r$ (Fig. 4.49).

Proof. $SD = SP - DP = R - DP = R - (1/2)(r_a - r)$

(as in the proof of Theorem 42)

Similarly $SE = R - (1/2)(r_b - r)$ and $SF = R - (1/2)(r_c - r)$

Therefore $SD + SE + SF = 3R - (1/2)(r_a + r_b + r_c - 3r)$.

$= 3R - (1/2)(r + 4R - 3r) = R + r.$ ☐

Theorem 43. (Euler's Theorem).

If d is the distance between the circumcentre and the incentre of a triangle then $SI^2 = d^2 = R^2 - 2Rr$.

Proof. We have $AI \cdot IP = IJ \cdot IK = (R + d)(R - d) = R^2 - d^2$. (Fig. 4.50).

As we have seen in the proof of Theorem 42, P is the centre of the circle IBI_aC and hence $PI = PB = PC$. Therefore $AI \cdot PC = R^2 - d^2$. Now, the right triangles AIZ and QPC are similar, since $\angle IAZ = \angle PAB = \angle PQC$ (Why?). This gives $AI/PQ = IZ/PC$ or $AI \cdot PC = PQ \cdot IZ$. Thus $R^2 - d^2 = 2Rr$ or $d^2 = R^2 - 2Rr$. ☐

Theorem 44. $(SI_a)^2 = R^2 + 2Rr_a$, $(SI_b)^2 = R^2 + 2Rr_b$ and $(SI_c)^2 = R^2 + 2Rr_c$

Proof. Exercise. ☐

Corollary. $(II_a)^2 = 4R(r_a - r)$, $(I_bI_c)^2 = 4R(r_b + r_c)$.

Proof. We use Fig. 4.49. For the triangle SII_a, SP is a median.

Therefore, $SI^2 + SI_a^2 = 2SP^2 + (1/2)II_a^2$.

This gives $R^2 - 2Rr + R^2 + 2Rr_a = 2R^2 + (1/2)\, II_a^2$

Hence $II_a^2 = 4R(r_a - r)$

Similarly, for the ΔSI_bI_c, SQ is a median and so

$$SI_b^2 + SI_c^2 = 2SQ^2 + (1/2)I_bI_c^2$$

Therefore $R^2 + 2Rr_b + R^2 + 2Rr_c = 2R^2 + (1/2)\, I_bI_c^2$

Hence $$I_bI_c^2 = 4R(r_b + r_c)$$ ☐

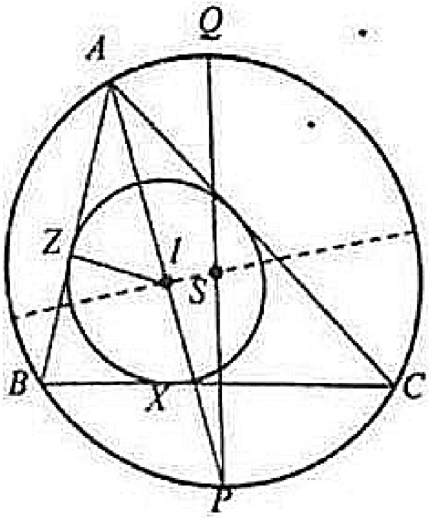


Fig. 4.50

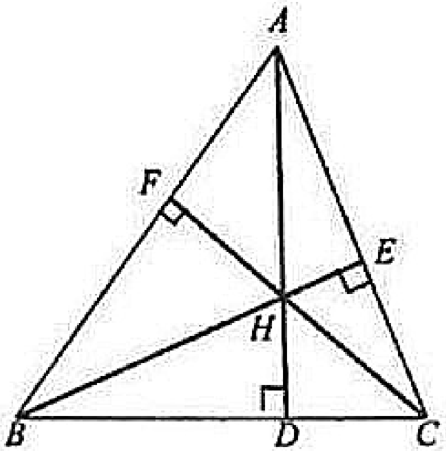


Fig. 4.51

Theorem 45. If the line of centres for the two circles (S, R) and (I, r) satisfy $SI^2 = R^2 - 2Rr$ then an infinite number of triangles may be found such that (I, r) is the incircle of each one of them and (S, R) is the circumcircle of each one of them.

Proof. From any point A on the circles (S, R) draw the two tangents AB, AC to the other circle (I, R) meeting (S, R) again at B, C (Fig. 4.50). If AI meets the circle (S, R) at P, then $SI^2 = R^2 - 2Rr$ implies that $AI \cdot IP = 2Rr$ (Why?). If Z is the foot of the perpendicular from I on AB, then the triangles AIZ and QPC are similar. Therefore $AI \cdot PC = PQ \cdot IZ = 2Rr$. This means that $AI \cdot IP = AI \cdot PC = 2Rr$ and hence $IP = PC$. As we have already seen in the proof of Theorem 42, for ΔABC, the point P must be the midpoint of the line joining the incentre and the excentre opposite to A; and we have $PB = PC = PI_a$. Now I lies on the internal bisector of $\angle A$ (Fig. 4.50) and $PC = PI$ implies that I must be the incentre of ΔABC. Since we have infinitely many choices available for A, there is an infinity of triangles having (I, r) as the incircle and (S, R) as the circumcircle. ☐

Theorem 46. Let ABC be a triangle with AD, BE, CF as the altitudes and H the orthocentre. Then $AH \cdot HD = BH \cdot HE = CH \cdot HF$.

Proof. The right triangles BHF and CHE are similar since $\angle FBH = \angle ECH$ (the quadrilateral $BCEF$ is cyclic) (Fig. 4.51). Therefore,

$$\frac{HF}{BH} = \frac{HE}{CH} \quad \text{or} \quad CH \cdot HF = BH \cdot HE$$

Similarly, $AH \cdot HD = BH \cdot HE$ ☐

Note. The same proof works when ΔABC is obtuse angled and H lies outside ΔABC.

Corollary. $HA \cdot HD = (1/2)(a^2 + b^2 + c^2) - 4R^2$.

Proof. We have $HA \cdot HD = HB \cdot HE =$ Power of H with respect to the circle on BC as diameter. (Fig. 4.52). Therefore, we get

$$HA \cdot HD = (a/2)^2 - A'H^2 \tag{1}$$

Now, HA' is the median through H for ΔHBC. Therefore,

$$HB^2 + HC^2 = (1/2)a^2 + 2(HA')^2 \tag{2}$$

If AP is the diameter of the circumcircle through A, then $CP \perp AC$ and hence $BH \parallel PC$. Similarly, $CH \parallel PB$. This means that $BPHC$ is a parallelogram and $BH = PC$. This gives,

$$\left\{\begin{array}{rl} & HB^2 = PC^2 = AP^2 - AC^2 = 4R^2 - b^2 \\ \text{Similarly} & HC^2 = PB^2 = AP^2 - AB^2 = 4R^2 - c^2 \end{array}\right\} \qquad (3)$$

From (1), (2) and (3) we get,

$$HA \cdot HD = a^2/4 - (HB^2/2 + HC^2/2 - a^2/4) = a^2/2 + b^2/2 + c^2/2 - 4R^2 \qquad ❑$$

Theorem 47. The chord CX of the circumcircle of ΔABC perpendicular to BC is equal to AH, where H is the orthocentre of ΔABC.

Proof. $CX \perp BC$ implies that BX is a diameter of the circumcircle of ΔABC. Therefore $\angle BAX = 90°$. This means that $AX \perp AB$ and so $AX \parallel HC$. Similarly, $AH \parallel XC$. Therefore $AHCX$ is a parallelogram and $CX = AH$. ❑

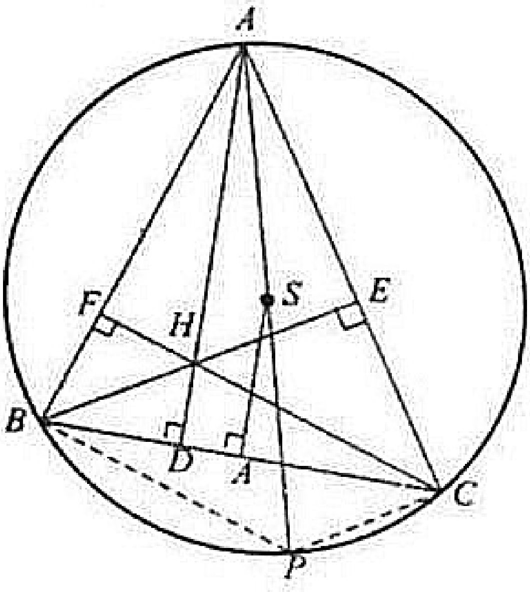


Fig. 4.52

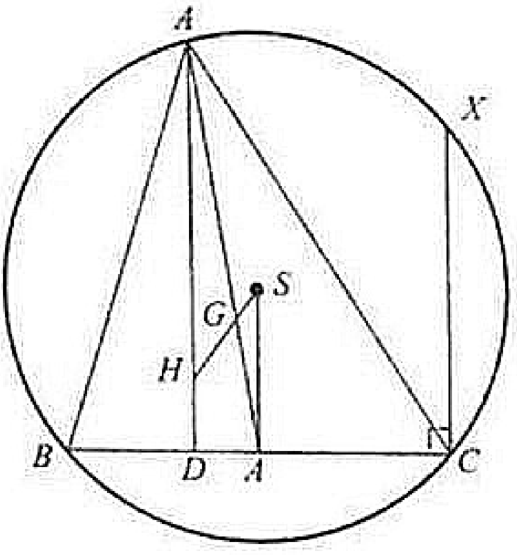


Fig. 4.53

Corollary. $AH = 2SA'$.

Proof. In ΔBCX, S and A' are the midpoints of the sides BX and BC respectively. Therefore $2SA' = CX$ and hence by Theorem 47, $AH = CX = 2SA'$. ❑

Theorem 48. In any triangle, the circumcentre, the orthocentre and the centroid are collinear. The centroid G trisects the line joining the circumcentre and the orthocentre. (This line is called the *Euler line* of the triangle).

Proof. Let the median AA' of ΔABC meet the line SH at G, where S is the circumcentre and H is the orthocentre of ΔABC (Fig. 4.53). The triangle AHG is similar to the triangle $A'SG$ and therefore,

$$\frac{AH}{SA'} = \frac{HG}{SG} = \frac{AG}{GA'}$$

But $2SA' = AH$ (Corollary to Theorem 47) and hence $AG/GA' = 2$. This means that G must be the centroid of ΔABC. Again, the above equation tells us that G trisects SH. Hence the theorem. ❑

Corollary 1.

(1) $SH^2 = 9R^2 - (a^2 + b^2 + c^2)$

(2) $GH^2 = 4R^2 - (4/9)(a^2 + b^2 + c^2)$

Proof. We have $SH = 3SG$ and $GH = 2SG$. By Cor. 3 to Theorem 28, Ch. 4 we have, $SG^2 = R^2 - (1/9)(a^2 + b^2 + c^2)$.

Therefore, $$SH^2 = 9SG^2 = 9R^2 - (a^2 + b^2 + c^2) \text{ and}$$ $$GH^2 = 4SG^2 = 4R^2 - (4/9)(a^2 + b^2 + c^2).$$ ☐

Corollary. $2\ HA^2 + HB^2 + HC^2 = 12R^2 - (a^2 - b^2 + c^2)$.

Proof. By Cor. 2, Thm. 28, Chapter 4, we get

$$\begin{aligned} HA^2 + HB^2 + HC^2 &= GA^2 + GB^2 + GC^2 + 3GH^2 \\ &= \frac{a^2 + b^2 + c^2}{3} + 3GH^2 \quad \text{(Cor. 1 to Thm. 28)} \\ &= \frac{a^2 + b^2 + c^2}{3} + 12R^2 - (4/3)(a^2 + b^2 + c^2) \\ &= 12R^2 - (a^2 + b^2 + c^2) \end{aligned}$$ ☐

Note. We have already seen in the proof of Corollary to Theorem 46 that $HB^2 = 4R^2 - b^2$, $HC^2 = 4R^2 - C^2$ and $HA^2 = 4R^2 - a^2$. Therefore, $HA^2 + HB^2 + HC^2 = 12R^2 - (a^2 + b^2 + c^2)$.

Theorem 49. If the altitude AD of ΔABC meets the circumcircle again at D', then D is the midpoint of HD' where H is the orthocentre of ΔABC. In other words, the line segment of the altitude extended between the orthocentre and the other point of intersection with the circumcircle is bisected by the corresponding side of the triangle.

Proof. We have BH and HD perpendicular to CA and BC respectively, and so $\angle BHD = \angle ACB$. Also, $\angle HD'B = \angle AD'B = \angle ACB$ (angles in the same segment). Therefore $\angle BHD = \angle HD'B$ and so $BH = BD'$. Now $BD \perp HD'$ implies that $HD = DD'$. ☐

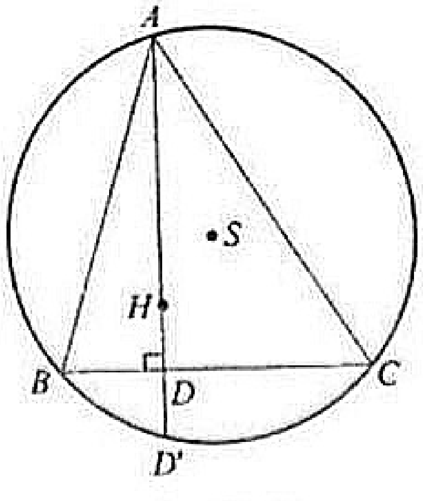


Fig. 4.54

Corollary 1. $BD \cdot DC = AD \cdot HD$

Proof. By the secant property of the circumcircle we get $BD \cdot DC = AD \cdot DD' = AD \cdot HD$ (by Theorem 49). ☐

Corollary 2. The circumcircle of ΔHBC is equal to the circumcircle of ΔABC (*i.e.*, they have the same radius).

Proof. The triangles HBC and $D'BC$ are congruent (Why?). Therefore, their circumcircles are equal circles. But the circumcircle of $\Delta D'BC$ is the same as that of ΔABC. Hence the result. ☐

Definition 4. If D, E, F are the feet of the altitudes of a triangle ABC on the corresponding sides then ΔDEF is *the orthic triangle* (or *the Pedal triangle*) of ΔABC.

Theorem 50. The three triangles cut off from a given triangle by the sides of its orthic triangle and the given triangle itself are mutually similar.

Proof. The quadrilateral $BCEF$ (Fig. 4.55) is cyclic (Why?) and hence $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Therefore $\Delta AEF \,|||\, \Delta ABC$. Likewise one can prove that $\Delta BDF \,|||\, \Delta BAC$ and $\Delta CDE \,|||\, \Delta CAB$. ❑

Theorem 51. A is the midpoint of the arc $F'E'$ of the circumcircle of ΔABC; B is the midpoint of the arc $F'D'$ and C is the midpoint of the arc $B'E'$, where D', E', F' are the points where the altitudes AD, BE, CF meet the circumcircle.

Proof. We note that $\angle ABE = \angle FBE = \angle FCE = \angle F'CA$ since the quadrilateral $BCEF$ is cyclic. Therefore arc $F'A$ = arc AE' or A is the midpoint of the arc $F'E'$ of the circle ABC. Similarly, we get the other results. ❑

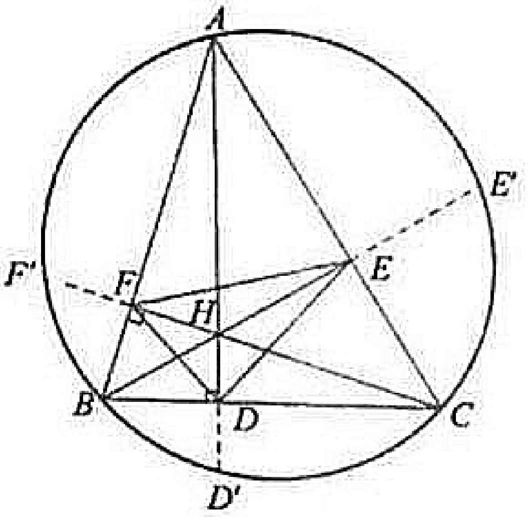


Fig. 4.55

Corollary. The radii of the circumcircle through the vertices of a triangle are perpendicular to the corresponding sides of the orthic triangle. In other words $SA \perp EF$, $SB \perp FD$ and $SC \perp DE$ (Fig. 4.55).

Proof. E and F are the midpoints of HE' and HF' (Thm 49) and therefore $EF \parallel E'F'$ and $EF = (1/2)E'F'$. Now by Theorem 51, SA bisects the chord $E'F'$ and $SA \perp E'F'$. This implies that $SA \perp EF$. ❑

Theorem 52. The orthocentre of an acute angled triangle is the incentre of the orthic triangle.

Proof. We again use Fig. 4.55. The line BE' bisects $\angle F'E'D'$ since arc BF' = arc BD'. Now $EF \parallel E'F'$ and $ED \parallel E'D'$ implies that BE bisects $\angle DEF$ of the orthic triangle. Similarly AD bisects $\angle FDE$ and CF bisects $\angle EFD$. Therefore, H is the incentre of ΔDEF. ❑

Corollary. The sides of a triangle bisect externally the angles of its orthic triangle.

Proof. The sides of a triangle ABC are perpendicular to the altitudes, which are the internal bisectors of the angles of the orthic triangle. Hence the sides of ΔABC are the external bisectors of the angles of its orthic triangle ΔDEF. ❑

Note. We observe that the quadrilaterals $BDHF$, $HDCE$, and $BCEF$ are all cyclic. Therefore, we get

$$\angle FDH = \angle FBH = \angle FBE = \angle FCE = \angle HCE = \angle HDE$$

(Angles in the same segment).

$\therefore$ DH bisects $\angle FDE$ of ΔDEF. Similarly, EH bisects $\angle DEF$ and hence H is the incentre of ΔDEF.

Theorem 53. With the usual notations, $AH + r_a = BH + r_b = CH + r_c = 2R + r$.

Proof. As in the proof of Theorem 42, we have $r_a - r = 2(SP - SA')$ where P is the point where SA' and AI meet on the circumcircle. But $2SA' = AH$ (Cor. to Theorem 47).

$$\therefore \quad AH + r_a - 2SP + r = 2R + r. \text{ Similarly we get}$$

$$BH + r_b = 2R + r = CH + r_c.$$ ❑

Theorem 54. If ABC is an acute-angled triangle and DEF is its orthic triangle then

$$\frac{EF}{BC} + \frac{FD}{CA} + \frac{DE}{AB} = \frac{R+r}{r}.$$

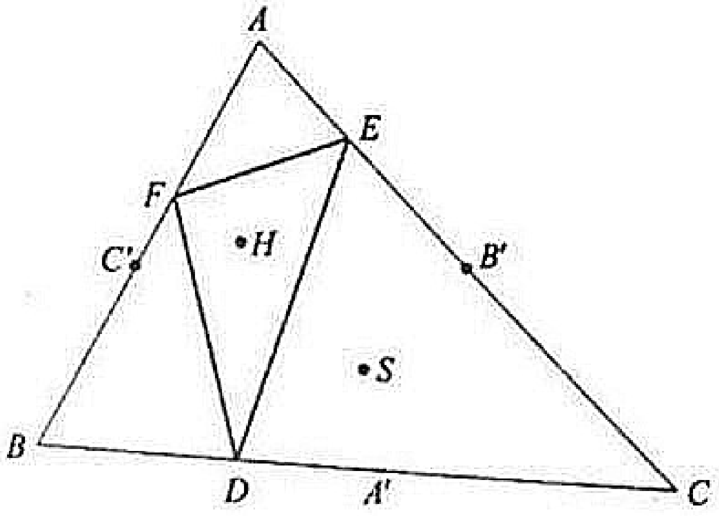


Fig. 4.56

Proof. The triangles AEF and ABC are similar.

$$\therefore \quad \frac{EF}{BC} = \frac{\text{Circumference of } \Delta AEF}{\text{Circumference of } \Delta ABC} = \frac{AH}{2R} = \frac{2SA'}{2R} = \frac{SA'}{R}$$

$$\text{Similarly} \quad \frac{FD}{CA} = \frac{SB'}{R}; \ \frac{DE}{AB} = \frac{SC'}{R}$$

$$\text{Adding} \quad \frac{EF}{BC} + \frac{FD}{CA} + \frac{DE}{AB} = \frac{SA' + SB' + SC'}{R} = \frac{R+r}{R}.$$ ❑

Theorem 55. Perimeter of the orthic triangle of ΔABC is

$$\frac{2 \text{ area of } \Delta ABC}{R}.$$

Proof. As in the proof of Theorem 54.

$$EF = \frac{SA'}{R} BC, \ FD = \frac{SB'}{R} CA \text{ and } DE = \frac{SC'}{R} AB$$

$$\therefore \quad \text{Perimeter of } \Delta DEF = \frac{1}{R}(SA' \,.\, BC + SB' \,.\, CA + SC' \,.\, AB)$$

$$= \frac{1}{R} 2(\Delta BSC + \Delta CSA + \Delta ASB)$$

$$= \frac{2}{R} \text{ area of } \Delta ABC.$$ ❑

Theorem 56. (The Nine-Point Circle Theorem).

The feet of the three altitudes of any triangle, the midpoints of the three sides and the midpoints of the segments from the orthocentre to the three vertices all lie on a circle of radius equal to half the circumradius. Further the centre of this circle bisects the line joining the orthocentre and the circumcentre.

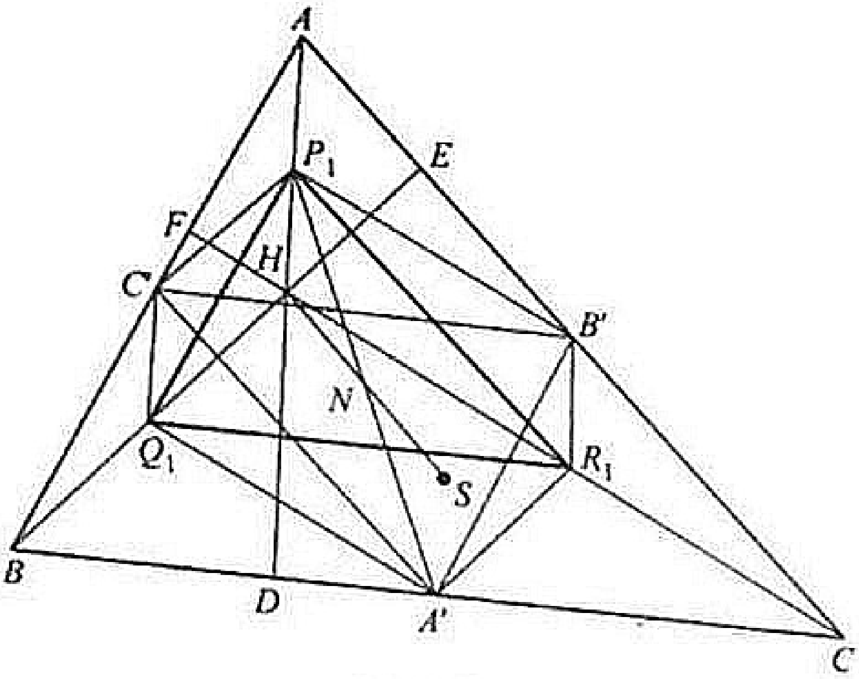


Fig. 4.57

Proof. Let D, E, F be the feet of the altitudes from A, B and C of ΔABC; let A', B', C' be the midpoints of BC, CA, AB respectively; H be the orthocentre, S the circumcentre and P_1, Q_1, R_1 be the midpoints of AH, BH and CH respectively. It is required to prove that D, E, F, A', B', C', P_1, Q_1, R_1 all lie on a circle with centre N the midpoint of the segment SH, and radius $R/2$. We have $C'B' \parallel BC$ and $Q_1R_1 \parallel BC$, since Q_1 and R_1 are the midpoints of the sides HB, HC of ΔHBC, Also $C'B' = Q_1R_1 = (1/2)BC$, Again from ΔABH, we get $C'Q_1 \parallel AH$ and $C'Q_1 = (1/2)\ AH$.

From ΔACH, we get $B'R_1 \parallel AH$ and $B'R_1 = (1/2)AH$.

Thus $C'Q_1 \parallel B'R_1$ and $C'Q_1 = B'R_1 = (1/2)AH$.

AH is perpendicular to BC implies that $C'Q_1$ is perpendicular to Q_1R_1. $\therefore$ $B'C'Q_1R_1$ is a rectangle. Similarly $C'A'R_1P_1$, and $A'B'P_1Q_1$ are also rectangles. Hence P_1, C', Q_1, A', R_1, B' all lie on the circle having $A'P_1$, $B'Q_1$ and $C'R_1$ as diameters.

Also $\angle A'DP_1 = 90°$ implies that this circle also passes through D. Similarly, it passes through E and F as well. Thus the nine-points A', B', C', P_1, Q_1, R_1, D, E, F, all lie on a circle. The radius of this nine-point circle is $(1/2)\ A'P_1$.

From the parallelogram $AP_1A'S$ we get $R = SA = A'P_1$

$\therefore$ The radius of the nine-point circle is $R/2$.

It remains to prove that the centre N of the nine-point circle is the midpoint of SH. We have (A', P_1), (B', Q_1), (C', R_1) as three pairs of diametrically opposite points on the nine-point circle. Therefore the triangle $A'B'C'$ can be obtained from $\Delta P_1Q_1R_1$ by a half turn (rotation through 180°) about the centre of the nine-point circle. Under this rotation, the triangles $A'B'C'$ and $P_1Q_1R_1$ get interchanged and hence their orthocentres also get interchanged. The orthocentre of $\Delta P_1Q_1R_1$ is H and the orthocentre of $\Delta A'B'C'$ is S.

Therefore H and S correspond with respect to a half turn about the centre of the nine-point circle. Hence the midpoint N of HS is the centre of the nine-point circle. ❑

Note. 1. The circumcentre, orthocentre, centroid and the nine-point centre all lie on the Euler line. G trisects SH and N bisects SH. S and N divide GH harmonically in the ratio $2:1$.

2. The fact that the nine-point centre bisects SH can also be seen as follows. $P_1HA'S$ is a parallelogram and hence the diagonals bisect each other. So the midpoint of SH must be the midpoint of $A'P_1$, which is the centre of nine-point circle of ΔABC.

Theorem 57. The sum of the powers of the vertices of a triangle ABC, with respect to its nine-point circle is $(1/4)\,(a^2+b^2+c^2)$.

Proof. The power of A with respect to the nine-point circle is

$$AE\cdot AB' = AF\cdot AC' \text{ (Fig. 4.57).}$$

$\therefore$ Power of A with respect to the nine-point circle is

$$= (1/2)\,(AE\cdot AB' + AF\cdot AC') = (1/2)\left(AE\cdot\frac{b}{2} + AF\cdot\frac{c}{2}\right)$$

$$= (1/4)\,(b\,AE + c\,AF)$$

The sum of the powers of A, B, C with respect to the nine-point circle

$$= (1/4)\,\{(b\,AE + c\,AF) + (c\,BF + a\,BD) + (a\,DC + b\,CE)\}$$

$$= (1/4)\,(b^2+c^2+a^2).$$ ❑

Corollary. $NA^2 + NB^2 + NC^2 + NH^2 = 3R^2$

Proof. Power of A with respect to the nine-point circle is

$$AN^2 - \left(\frac{R}{2}\right)^2 = AN^2 - \frac{R^2}{4}$$

Sum of the powers of the vertices with respect to the circle $(N, R/2)$

$$= (1/4)\,(a^2+b^2+c^2)$$

$$= NA^2 + NB^2 + NC^2 - 3\frac{R^2}{4}.$$

$$\therefore \quad NA^2 + NB^2 + NC^2 + NH^2 = (1/4)(a^2+b^2+c^2) + 3\,\frac{R^2}{4} + \frac{SH^2}{4}$$

$$= \frac{1}{4}\,(a^2+b^2+c^2) + \frac{3R^2}{4} + \frac{9R^2-(a^2+b^2+c^2)}{4} = 3R^2$$

(From Cor. 1 to Theorem 48) ❑

Theorem 58. All triangles inscribed in a given circle and having a given point as the orthocentre, have the same nine-point circle.

Proof. All these triangles have the same circumcentre S and the same orthocentre H. Therefore they have the same nine-point centre N, namely the midpoint of SH. Further the radius of the nine-point circle is $R/2$ where R is the radius of the given circle. ❑

Theorem 59 (Feuerbach's Theorem). In any triangle, the nine-point circle touches the incircle and the three described circles.

Proof.

First Proof. We assume $\angle B > \angle C$. Let $A'T$ be the tangent (Fig. 4.58) to the nine-point circle at A'. Then we have $\angle CA'T = \angle CA'B' - \angle B'A'T = \angle B - \angle B'C'A'$ (angle between a chord and a tangent = angle in the alternate segment). $= \angle B - \angle C$ (Note that $\Delta A'B'C'$ is the medial triangle) (1)

Let AI meet BC at U. We have $\dfrac{IU}{UI_a} = \dfrac{r}{r_a}$ (Fig. 4.59)

$\therefore$ The transverse common tangents to the incircle and the excircle (I_a, r_a) meet at U (Theorem 14).

Hence if UP_1 and UP_1' are the tangents to the incircle and the excircle opposite to A from U then P_1UP_1' is a straight line.

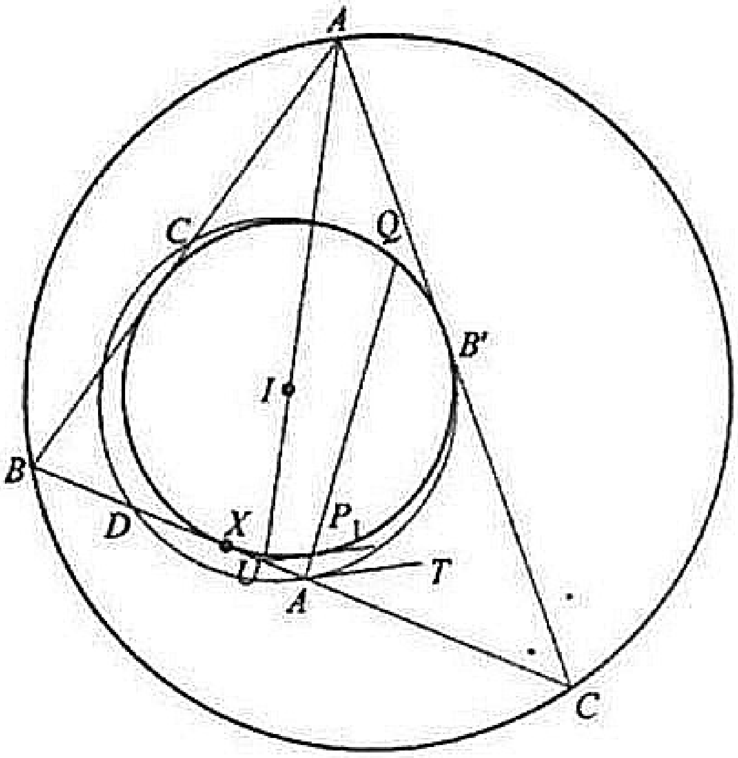


Fig. 4.58

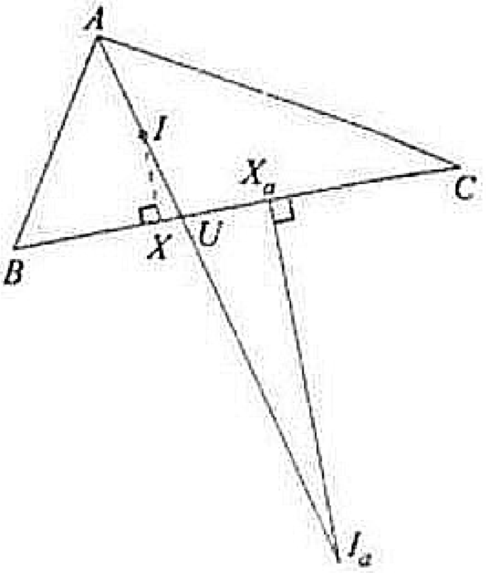


Fig. 4.59

$$\angle AUP_1 = \angle BUA \text{ (since } UP_1,\ UB \text{ are tangents to the in circle).}$$

$$= \angle CAU + \angle BCA = \frac{\angle A}{2} + \angle C.$$

$$\angle P_1UC = 180^\circ - \angle BUP_1 = 180^\circ - 2\angle BUA$$

$$= 180^\circ - (\angle A + 2\angle C) = \angle B - \angle C \qquad (2)$$

$\therefore$ $UP_1 \parallel A'T$ from (1) and (2).

Let $A'P_1$ meet the incircle again at Q . AU divides I_a harmonically implies that the feet of the perpendiculars on BC from A, U divide the feet of the perpendiculars on BC from I, I_a harmonically.

$\therefore$ DU divides XX_a harmonically and A' is the midpoint of XX_a implies that

$$A'P_1 \cdot A'Q = A'X^2 = A'U \cdot A'D.$$

$\therefore$ P_1, Q, U, D are concyclic and this gives

$$\angle A'QD = \angle P_1UA' = \angle P_1UC = \angle B - \angle C \text{ from (2)}$$

From (1) we get $\angle A'QD = \angle B - \angle C = \angle TA'C$

$\therefore$ For the nine-point circle $A'T$ is a tangent and $A'D$ is a chord with $\angle A'QD = \angle TA'C$ implies that Q lies on the nine-point circle. Now Q is a common point to the incircle and the nine-point circle. The tangent at P_1 to the incircle is parallel to the tangent at A' to the nine-point circle. Therefore the tangent at Q to the nine-point circle makes the same angle $\angle TA'Q$, with QP_1 and the tangent at Q to the incircle also makes the same angle with QP_1. This means that the incircle and the nine-point circle have the same tangent at Q or the two circles touch at Q.

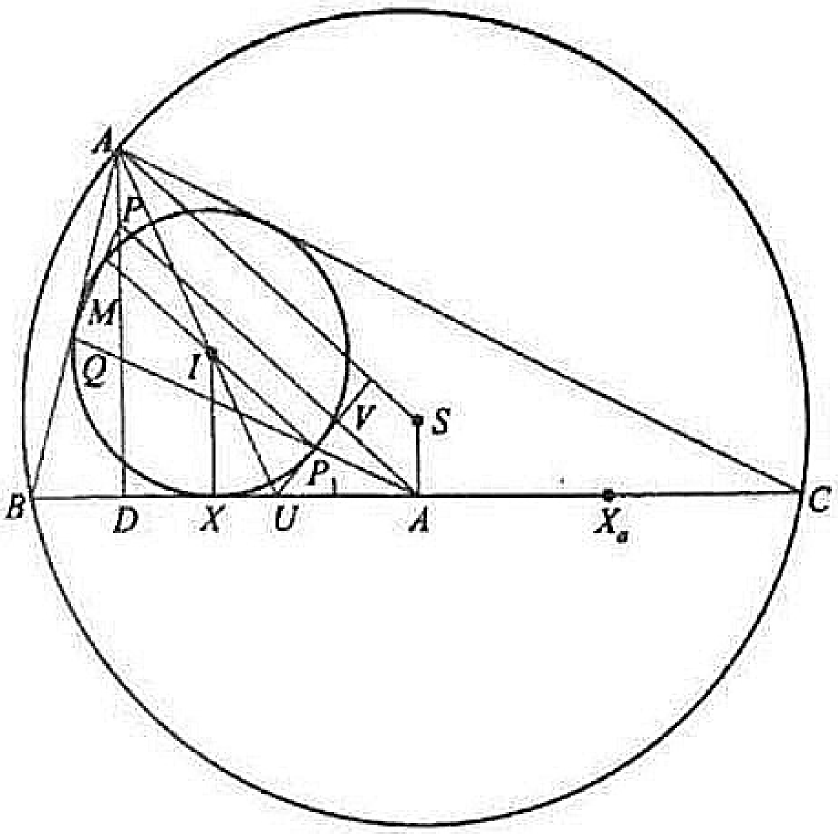


Fig. 4.60

Second Proof. As we have already seen in the first proof, we have $A'U \cdot A'D = A'X^2$. UP_1 is a tangent to the incircle and UP_1, UX are symmetric with respect to AU. Also, AS and AD are symmetric with respect to AU. Therefore $UP_1 \perp AS$ and hence $UP_1 \perp A'P$, a diameter of the nine-point circle. $A'P_1$ meets the incircle at Q. Let $A'P$ and UP_1 meet at V. Then $\Delta A'DP \,|||\, \Delta A'VU$ and we have

$$A'V \cdot A'P = A'D \cdot A'U = A'X^2 = A'P_1 \cdot P_1Q.$$

This means that V, P, P_1 and Q are concyclic.

So, $\angle PQP_1 = \angle P_1VA' = 90°$ and $PQ \perp QP_1A'$. Hence Q lies on the nine-point circle for which $A'P$ is a diameter. The lines P_1IM and $A'VP$ are parallel and hence Q is

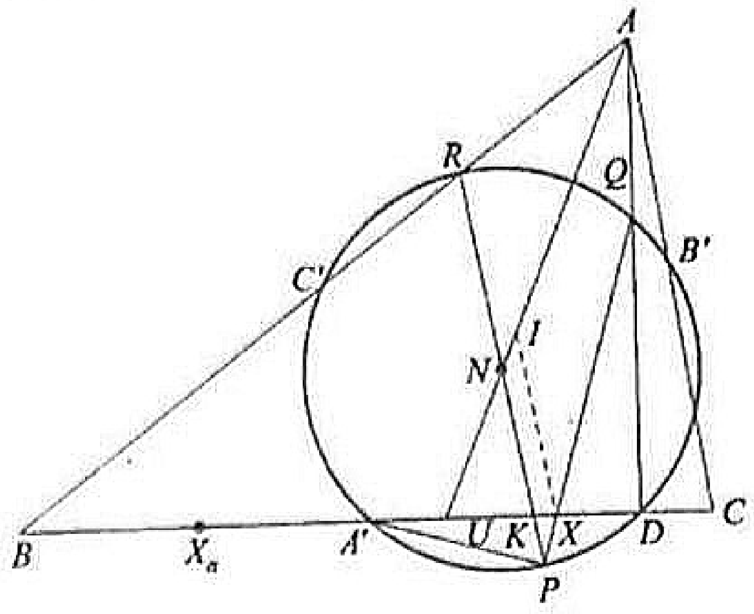


Fig. 4.61

collinear with the midpoints of P_1M and $A'P$. But P_1M and $A'P$ are diameters of the incircle and the nine-point circle; and Q lies on them implies that the two circles touch at Q.

Third Proof. Let PR be the diameter of the nine-point circle perpendicular to BC (Fig. 4.61). Let AI meet BC at U; X be the point of contact of the incircle with BC and let PX meet the nine-point circle again at Q. Let AD be the altitude through A. Now PR is the perpendicular bisector of DA' and hence arc DP = arc PA'. This implies that

$$\angle PA'D = (1/2)\angle A'C'D = (1/2)(\angle A'C'B - \angle DC'B)$$
$$= (1/2)(\angle A - (180° - 2\angle B))$$

(since $DA'B'C'$ is cyclic). Therefore,

$$\angle PA'D = (1/2)(\angle B - \angle C)$$

But $$\angle UIX = \angle UAD \; (\because \quad IX \mid AD)$$
$$= \frac{\angle A}{2} - (90° - \angle B) = \frac{\angle B}{2} - \frac{\angle C}{2} \qquad (2)$$

From (1) and (2), $\angle PA'D = \angle UIX$.

This implies that $\Delta UIX \;|||\; \Delta PA'K$ and hence

$$XI \cdot PK = UX \cdot KA'$$

This gives $$2r \cdot PK = 2KA' \cdot UX = DA' \cdot UX \qquad (3)$$

But DU divides XX_a harmonically (as already seen in the earlier proofs) and A' is the midpoint of XX_a.

$\therefore$ $$A'U \cdot A'D = A'X^2 \text{ or } A'D\,(A'X - UX) = A'X^2.$$

Equivalently $$A'D \cdot UX = A'X\,(A'D - A'X) = A'X \cdot XD \qquad (4)$$

Substituting in (3), we get $2r \cdot PK = A'X \cdot XD = PX \cdot XQ$ (5)

Also $\angle XKR = 90° = \angle RQX$ implies that $XKRQ$ is concyclic.

$$PX \cdot XQ = PK \cdot KR \text{ and so } PX \cdot PQ = PK \cdot PR = PK \cdot 2p \qquad (6)$$

(where p is nine-point radius of ΔABC).

From (5) and (6) we get $\dfrac{r}{p} = \dfrac{XQ}{PQ}$.

This implies that $\Delta IXQ \;|||\; \Delta NPQ$. Therefore, Q, I, N are collinear and

$$\frac{XI}{IQ} = \frac{NP}{NQ} = 1.$$

This says that Q lies on the incircle as well; and this together with the fact that Q, N, I are collinear implies that the nine-point circle and the incircle touch at Q.

Fourth Proof. We have $SI^2 = R^2 - 2Rr$

$$IH^2 = 2r^2 - 2R\,r_p \text{ (where } r_p \text{ is the inradius of the orthic } \Delta DEF)$$
$$SH^2 = R^2 - 4Rr_p \text{ (Exercise)}$$
$$NI^2 = (1/2)(SI^2 + IH^2) - NH^2 \quad (\because \quad IN \text{ is a median of } \Delta IHS)$$
$$= (1/2)(R^2 - 2Rr + 2r^2 - 2Rr_p) - (1/4)SH^2$$
$$= (1/2)(R^2 - 2Rr + 2r^2 - 2Rr_p - (1/2)R^2 + 2Rr_p)$$
$$= \left(\frac{R}{2} - r\right)^2$$

∴ The incircle and the nine-point circle touch each other. □

Note. 1. We have proved that the nine-point circle touches the incircle. In all the four proofs, some simple modifications and similar reasonings imply that the nine-point circle also touches each of the three escribed circles.

2. In the third proof, we actually give a construction of the point of contact of the incircle and the nine-point circle.

Theorem 60. (Pedal line Theorem). The feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumcircle.

Proof. Let A_1, B_1, C_1 be the feet of the perpendiculars from P a point on the circumcircle ABC on the sides BC, CA, AB respectively of ΔABC. P lies on the circles A_1BC_1, A_1B_1C, and AB_1C_1. Also PB, PC and PA are diameters of the circles A_1BC_1, A_1B_1C and AB_1C_1 respectively (Fig. 4.62). Therefore, $\angle C_1PA_1 = 180° - \angle B = \angle APC$. This gives

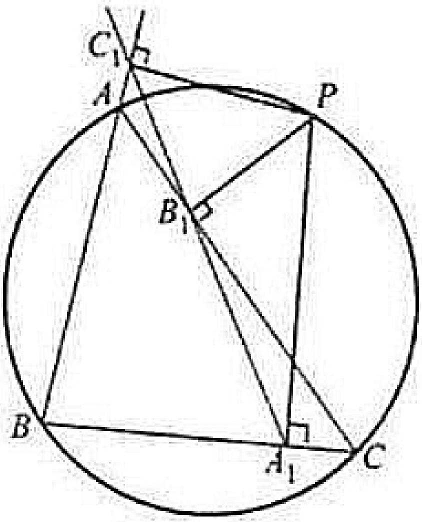


Fig. 4.62

$$\angle C_1PA = \angle C_1PA_1 - \angle APA_1 = \angle APC - \angle APA_1 = \angle A_1PC \qquad (1)$$

Also, quadrilateral A_1CPB_1 is cyclic implies that $\angle A_1PC = \angle A_1B_1C$ (2)

and quadrilateral AB_1PC_1 is cyclic gives $\angle C_1PA = \angle C_1B_1A$ (3)

(1), (2) and (3) imply that $\angle A_1B_1C = \angle C_1B_1A$ and hence A_1, B_1, C_1 are collinear.

Converse. If P is a point such that the feet of the perpendiculars from P onto the sides BC, CA, AB of ΔABC are collinear, then P must lie in the open region which is inside one of the angles of the triangle and P must be outside ΔABC.

Assuming that P is within $\angle B$ of ΔABC (Fig. 4.62), we retrace our steps in the above proof. A_1, B_1, C_1 are collinear implies that $\angle A_1B_1C = \angle C_1B_1A$. Now AB_1PC_1 is cyclic implies that $\angle C_1B_1A = \angle C_1PA$ and the quadrilateral A_1CPB_1 is cyclic gives $\angle A_1B_1C = \angle A_1PC$. Therefore, we get $\angle A_1PC = \angle C_1PA$. Adding $\angle APA_1$ we get $\angle APC = \angle C_1PA_1$. Now BA_1PC_1 is cyclic implies that $\angle C_1PA_1 = 180° - \angle B$. Hence $\angle APC = 180° - \angle B$ or P lies on the circle ABC. □

Definition. If P_1, P_2, P_3 are the feet of the perpendiculars from a point P onto the sides of a triangle ABC, then $\Delta P_1P_2P_3$ is called the *pedal triangle* of P, with respect to ΔABC.

Theorem 60 says that if P lies on the circumcircle then the pedal triangle of P gets degenerated into a straight line also known as the *Simson line* of P.

Theorem 61. The sides of the pedal triangle of a point P with respect to the ΔABC are given by

$$P_2P_3 = a\left(\frac{AP}{2R}\right),\ P_3P_1 = b\left(\frac{BP}{2R}\right) \text{ and } P_1P_2 = c\left(\frac{CP}{2R}\right)$$

where R is the circumradius of ΔABC.

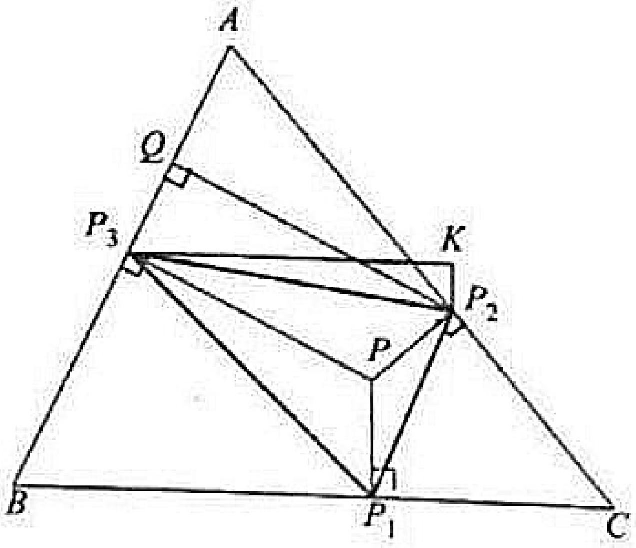


Fig. 4.63

Proof. Let $P_1P_2P_3$ be the pedal triangle of P with respect to ΔABC. Draw $P_2Q \perp AP_3$ (Fig. 4.63) and P_3K be the diameter through P_3 of the circle AP_3PP_2. Then $\angle P_3AP_2 = \angle P_3KP_2$ (angles in the same segment) and further $\angle AQP_2 = \angle KP_2P_3 = 90°$ ($\because$ P_3K is a diameter).

$\therefore$ $$\Delta AQP_2 \mathrel{|||} \Delta KP_2P_3.$$

This gives $$P_2P_3 = \frac{QP_2}{AP_2}\cdot KP_3 = \frac{QP_2}{AP_2}\cdot AP \text{ (as } KP_3 = AP)$$

By the same argument applied to ΔABC we get

$$BC = \left(\frac{CF}{AC}\right)2R$$

Also, from the similar triangles AQP_2 and AFC we get

$$\frac{QP_2}{AP_2} = \frac{FC}{AC} \text{ and thus } \frac{P_2P_3}{BC} = \frac{AP}{2R} \quad \text{or} \quad P_2P_3 = a\left(\frac{AP}{2R}\right)$$

Similarly, we get the other sides of the pedal triangle. ☐

Corollary 1. (Ptolemy's Theorem. See Theorem 22) If $ABCD$ is a cyclic quadrilateral then

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Proof. Take $D = P$ in the 'Pedal line Theorem'. Then the pedal triangle of P degenerates into a straight line $A_1B_1C_1$. By the formula derived in Theorem II.

we have $$B_1C_1 = a\left(\frac{AP}{2R}\right),\ C_1A_1 = b\left(\frac{BP}{2R}\right),\ A_1B_1 = c\left(\frac{CP}{2R}\right).$$

Now $$A_1B_1 + B_1C_1 = A_1C_1 \text{ and hence } \frac{c\cdot CP + a\cdot AP}{2R} = b\left(\frac{BP}{2R}\right)$$

or $a\,.\,AP + c\,.\,CP = b\,.\,BP$. In other words $BC\,.\,AD + AB\,.\,CD = AC - BD$ which is Ptolemy's theorem. ☐

Corollary 2. If ABC is a triangle and P is not on the arc CA of the circumcircle of ΔABC, then $AB\,.\,CP + BC\,.\,AP > AC\,.\,BP$.

Proof. By the converse of 'Pedal line Theorem', if P is not on the circumcircle, its pedal triangle $A_1B_1C_1$ is non-degenerate, but is a genuine triangle. Therefore $A_1B_1 + B_1C_1 > A_1C_1$ which gives

$$AB \cdot CP + BC \cdot AP > AC \cdot BP \text{ (See the proof of Corollary 1).} \quad \square$$

Note. The Simson line of any vertex of a triangle is the altitude through that vertex and the Simson line of the point diametrically opposite to a vertex is the corresponding side. This is immediate from the definition of 'Simson lines'.

Theorem 62. If $A_1B_1C_1$ is the Simson line of a point P on the circumcircle of ΔABC, then the triangles PA_1B_1 and PBA are similar. Further,

1. $PA \,.\, PA_1 = PB \,.\, PB_1 = PC \,.\, PC_1$
2. $\dfrac{PA_1 \,.\, B_1C_1}{a} = \dfrac{PB_1 \,.\, C_1A_1}{b} = \dfrac{PC_1 \,.\, A_1B_1}{c}$

Proof. In Fig. 64, the quadrilateral PCA_1B_1 is cyclic.

$$\therefore \qquad \angle A_1PB_1 = \angle A_1CB_1 = \angle BCA = \angle BPA.$$

Also, $\dfrac{PA_1}{PB_1} = \dfrac{PB}{PA}$ (why?) and hence $\Delta PA_1B_1 \,|||\, \Delta PBA$.

This gives $PA \,.\, PA_1 = PB \,.\, PB_1$. Similarly $PB \,.\, PB_1 = PC \,.\, PC_1$.

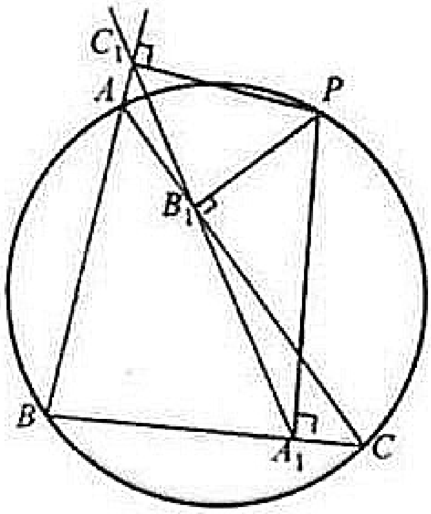


Fig. 4.64

This proves (1).

From Theorem 61, we have $B_1C_1 = a\left(\dfrac{AP}{2R}\right)$, $C_1A_1 = b\left(\dfrac{BP}{2R}\right)$ and $A_1B_1 = c\left(\dfrac{CP}{2R}\right)$.

Therefore, $\dfrac{PA_1 \,.\, B_1C_1}{a} = \dfrac{PA_1 \,.\, AP}{2R}$ and similarly

$$\frac{PB_1 \,.\, C_1A_1}{b} = \frac{PB_1 \,.\, PB}{2R}, \quad \frac{PC_1 \,.\, A_1B_1}{c} = \frac{PC_1 \,.\, PC}{2R}$$

Using (1), we get $\dfrac{PA_1 \,.\, B_1C_1}{a} = \dfrac{PB_1 \,.\, C_1A_1}{b} = \dfrac{PC_1 \,.\, A_1B_1}{c}$ $\square$

Theorem 63. If P is a point on the circumcircle of a triangle ABC, then the Simson line of P bisects the line joining P and the orthocentre of the triangle.

Proof. Let $A_1B_1C_1$ be the Simson line of P and let the altitude BE meet the circumcircle again at T. Suppose that PB_1 meets the circumcircle again at K. Let $HL \parallel BK$ meet PK at L (Fig. 4.65). Now, $BHLK$ is a parallelogram and $PTBK$ is an isosceles trapezium (note that $PTBK$ is a cyclic quadrilateral). This means that $HL = BK$, $PT = BK$; also $HE = ET$ (Theorem 49, Chapter 4) and EA is the perpendicular bisector of HT.

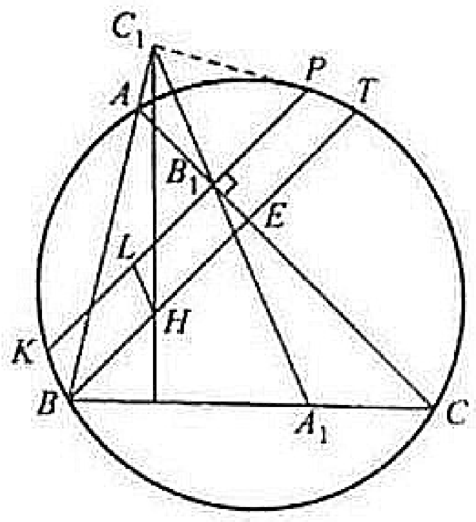


Fig. 4.65

$\therefore$ EA bisects perpendicularly the opposite side LP of the isosceles trapezium $LHTP$. This means that the Simson line $A_1B_1C_1$ passes through the midpoint B_1 of PL and it is parallel to LH. This implies that $A_1B_1C_1$ bisects PH. (See Fig. 4.65). ❑

EXERCISE 4.4

1. Prove that the angle which the external bisector of $\angle A$ of ΔABC makes with BC is half the difference of $\angle B$ and $\angle C$.
2. If the tangent at A to the circumcircle ABC meets BC at X then prove that $XA = XD = XD'$ where D, D' are the points where the internal and external bisectors of $\angle A$ meet BC respectively.
3. Let m_a, m_b, m_c be the medians through A, B, C of ΔABC. If XYZ is the triangle whose sides are of lengths m_a, m_b, m_c, prove that the medians of ΔXYZ are of lengths $3a/4$, $3b/4$, $3c/4$ respectively (where $a = BC$, $b = CA$, $c = AB$).
4. Let S be a given circle and G be a given point. How many triangles are there inscribed in S and having G as their centroid?
5. Show that a parallel to a side of ΔABC through its centroid G divides the area ΔABC in the ratio 4 : 5.
6. Show that in any ΔABC, $3s/2 < m_a + m_b + m_c < 2s$.
7. If X is the harmonic conjugate of the centroid G of ΔABC with respect to A, D (where D is the midpoint of BC) show that $XD = AD$.
8. Find the locus of the centroid of a triangle on a fixed base and inscribed in a fixed circle.
9. If two point P, Q are equidistant from the centroid of a triangle ABC show that $PA^2 + PB^2 + PC^2 = QA^2 + QB^2 + QC^2$; and conversely.
10. ABC is a triangle. Find the locus of P if $PA^2 + PB^2 = PC^2$.
11. S is a given circle and O is a given point. If a variable chord AB subtends a right angle at O, find the locus of the midpoint of AB.
12. If X, Y, Z are the feet of the perpendiculars from the centroid G of ΔABC upon the sides BC, CA, AB prove that area $\Delta XYZ = 4\Delta^2(a^2 + b^2 + c^2)/9a^2b^2c^2$ where Δ = area of ΔABC.

13. Is the corollary to Theorem 29 true for external bisectors?

14. With notations as in Theorem 31, prove that any two points X, Y of $\{I, I_a, I_b, I_c\}$ are the extremities of a diameter of a circle passing through the other two vertices of ΔABC, not in line with X, Y.

15. Show that an external bisector of $\angle A$ of ΔABC is parallel to the line joining the points where the external (internal) bisectors of $\angle B, \angle C$ meet the circumcircle.

16. $DP = r$ is a given line segment. DQ is another line segment in line with PD such that D is in between P and Q: and further $DQ = r_a$. Let A be the harmonic conjugate of D with respect to P, Q. Then prove that AD is equal to the altitude h_a of ΔABC for which the inradius is r and exradius opposite to B is r_a.

17. Deduce from Exercise 16 that $h_a = rr_a/(r_a - r)$

$$h_b = rr_b/(r_b - r)$$

$$h_c = rr_c/(r_c - r).$$

18. In a variable triangle inscribed in a fixed circle and circumscribing a fixed circle prove that the sum of the exradii is constant.

19. Prove (a) $SI^2 + SI_a^2 + SI_b^2 + SI_c^2 = 12R^2$

(b) $II_a^2 + II_b^2 + II_c^2 = 8R(2R - r)$

(c) $I_aI_b^2 + I_bI_c^2 + I_cI_a^2 = 8R(4R + r)$.

20. Let ABC be a triangle with incentre I and circumcentre S. If XY is the diameter of the incircle perpendicular to the diameter PQ of the same circle (I, r) passing through S, show that the perimeter of $\Delta SXY = 2R$.

21. Let ABC be a triangle with X, Y, Z being the points of contact of the incircle with the sides of ΔABC. Show that the circles (A, AY), (B, BZ) and (C, CX) are tangent to each other.

22. ABC is a triangle; the excircle opposite to A touches BC at X_a. Prove that AX_a bisects the perimeter of the ΔABC.

23. XY is a straight line parallel to BC through the incentre I of ABC meeting AB, AC at X, Y respectively. Prove that $XY = XB + YC$.

24. ABC is a triangle; PQ, RS, TV are the tangents to the incircle parallel to BC, CA, AB respectively meeting AB, AC at P, Q; CA, CB at R, S; BC, BA at T, V respectively. Prove that the sum of the perimeters of ΔAPQ, ΔBTV and ΔCRS is equal to the perimeter of ΔABC.

25. With usual notations as in the text, prove that $AZ \,.\, BX \,.\, CY = r\Delta$.

26. ABC is a right triangle with $\angle A = 90°$. If the incircle of ΔABC touches BC at X, prove that, area $\Delta ABC = BX \,.\, XC$.

27. From any point inside a regular polygon perpendiculars are drawn to the sides of the polygon, prove that the sum of their lengths is a constant.

28. Prove that area of ΔABC = area of quadrilateral I_aYAZ.

29. If $\angle A = 60°$ in ΔBAC, prove that S, H, I, I_a, B, C all lie on a circle.

30. The incircle of ΔABC touches the sides BC, CA AB at X_1, Y_1, Z_1; the incircle of $\Delta X_1 Y_1 Z_1$ touches the sides of $\Delta X_1Y_1Z_1$ at X_2, Y_2, Z_2 and likewise points X_n, Y_n, Z_n are defined for $n > 2$. Prove that $\angle Y_n X_n Z_n = 60° + (-2)^{-n} (\angle A - 60°)$.

31. A variable straight line PQ cuts two fixed straight lines AB, CD at P, Q; the bisectors of $\angle AXY$ and $\angle CYX$ meet at X; find the locus of X.

32. In ΔABC, AD is the altitude through A; x, y, z are the inradii of ΔADC, ΔADB and ΔABC. Prove that $x^2 + y^2 = z^2$.

33. $ABCD$ is a quadrilateral circumscribing a circle. Prove that the incircles of ΔABC and ΔADC touch each other.

34. A circle of constant radius passes through a fixed point A and intersects two fixed straight lines AB, AC in B, C. Prove that the locus of the orthocentre of ΔABC is a circle.

35. Two rectangular chords, AB, CD of a circle revolve about a fixed point P. Show that the orthocentres of ΔABC, ΔABD describe the same circle.

36. Prove that the Euler line of ΔABC passes through A if and only if the triangle is either isosceles or right angled.

37. Let PQ be a diameter of the circumcircle of ΔABC whose centroid is G. Prove that PG bisects QH where H is the orthocentre of ΔABC.

38. ABC is a triangle: the tangents at A, B, C to the circumcircle form ΔPQR. Prove that the circumcentre of ΔPQR lies on the Euler line of ΔABC.

39. Prove that the circumcentre of ΔABC lies on the Euler line of ΔXYZ where X, Y, Z are the points of contact of the incircle with the sides of ΔABC.

40. If P, Q, R are the midpoints of AH, BH, CH show that $\Delta PQR \equiv \Delta A'B'C'$ (medial traingle) where H is the orthocentre of ΔABC.

41. Prove that $\Delta DB'C'$ is congruent to ΔPQR (notations as in 40; D is the foot of the altitude from A).

42. With notations as in 40, prove that SP is bisected by AA' where S is the circumcentre.

43. If a variable triangle has a fixed base and a constant circumradius, show that its nine-point circle is tangent to a fixed circle.

44. A variable ΔABC has its vertex A fixed and has a fixed nine-point circle: prove that the locus of its orthocentre is a fixed circle.

45. If the pedal line of P with respect to ΔABC is parallel to AS, prove that $PA \parallel BC$.

46. Let the altitude AD of ΔABC meet the circumcircle of ΔABC at D'. Prove that the pedal line of D' is parallel to the tangent at A to the circle.

47. Prove that the feet of the perpendiculars from the midpoint A' of BC to the sides of the orthic triangle are collinear.

48. P is a point on the circle ABC and H is the orthocentre of ΔABC, prove that the pedal line of P bisects PH.

49. The perpendiculars from P, a point on circle ABC, to BC, CA, AB meet the circle again at X, Y, Z. Prove that $\Delta ABC \equiv \Delta XYZ$.

50. Let X be any point on the circumcircle of ΔABC. Let the altitudes AD, BE, CF meet the circle again at D', E', F'; if XA', XB', XC' cut BC, CA, AB at P, Q, R, prove that P, Q, R are collinear with the orthocentre H of ΔABC and parallel to the pedal line of X.

4.5 CONSTRUCTIONS

Let us begin with some very basic constructions.

Construction 1. From a given a point in a given straight line, construct a straight line, making with the given line an angle equal to a given angle.

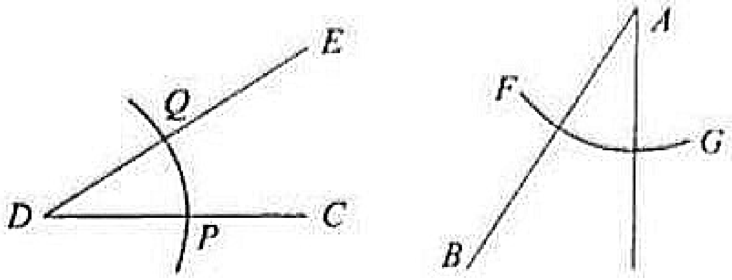


Fig. 4.66

Let A be the given point, AB the given straight line and $\angle CDE$ be the given angle. With D as centre and any convenient radius, draw an arc of the circle meeting DC at P and DE at Q. With A as centre and same radius draw an arc Γ of a circle meeting AB at F. With centre F and radius equal to PQ draw an arc meeting the arc Γ at G. Then AG is the required straight line. (Fig. 4.66).

Proof. By construction, the three sides of ΔAFG are equal in length to the corresponding sides of ΔDPQ. Hence the two triangles are congruent. $\angle FAG = \angle PDQ = \angle CDE$ and AG is the required line. □

Construction 2. Let $\angle BAC$ be a given angle. Bisect the angle $\angle BAC$. With centre A and any radius draw an arc of a circle meeting BA at D and AC at E. With D, E as centres and suitable radius draw two arcs meeting at F. Then AF bisects the given angle $\angle BAC$ (Fig. 4.67).

Proof. By construction, $DF = EF$ and hence the three sides of ΔADF are equal to the corresponding sides of ΔAEF, $\therefore$ $\Delta ADF \equiv \Delta AEF$ and this means that $\angle DAF = \angle EAF$ or AF bisects $\angle BAC$. □

Construction 3. Given a line segment AB, bisect AB.

Choose a suitable radius and draw arcs of circles with centres A and B to meet at X and Y. Let XY meet AB at C. Then C bisects AB. (Fig. 4.68).

Proof. By construction, $AX = BX = AY = BY$. Therefore $\Delta AXY \equiv \Delta BXY$. Now $\angle AXY = \angle BXY$ implies $\Delta AXC = \Delta BXC$ and so $AC = BC$. □

Construction 4. To draw the perpendicular to a given straight line from a given point on it. C is the given point on a given line AB. Cut off equal lengths CD, CE on AB as in Fig. 4.69. With any radius bigger than CD, draw arcs with centres D and E meeting at X. Then $CX \perp AB$.

Proof. By construction $DX = EX$, $DC = CE$ and hence $\Delta DCX = \Delta ECX$.

Therefore $\angle DCX = \angle ECX = 90°$ □

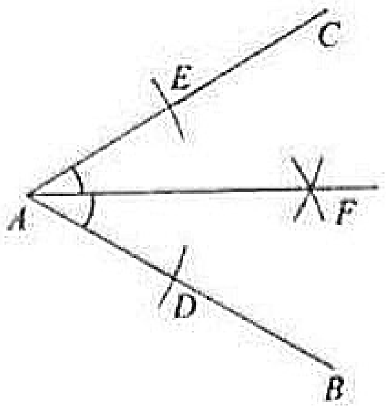


Fig. 4.67

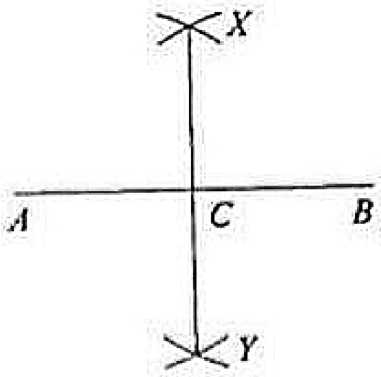


Fig. 4.68

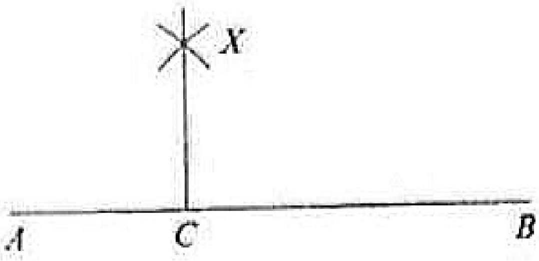


Fig. 4.69

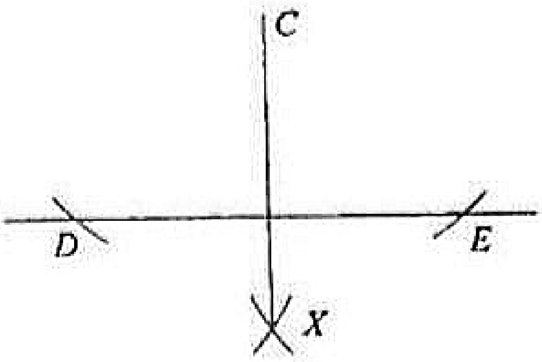


Fig. 4.70

Construction 5. To draw the perpendicular to a given straight line from a given point not on it.

C be the given point and AB the given straight line. With C as centre draw a convenient arc of a circle meeting AB at D, E. With suitable radius, draw arcs of circles with centres D and E meeting at X (Fig. 4.70). Then $CX \perp AB$.

Proof. By construction, $DX = EX$, $CD = CE$. So $\Delta CDX \equiv \Delta CEX$ and $\angle DCX = \angle ECX$. Therefore $\Delta CDG \equiv \Delta CEG$. Hence $\angle CGD = \angle CGE = 90°$ or $CX \perp AB$. ❑

Construction 6. Draw a circle passing through three non-collinear points.

Using the earlier constructions, draw the perpendicular bisectors of AB, AC to meet at S. Then the circle with S as centre and radius SA is the required circle.

Proof. S is on the perpendicular bisector of AB and AC implies that $SA = SB$ and $SA = SC$. Thus $SA = SB = SC$ or the circle with centre S and radius SA passes through A, B and C. ❑

Construction 7. To draw a straight line parallel to a given straight line through a given point.

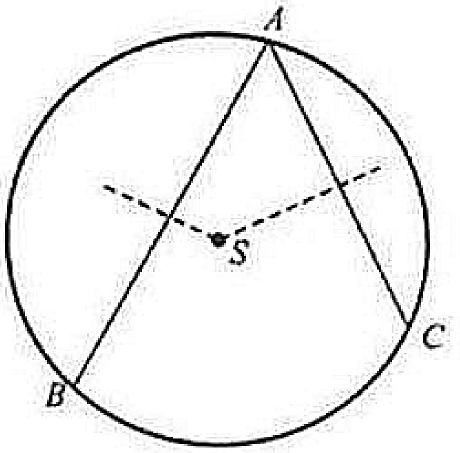


Fig. 4.71

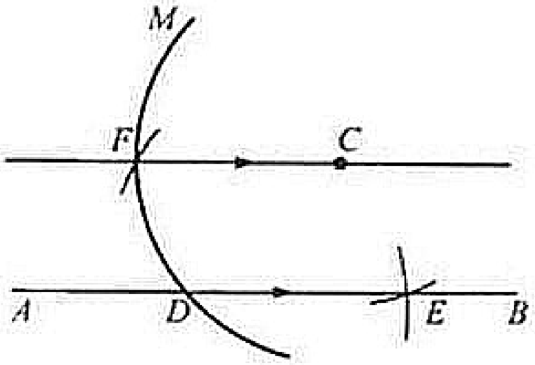


Fig. 4.72

Let AB be the given straight line and C the given point. With C as centre draw an arc Γ meeting AB at D. With the same radius draw an arc with D as centre to meet AB at E. Now with D as centre and radius equal to CE, draw an arc of a circle cutting Γ at F. Join CF and CF is the required straight line parallel to AB.

Proof. By construction $CF = CD = DE$. Therefore $\Delta CDF \equiv \Delta DCE$. This means that $\angle FCD = \angle EDC$. These are alternate angles for the transversal CD cutting AB and CF. Therefore $AB \parallel CF$. ❑

Construction 8. Divide a given segment into a given number of equal parts.

Let AB be the given segment to be divided into n equal parts. Draw through A any convenient ray AC. Along AC cut off equal segments AA'_1, $A'_1A'_2$, $A'_2A'_3$, ... $A'_{n-1}A'_n$. Draw straight lines through A'_1, A'_2 ..., A'_{n-1} parallel to A'_nB meeting AB at A_1, A_2 ..., A_{n-1} respectively. Then A_1, A_2..., A_{n-1} are the required points of division.

Proof. By the equal intercepts theorem on parallel lines, we have ❑

$$AA_1 = A_1A_2 = A_2A_3 = \ldots = A_{n-1}B.$$

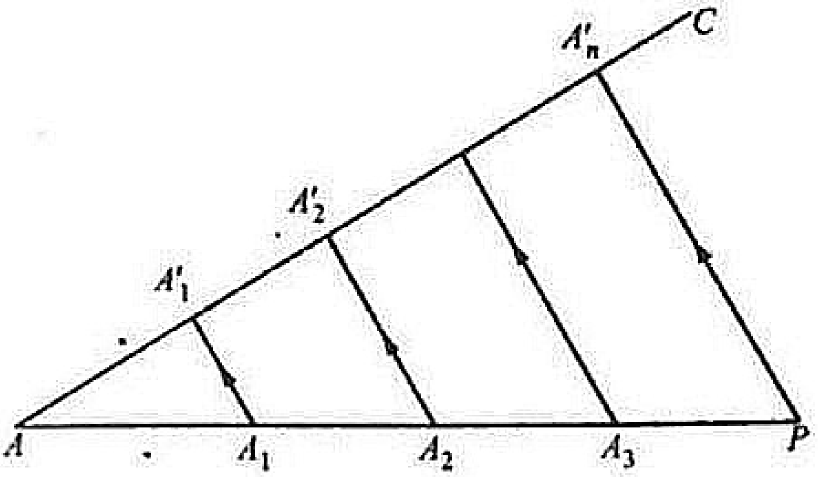


Fig. 4.73

Construction 9. To construct *the fourth proportional* to three given segments. Given three segments of lengths x, y, z we wish to find a segment AG such that $x : y = z : AG$.

Draw any convenient angle $\angle BAC$. On AB cut off AD, AE equal to x, y respectively. On AC cut off $AF = z$. See Fig. 4.74. Draw $EG \parallel DF$ meeting AC at G. Then ΔADF ||| ΔAEG and hence

$$\frac{AD}{AE} = \frac{AF}{AG} \quad \text{or} \quad \frac{x}{y} = \frac{z}{AG}$$

Thus AG is the required fourth proportional to x, y, z. ❑

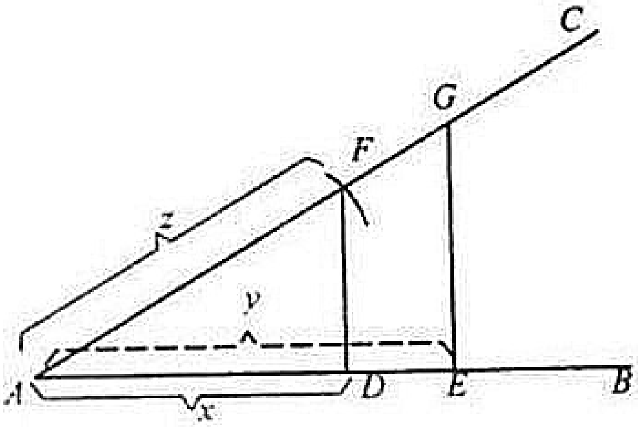


Fig. 4.74

Construction 10. To construct *the mean proportional* to two given segments. Let x and y be the lengths of the two given segments. It is required to find a segment of length z such that $x/z = z/y$ or $z^2 = xy$.

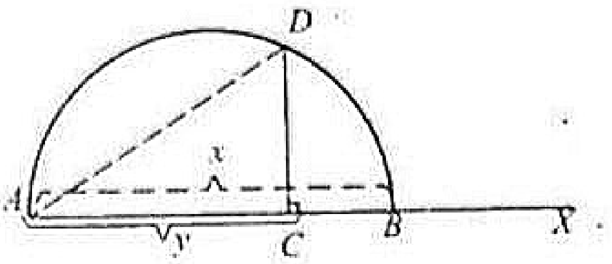


Fig. 4.75

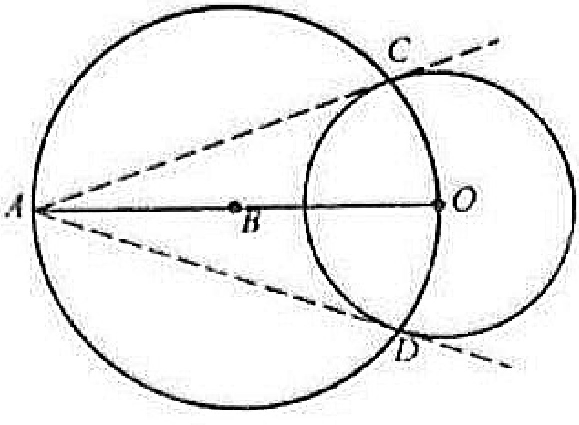


Fig. 4.76

Draw a straight line AX. On AX, cut off AB equal to x and AC equal to y. (Fig. 4.75). In our figure we have taken $y < x$. On AB draw the semicircle and draw $CD \perp AB$

meeting this semicircle at D. (All these constructions can be done using our earlier constructions). $AD = z$ is the required mean proportional.

Proof. By construction, D lies on the semicircle on AB and hence $\angle ADB = 90°$. And $\Delta DAB \,|||\, \Delta CAD$ and hence $AB/AD = AD/AC$ or $x/z = z/y$. □

Construction 11. Draw a tangent to a circle from an external point.

Let A be the given point and O the centre of the given circle. Bisect AO and find the midpoint B of AO. With B as centre and radius $BA = BO$ draw a circle cutting the given circle at C, D. Then AC, AD are the tangents from A to the given circle. (Fig. 4.76).

Proof. By construction ACO is a semicircle and hence $\angle ACO = 90°$. Similarly $\angle ADO = 90°$. Hence AC, AD are the tangents from A. □

Construction 12. To draw a direct common tangent to two given circles.

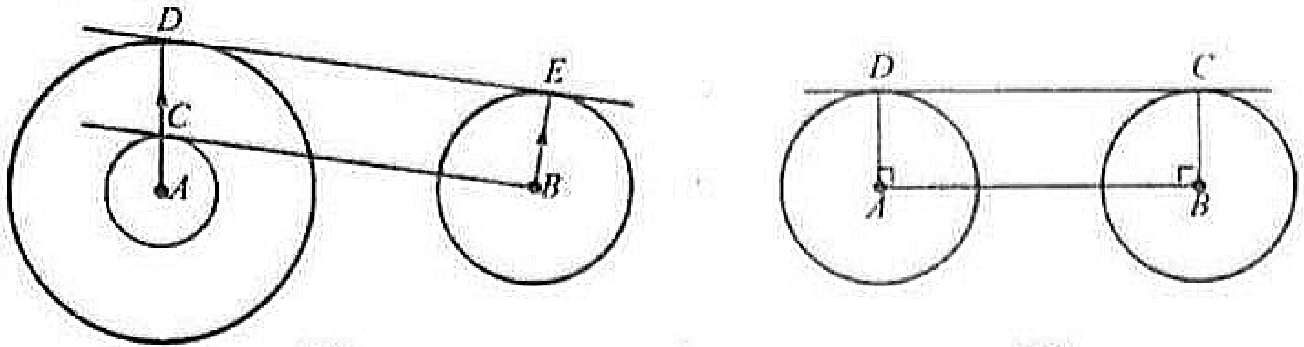
Let A, B be the centres of the two given circles with radii, r_1, r_2 respectively. We first assume that $r_1 > r_2$. Draw the circle with centre A and radius $r_1 - r_2$. Draw a tangent BC to this circle [Fig. 4.77*a*]. Let AC cut the circle (A, r_1) at D. Draw $BE \parallel AD$ cutting the circle (B, r_2) at E. Then DE is a direct common tangent to the two given circles.

Proof. We have $CD = r_1 - (r_1 - r_2) = r_2 = BE$. By construction $CD \parallel BE$ and hence $CBED$ is a parallelogram. BC is a tangent to $(A, r_1 - r_2)$ implies that $BC \perp AC$. Therefore $CBED$ is a rectangle and hence $\angle CDE = \angle DEB = 90°$. Therefore DE is a common tangent to the two circles. When $r_1 = r_2$, construct the rectangle $ABED$ on AB to get DE as a direct common tangent [Fig. 4.77*b*]. □

Construction 13. Draw a transverse common tangent to two given non-intersecting circles.

We imitate what we did in the last construction.

With A as centre draw the circle of radius $r_1 + r_2$. Draw the tangent BC to this bigger circle (Fig. 4.78). Let AC cut the circle (A, r_1) at D. Draw $BE \parallel DA$ meeting the circle (B, r_2) at E. DE is a transverse common tangent.



(*a*) (*b*)

Fig. 4.77

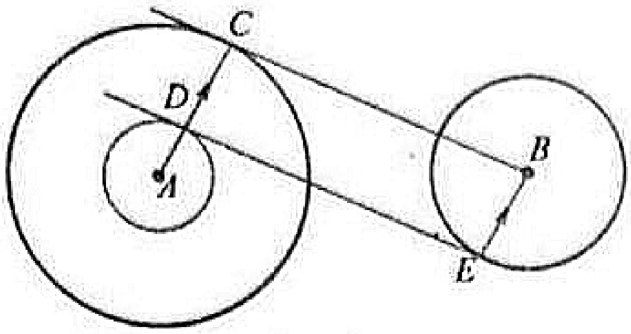


Fig. 4.78

Proof. Exercise (similar to the previous construction). □

Construction 14. Through a given point outside a given circle draw a secant so that the chord determined by it subtends an angle at the centre equal to the acute angle between the secant and the diameter through the given point.

Let O be the centre of the given circle and A be the given point outside the circle. Draw the circle with centre A and radius AO meeting the given circle at Q, Q' (Fig. 4.79). Join AQ. Then AQ is a secant satisfying our requirements.

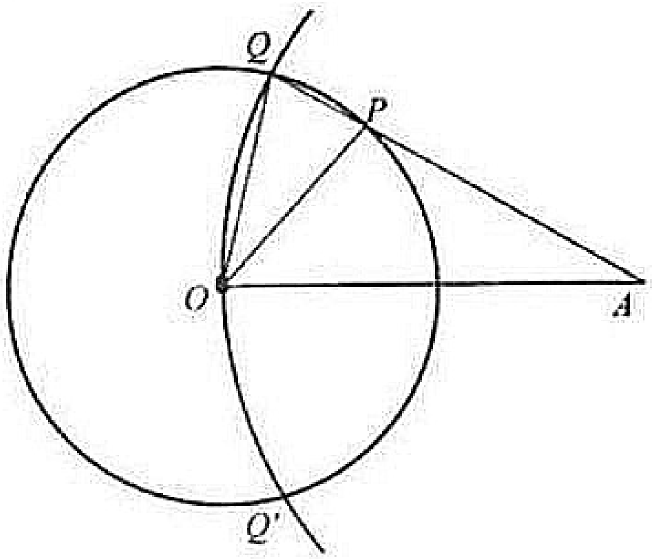


Fig. 4.79

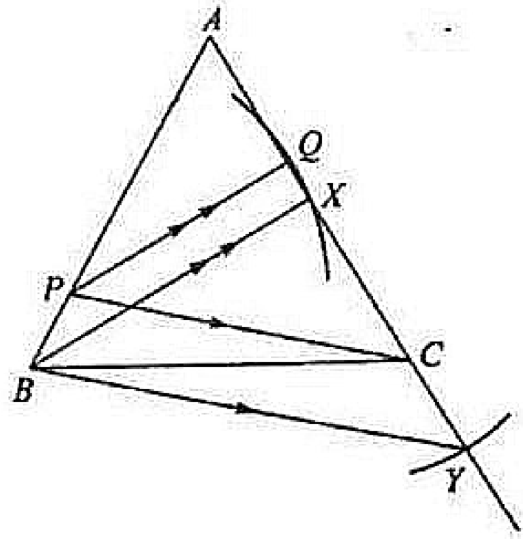


Fig. 4.80

Proof. Let AQ meet the circle again at P. We have $OQ = OP$ and $AQ = AO$ (by construction) and $\angle OQP = \angle OQA$. Therefore the vertical angles in the isosceles triangles QOP and QOA are also equal.

So, $\angle QOP = \angle OAQ$ and AQ is a secant satisfying our requirements. □

Question. Can you replace the adjective 'acute' by 'obtuse' in the above problem? Can you draw a secant AQ so that $\angle QOP$ is the obtuse angle between AQ and AO?

Remark. The above problem has two solutions symmetrical about AO.

Construction 15. Let ABC be a triangle. Find two points P, Q on AB, AC,. produced if necessary such that the line segments AP, PQ, QC are all equal.

Draw the circle with radius BA and centre B to cut AC at X. Draw the circle with centre X and radius BA to cut AC at Y. Draw the parallel CP to YB meeting AB at P (Fig. 4.80). Let the parallel to BX through P meet AC at Q. Then P, Q are the required points.

Proof. By construction $\Delta PQC \,|||\, \Delta BXY$. Again by construction $XB = XY$. So, ΔPQC is also isosceles and $PQ = QC$. The triangles APQ and ABX are similar and $AB = BX$ by construction. Therefore $AP = PQ$. Thus $AP = PQ = QC$.

Remark

1. The point X is uniquely fixed on AC. We have two symmetric positions for Y on AC and we get two solutions.
2. What happens if $\angle A = 90°$ in the above construction?

Construction 16. Let S, S' be two given circles. Find points P, Q on S, S' respectively such that PQ is equal to a given length and PQ is parallel to a given direction.

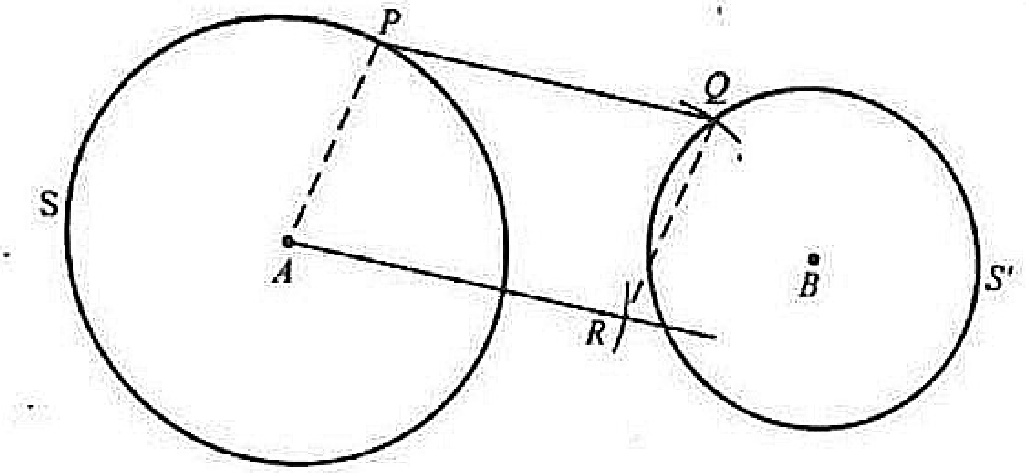


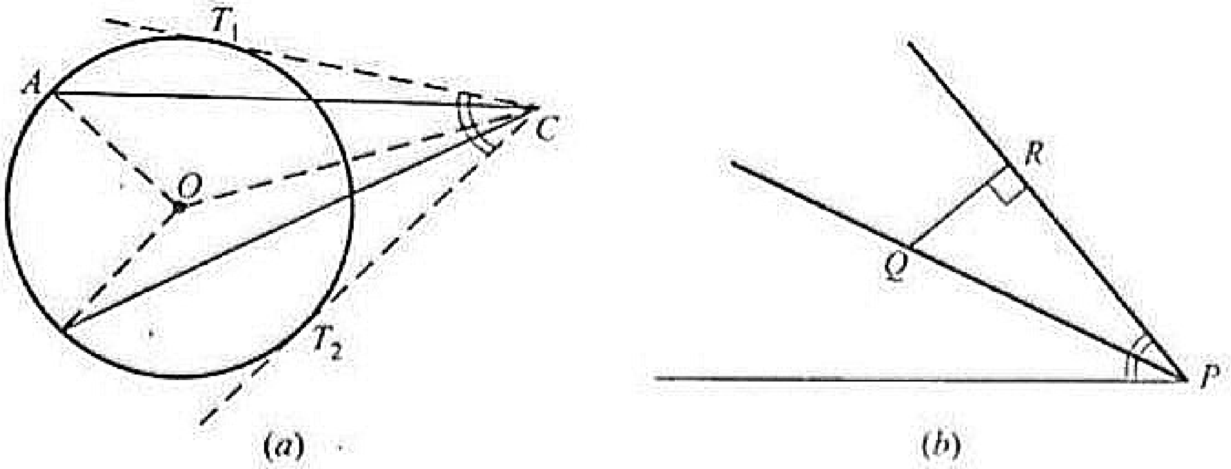
Fig. 4.81

Let A be the centre of circle S with radius r and B be the centre of circle S' with radius r'. Draw the line through A parallel to the given direction l. Cut off AR on this straight line such that AR is equal to the given length m. With R as centre and radius r draw a circle meeting S' at Q. Draw the straight line through Q parallel to AR and lay off $QP = QR$ so as to form a parallelogram in which QR is a side and not a diagonal. The points P, Q are the required points.

Proof. $PQ \parallel l$ and $|PQ| = m$ are true by construction. The only thing remains to be proved is that P is on the circle S. In the parallelogram $ARQP$ we have $AP = RQ$. But RQ = radius of S by construction. Hence, P lies on S. ☐

Note. We have two positions of R on either side of A. The circle with each one of these points as centre and radius equal to that of S(1) may cut S' at two points (2) may touch S' (3) may not cut S' at all. Therefore, the problem has four solutions, three solutions, two, one or no solutions.

Construction 17. Draw a circle passing through two given points and subtending a given angle at a third given point.



(a) (b)

Fig. 4.82

It is required to find a circle through the two given points A, B and subtending a given angle at C; *i.e.*, if CT_1, CT_2 are the tangents to the circle then $\angle T_1CT_2$ is the given angle, $\angle P$. See Fig. 4.82(a). Take any point Q on the internal bisector of $\angle P$ and draw QR perpendicular to one of the sides of the angle [Fig. 4.82(b)]. Divide AC internally and externally in the ratio $QR : QC$ to get the points E, F. Similarly divide BC in the ratio $QR : QC$ internally and externally at G, H. Let the circles on EF and GH as diameters meet at O. Then O is the centre of the required circle.

Proof. By construction $OA/OC = QR/QC = OB/OC$. If CT_1, CT_2 are the tangents to the circle from C then we have

$$\frac{OT_1}{OC} = \frac{OA}{OC} = \frac{QR}{QC} \quad \text{(since } OA = OT_1\text{)}.$$

Now in the right angled triangles OT_1C and QRC we have $OT_1/QR = OC/QC$ and hence the two triangles are similar. Therefore $\angle OCT_1 = \angle QCR = \angle P/2$. Hence $\angle T_1CT_2 = \angle P$ = given angle. ☐

Construction 18. Through two given points on a circle, draw two parallel chords whose sum will have a given length.

Suppose AD, BC are the two required chords passing through the two given points A and B on the circle S with centre O. Then the trapezium $BCDA$ being cyclic, is isosceles. Therefore $CD = BA$. Now CD and BA are chords of the same length. So, CD touches the circle with centre O and radius OE (E is the midpoint of AB); and the point of contact is F the midpoint of CD. Also $2EF = AD + BC = 2s$ (given length). Thus we have the following construction. Find the midpoint E of AB and draw the circle with centre O and radius OE; also draw the circle with centre E and radius s (where $AD + BC = 2s$ = the given length) to cut the circle (O, OE) at F. Let the tangent to the circle (O, OE) at F meet the given circle S at C, D (Fig. 4.83). Then AD, BC are the required chords.

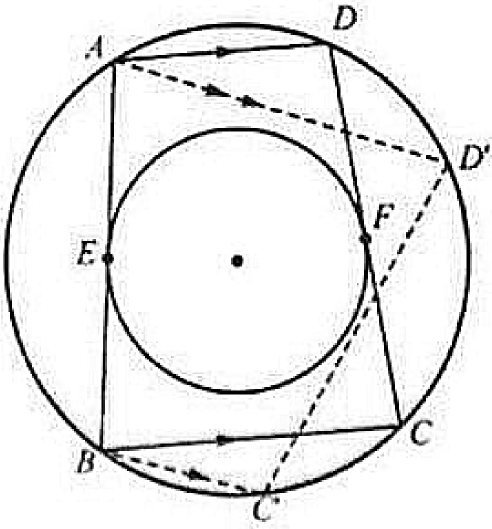


Fig. 4.83

Proof. Since by construction, the chords AB and CD are equidistant from the centre, we have $AB = CD$. Therefore $ABCD$ becomes an isosceles trapezium, and $AD + BC = 2EF = 2s$ = given length. ☐

Note. If $s > 2OE$ then the circle with centre E and radius s will not cut the circle (O, OE). If $s < 2OE$, we get two points of intersection F, F' giving rise to two solutions.

Construction 19. Given a line segment AB and an angle α, construct a segment of a circle such that AB subtends angle α at any point on this circular arc.

Draw AT such that $\angle BAT = \alpha$ the given angle. Let the perpendicular bisector of AB and the perpendicular to AT at A meet at O. Then the arc Γ of the circle with centre O and radius OA is the required arc (Fig. 4.84).

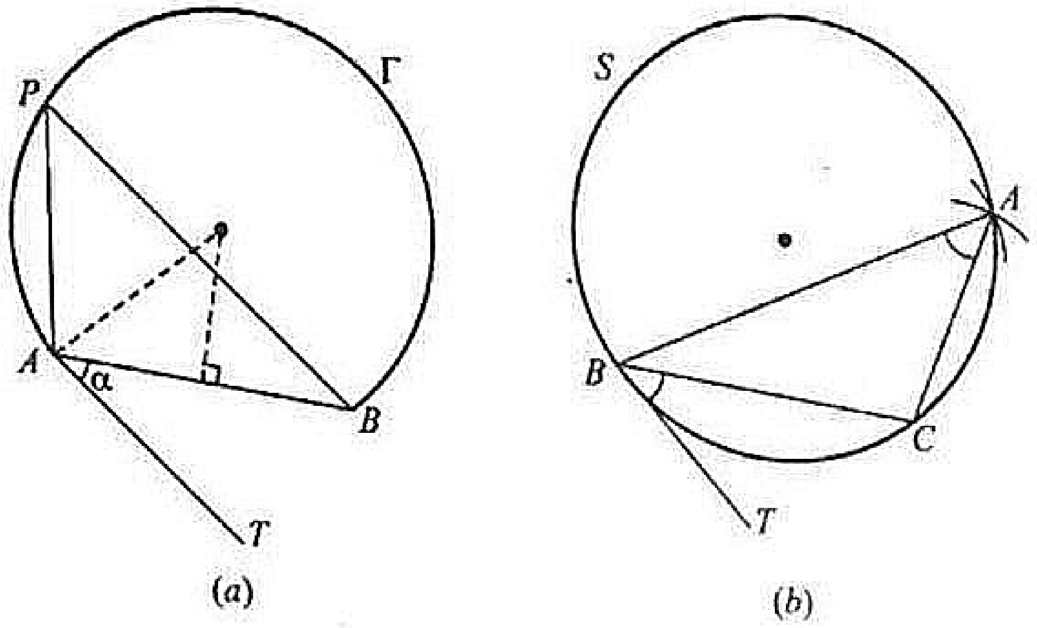


Fig. 4.84

Proof. If P is any point on this arc (Fig. 4.84) we have $\angle APB = \angle BAT = \alpha$ (by construction AT is a tangent to the circle (O, OA) at A). ❑

Construction 20. Given the base BC, the vertical angle A and the side AC, construct ΔABC.

As in construction 19, draw the segment S of the circle on BC such that BC subtends $\angle A$ at every point on this arc. With C as centre and radius AC draw an arc cutting the circular arc S at A. [Fig. 4.84(b)]. Then it is clear that ΔABC satisfies our requirements. ❑

Construction 21. Construct a triangle given the base, the opposite angle and the difference of the other two sides.

Let ABC be the required triangle. Suppose we are given a, A and $b - c$ Take D on AC such that $AD = AB$. This makes $CD = b - c$. Then ΔADB is isosceles and $\angle ADB = \angle ABD = 90° - A/2$. See Fig. 4.85. Therefore the exterior angle $BDC = 90° + A/2$. Now in ΔBDC, we know the base $BC = a$, $CD = b - c$ and $\angle BDC = 90° + A/2$. Therefore we may construct ΔBDC. Third vertex A lies on CD produced and on the perpendicular bisector of BD. ❑

Note. If $a > b - c$, the problem has no solution.

Construction 22. Construct a triangle given the base, the difference of the other two sides and the altitude to one of these sides.

Suppose we are given $BC = a$, $b - c$ and the altitude $CF = hc$.

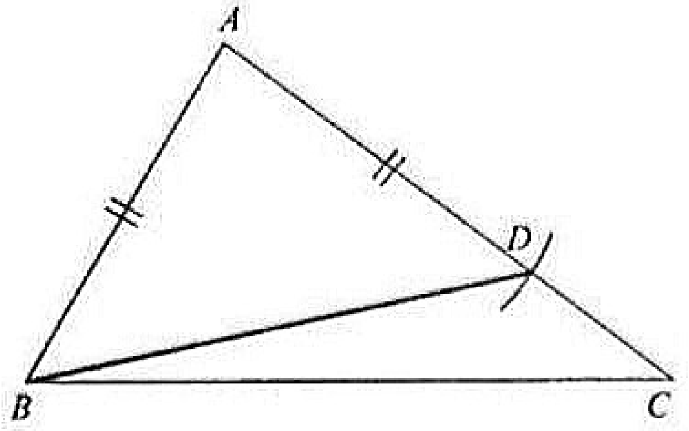


Fig. 4.85

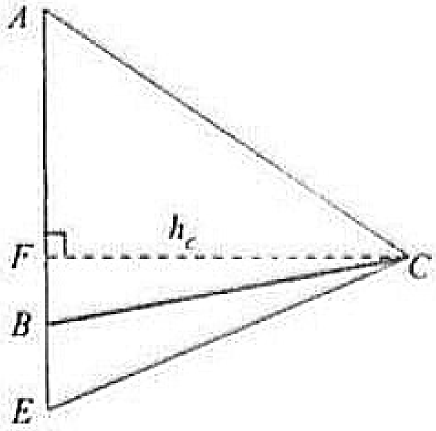


Fig. 4.86

Let ABC be a triangle with the required properties. Let E be a point on AB produced such that $BE = b - c$ or $AE = AC$. In ΔBEC, we know BE, BC and the altitude through C (Fig. 4.86). Therefore the ΔBEC may be constructed. (Note that C must be at a distance h_c from BE and hence lies on the straight line parallel to BE at a distance h_c, from BE). Once ΔBEC is known, we can fix the third vertex A on EB.

Construction 23. Construct ΔABC given a, A and $b + c$ (with the usual notations).

Let ABC be a triangle satisfying our requirements. Produce BA to D such that $AD = AC$. Now, ΔACD is isosceles and $\angle ADC = \angle ACD = (1/2)\angle BAC = A/2$. In ΔBCD, we know $BC = a$, $BD = b + c$ and $\angle BDC = A/2$. Therefore we can construct ΔBDC. A lies on the perpendicular bisector of CD and on BD. ❑

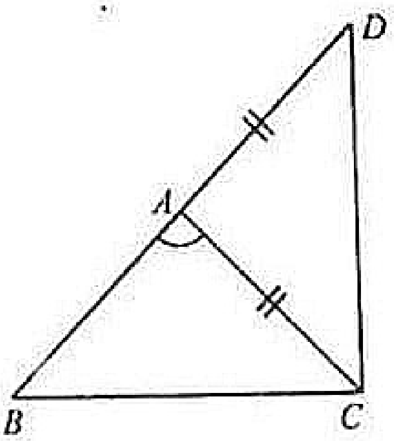


Fig. 4.87

Note. If $a > b + c$, the problem has no solution. Prove that the problem has two, one or no solutions.

Construction 24. Construct a triangle given its perimeter, the angle opposite the base and the altitude to the base.

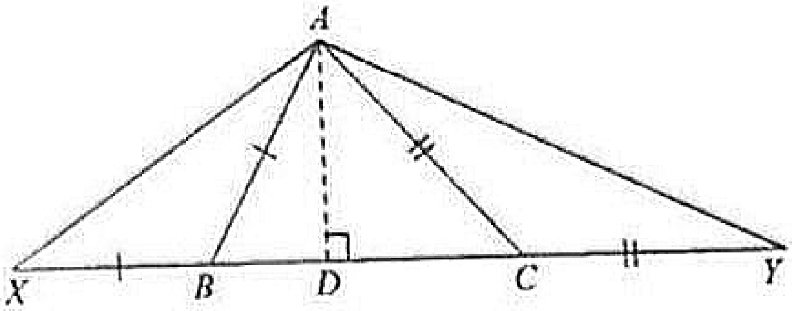


Fig. 4.88

Suppose we are given $2s$, A, h_a (usual notations). Let ABC be the required triangle. Produce BC either way and let $BX = BA$ and $CY = CA$. Then $XY = XB + BC + CY = a + b + c = 2s$.

ΔXAB and ΔYAC are isosceles triangles.

Therefore $\angle AXB = \angle XAB = (1/2)\angle ABC$ and similarly $\angle AYC = (1/2)\angle ACB$.

Therefore $\angle XAY = (1/2)\angle B + \angle A + (1/2)\angle C = 90° + \angle A/2$.

Now ΔXAY may be constructed since we know the base $XY = 2s$, the vertical angle $\angle XAY = 90° + \angle A/2$ and the altitude h_a through A. Also $BA = BX$ and $CA = CY$ imply that B and C lie on the perpendicular bisectors of AX and AY respectively.

Thus ΔABC may be constructed. ❑

Note. This problem may have two solutions symmetric about the perpendicular bisector of XY or just one solution or none at all.

Construction 25. Given a, A, $h_b + h_c$ (with the usual notations) construct the triangle ABC.

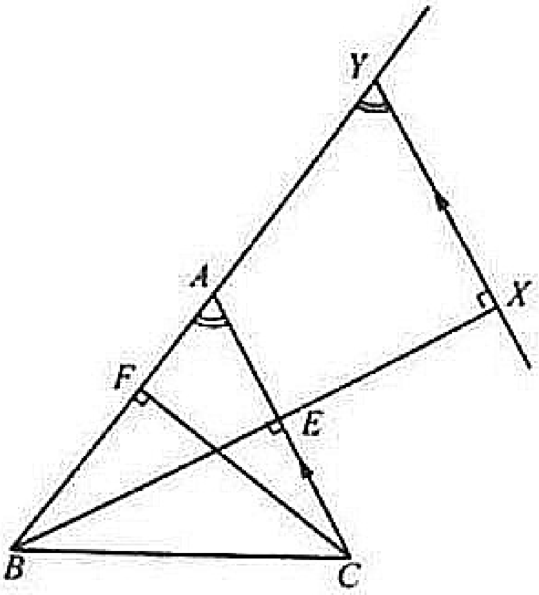


Fig. 4.89

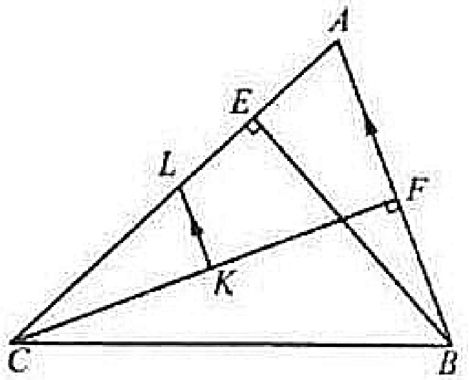


Fig. 4.90

Let ABC be a triangle meeting our requirements. Produce BE to X such that $EX = h_c = CF$. Draw XY parallel to CA meeting BA at Y. Then we have

$$\angle BYX = \angle BAC = \angle A \quad \text{(corresponding angles).}$$

$\angle BXY = \angle BEA = 90°$. Now, in the right angled triangle BXY we know $BX = h_b + h_c$ and the acute angle $\angle BYX$. Therefore ΔBXY may be constructed. We observe that if $AZ \parallel BE$ meeting XY at Z (Fig. 4.89) then $AZXE$ is a rectangle and $AZ = EX$. Therefore $AZ = EX = h_c$, In the right angled triangles ACF and AYZ, we have $AZ = EX$. Therefore $AZ = EX = h_c$. In the right angled triangles ACF and AYZ, we have $AZ = CF$, $\angle AYZ = \angle CAF = \angle A$.

Therefore $\Delta ACF \equiv \Delta AYZ$; $\quad \therefore \quad AY = AC = b$ and hence

$$BY = BA + AY = c + b.$$

In the isosceles triangle AYC, $\angle AYC = \angle ACY = \angle A/2$. Also $\angle A = \angle AYX$ implies that YC bisects $\angle Y$. Therefore C is the intersection of this angular bisector of $\angle AYX$ and the circle (B, a). Once C is fixed, A is determined on BY by the perpendicular bisector of YC. □

Construction 26. Given a, A, $h_c - h_b$, construct ΔABC. Again let ABC be the triangle satisfying our requirements.

Let BE, CF be the altitudes through B and C respectively. Let K be the point on CF such that $KF = BE$. This gives $KC = h_c - h_b$. Let $KL \parallel AB$ meeting AC at L (Fig. 4.90). Now the right triangle CKL may be constructed since we know CK and $\angle KLC = \angle A$. The triangle ABL is isosceles (why?) and therefore $\angle ALB = 90° - A/2$. Also $\angle ALK = 180° - A$ (Fig. 4.90). Therefore LB is the bisector of $\angle ALK$. Hence B lies on the angular-bisector of $\angle ALK$ and the circle (C, a). This determines B. To fix A, we note that A is the intersection of CL and the perpendicular bisector of BL. □

Construction 27. Find points, D, E on AB and AC of ΔABC so that $BD = DE = EC$.

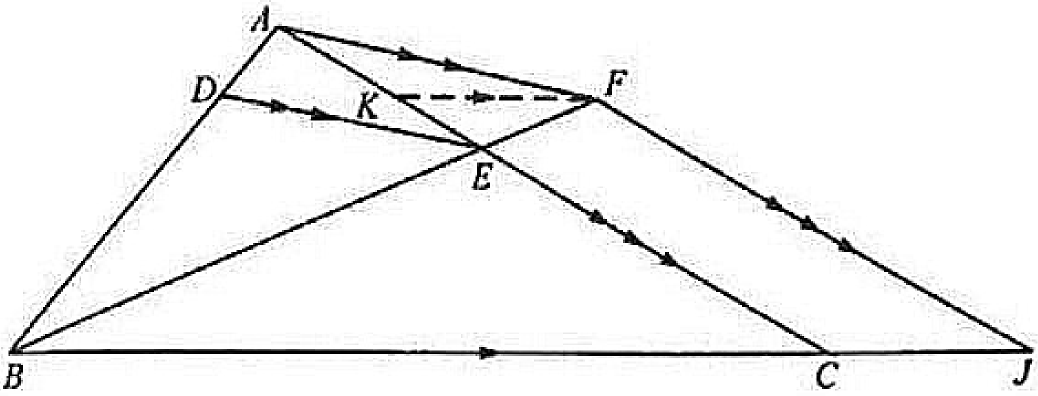


Fig. 4.91

Suppose D, E are the required points (Fig. 4.91). Draw $AF \parallel DE$ meeting BE at F and $FJ \parallel AC$ meeting BC at J. From the similar triangles BDE and BAF we get $DE/AF = BD/BA = BE/BF$. But $BD = DE$ and hence $AF = AB$. Again from the similar triangles BEC and BFJ we get $EC/FJ = BE/BF = BC/BJ$; and therefore $DE/AF = BE/BF = EC/FJ$. But $DE = EC$ and hence $AF = FJ$. Thus $BA = AF = FJ$. The quadrilateral $BAFJ$ is constructed as follows. Take K on CA such that $CK = AB$. Draw the parallel through K to BC meeting the circle with centre A and radius AB at F. Let the parallel to AC through F meet BC at J. This completes the construction of the quadrilateral $BAFJ$. AC and BF determines E. The parallel to AF through E determines D. ❑

Construction 28. Given A, $a + b$, $a + c$ construct ΔABC.

Let ΔABC be the required triangle. Produce AB and AC and take points D, E on AB, AC such that $BD = BC = CE$ (Fig. 4.92). This gives $AD = c + a$, $AE = b + a$. This means that ΔADE can be constructed, as we know AD, AE and the included angle $\angle A$. Now find points B, C on the sides AD, AE of ΔADE such that $DB = BC = CE$ using the previous construction. ❑

Construction 29. Construct ΔABC given the perimeter $2s$ and two angles B and C.

See Fig. 4.93. Draw $XY = 2s$ and $\angle YXK = \angle B$ and $\angle XYL = \angle C$. Bisect the angles KXY and LYX and let the bisectors meet at A. Draw $AB \parallel KX$ and $AC \parallel LY$ meeting XY at B, C respectively. Then ΔABC is the required triangle.

Proof. By construction $\angle AXB = \angle B/2$ and $\angle XBA = 180° - B$: $\angle BAX = \angle B/2$ which means that $XB = AB$ and similarly $AC = CY$. Thus $AB + BC + CA = XY = 2s$. ❑

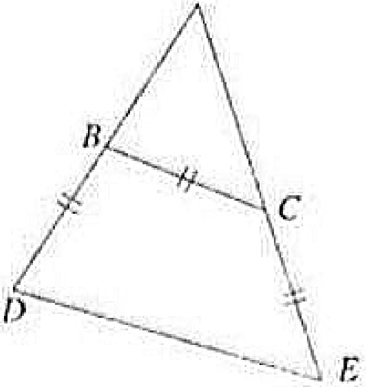


Fig. 4.92

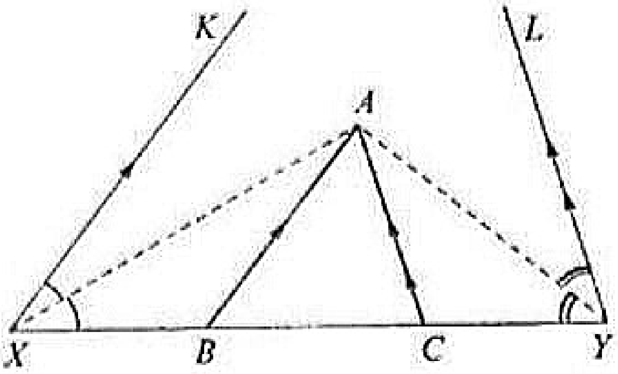


Fig. 4.93

Construction 30. Let $\angle XOY$ be a given angle and A a given point. Draw a straight line through A such that the segment intercepted on it by the sides of the given angle is divided by A in a given ratio $\lambda : \mu$.

Take any point C on OY. Find D on AC such that $AC : AD = \lambda : \mu$. Draw $DP \parallel YO$ meeting OX at P. Then AP is the required line (Fig. 4.94). The triangles AQC and APD are similar. Therefore $AQ/AP = AC/AD = \lambda/\mu$. ❑

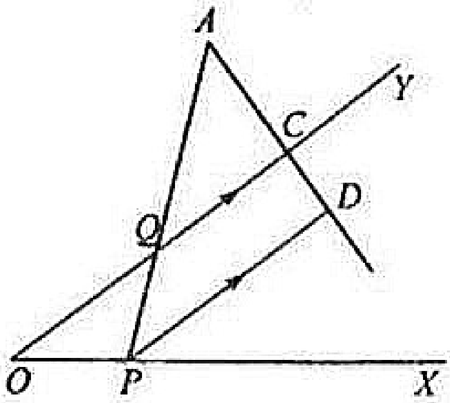


Fig. 4.94

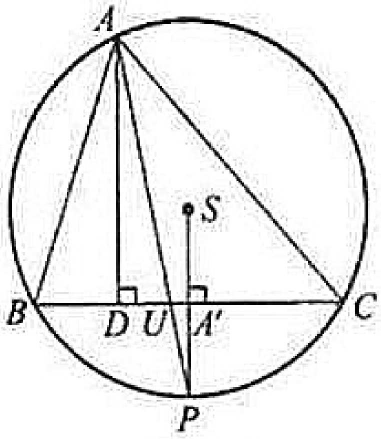


Fig. 4.95

Construction 31. Construct ΔABC given a, A, $A'D/A'U$ where D is the foot of the altitude from A and U is the point where the internal bisector of $\angle A$ meets BC and A' is the midpoint of BC.

Let ABC be a triangle with the required properties and S be the circumcentre of ΔABC. SA' meets the circumcircle at P. Then AU also meets the circumcircle at P (Why?). P bisects the arc BC. (Fig. 4.95) $\Delta ADU \,|||\, \Delta PA'U$ and hence $A'U/UD = A'P/AD$. Now, BC and $\angle A$ are given and therefore the circumcircle of ΔABC can be constructed. So, $A'P$ is determined. Also $A'U/UD$ is known from the hypothesis. Therefore $A'U/UD = A'P/AD$ gives $AD \cdot \Delta ABC$ is readily constructed now. ❑

Construction 32. Construct ΔABC given R, $b + c$, $\angle B - \angle C$.

Let ΔABC be the required triangle. Let AP be parallel to BC meeting the circumcircle again at P. (Fig. 4.96). Now $APCB$ is an isosceles trapezium. $\angle ABP = \angle ABC - \angle PBC = \angle B - \angle BPA = \angle B - \angle BCA = \angle B - \angle C$. The circumradius R and $\angle ABP = \angle B - \angle C$ determine AP (Why?).

Now in ΔABP, we know base AP, the vertical angle $\angle ABP = \angle B - \angle C$ and the sum $AB + BP = AB + AC = b + c$. Hence we may construct ΔABP. The straight line through B parallel to AP meets the circumcircle at the vertex C of ΔABC. ❑

Construction 33. Construct a triangle given the median, the altitude and the bisector from the same vertex (Given h_a, m_a, t_a)

Let AD be the altitude h_a, AU the bisector t_a and AA' the median m_a. We can construct the right angled triangles ADU and ADA' using the given data. (Fig. 4.97).

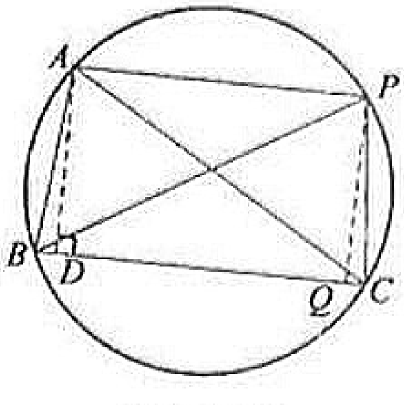


Fig. 4.96

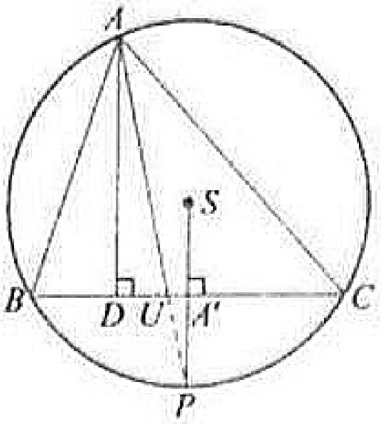


Fig. 4.97

The circumcentre S lies on the perpendicular to DA' through A' such that $\angle UAS = \angle DAU$. (why?). Once S is determined draw the circle with centre S and radius SA to cut DA' at B and C. ABC is the required triangle. ❑

Construction 34. Construct ΔABC given the medians m_a, m_b and m_c.

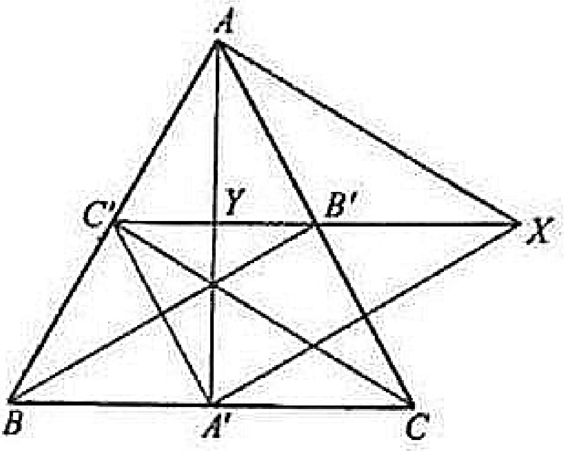


Fig. 4.98

Suppose ABC is the required triangle with A', B', C' as the midpoints of the sides BC, CA, AB respectively. Produce $C'B'$ to X such that $B'X = C'B'$. Then in the quadrilateral $AXCC'$, the diagonals AC and $C'X$ bisect each other. Therefore $AXCC'$ is a parallelogram. This gives $AX = CC' = m_c$. From the parallelogram $BA'XB'$ we get $A'X = BB' = m_b$. This means that the sides AX, XA', $A'A$ of the triangle AXA' are m_c, m_a, m_b respectively, and can be constructed. To go from $\Delta AXA'$ to ΔABC, we observe that XY is a median of $\Delta AXA'$ (See Fig. 4.98). Therefore the point Y can be determined on AA' and further B', C' satisfy $B'Y = YC' = (1/3)XY$. This means that B', C' can be located on XY. From $\Delta AB'C$, the ΔABC is readily constructed.

Construction 35. Given the altitude h_a, a and the bisector t_a, construct ΔABC.

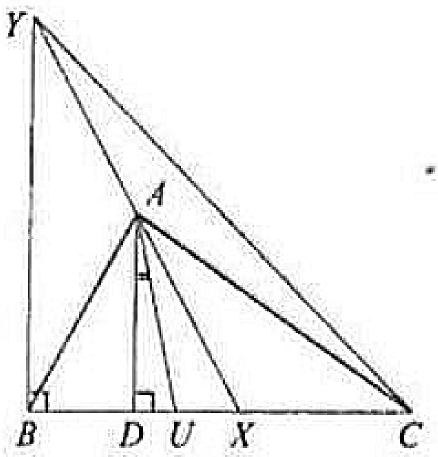


Fig. 4.99

Suppose ΔABC satisfies our requirements. See Fig. 4.99. Let X be a point on DC such that $BD = DX$. Then $\angle XAC = \angle AXD - \angle ACB = \angle B - \angle C$ (since ΔABX is isosceles). As we have seen earlier $\angle DAU = (1/2)(\angle B - \angle C)$ (Theorem 24) which is a known angle from the right angled triangle ΔADU. Let the perpendicular at B to BC meet XA at Y. Then in the right angled triangle BYC we know $BY = 2AD = 2h_a$ and $BC = a$ is given. Now A lies on the perpendicular bisector of BY and also on the segment of the circle on YC subtending $180° - (B - C)$ at the circumference (note that $\angle YAC = 180° - \angle CAX = \angle B - \angle C$). This completes the construction of ΔABC.

Construction 36. Construct ΔABC given a, m_a, $B - C$.

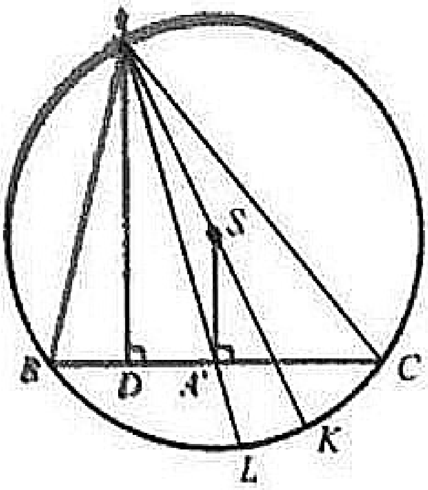


Fig. 4.100

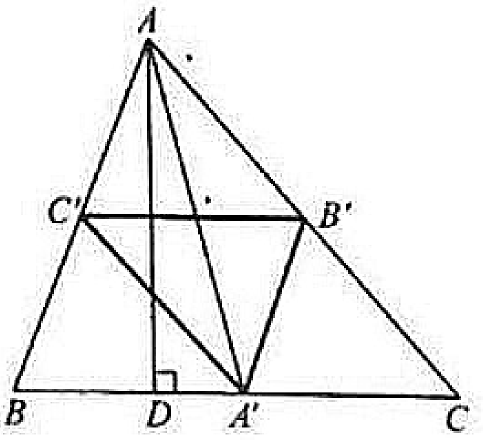


Fig. 4.101

See Fig. 4.100. Suppose ΔABC is the required triangle.

Let AS meet the circumcircle again at K (Fig. 4.100). We have $\angle A'SK = \angle DAK = \angle B - \angle C$ (Theorem 24). Therefore $\angle ASA' = 180° - \angle DAS = 180° - (B - C)$. If AA' meets the circumcircle again at L then $A'A \,.\, A'L = A'B \,.\, A'C$ (secant property). Therefore $A'L = (A'B \,.\, A'C)/(A'A) = (a^2/4m_a)$. $A'L$ is the third proportional to a and $4m_a$ which can be constructed. Now, construct the segment $AA' = m_a$ and on it construct the segment of a circle at which AA' subtends $180° - (B - C)$. S must lie on this arc. Produce AA' to L such that $A'L - (a^2/4m_a)$. S also lies on the perpendicular bisector of AL. Now the circle with centre S and radius SA cuts the perpendicular to SA' at the vertices B and C. ABC is the required triangle.

Construction 37. Given h_a, m_a, and A construct ΔABC.

As h_a and m_a are known, $\Delta ADA'$ may be constructed (Fig. 4.101). The median AA' subtends $\angle AB'A' = 180° - \angle A$ at B' as $A'B' \parallel AB$. Therefore B' lies on the circular arc on AA' at which AA' subtends $180° - \angle A$. Further the line joining the midpoints of AD and AA' should pass through B'. These informations determine B', from which ΔABC is easily constructed. ❑

Construction 38. Construct triangle ΔABC given A, m_b, m_c.

See Fig. 4.102. Now $\angle BAB' = \angle A$ and hence A lies on the segment of a circle on BB' at which BB' subtends $\angle A$. Produce BB' to X such that $B'X = BB'$. Then $B'CX = \angle A$ since

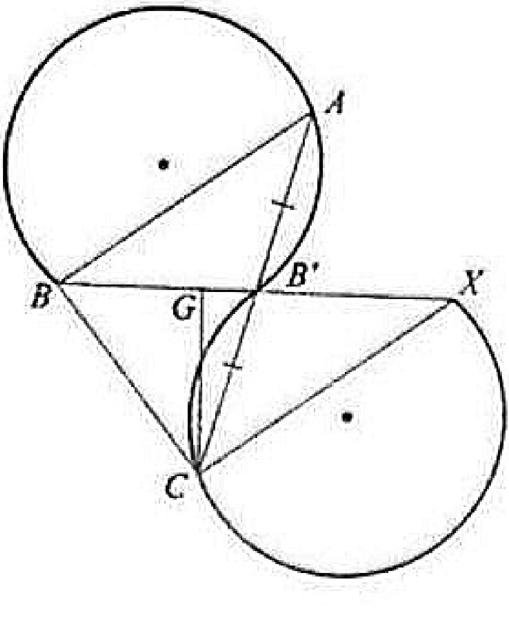


Fig 4.102

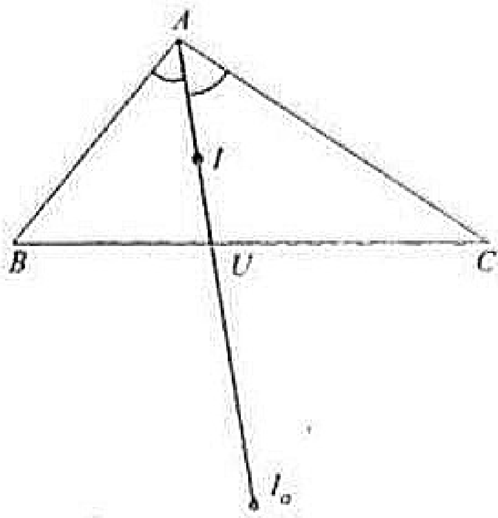


Fig. 4.103

$BCXA$ is a parallelogram (see construction 34). Therefore C lies on the segment of the circle on $B'X$ at which $B'X$ subtends $\angle A$. We find G on BB' from $BG : GB' = 2 : 1$. Having determined G, we note that C also lies on the circle with centre G and radius $(2/3)m_c$. Thus we can determine C. Now ΔABC is readily constructed. □

Construction 39. Given a, t_a, $b + c$, construct ΔABC.

Let $b + c = k$. (See Fig. 4.103). Divide AU internally and externally in the ratio $k : a = (b + c) : a$. Then we get the points I and I_a. The circle on II_a as diameter passes through B and C. Therefore BC is a chord through U of the circle on II_a as diameter, having the given length a, (Why?). This gives the required triangle ABC.

Note. The above problem has two solutions. □

Construction 40. Construct a triangle ABC, given the three points of intersection of the internal bisectors produced with the circumcircle.

We are given in position the three points P, Q and R (Fig. 4.104). The circumcircle of ΔPQR is the same as the circumcircle of ΔABC. As we have already seen P, Q, R are the midpoints of II_a, II_b and II_c respectively. We observe that AI is the common chord of the two circles (Q, QA) and (R, RI). So, AIP is perpendicular to QR and hence A is the second point of intersection of the altitude through P of the known triangle PQR. This determines A and similarly we find the other two vertices B and C. □

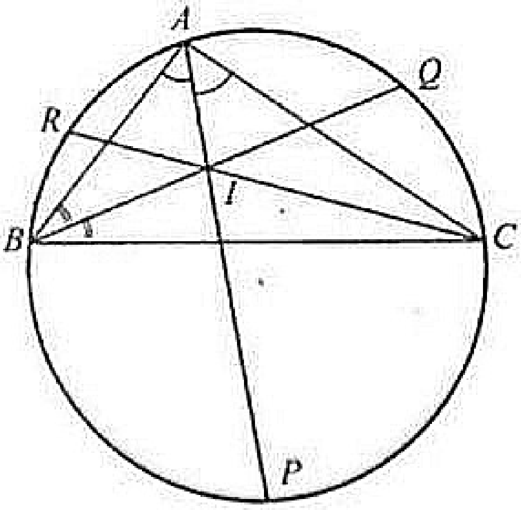


Fig. 4.104

Construction 41. Construct ΔABC given a, R, r.

Draw a circle of radius R and construct a chord BC of this circle having length a. Then $\angle A = \angle BPC$ for any point P on the corresponding segment of the circle. We know that $\angle BIC = 180° - (B/2 + C/2) = 90° + A/2$. This says that I should lie on the arc on BC at which BC subtends $90° + A/2$. Further I is at a distance r from BC. This determines I. Once $\angle BIC$ is known, ΔABC is easily constructed.

Note. The above problem has two solutions or one solution. □

Construction 42. Given A, $c + a$, r construct ΔABC.

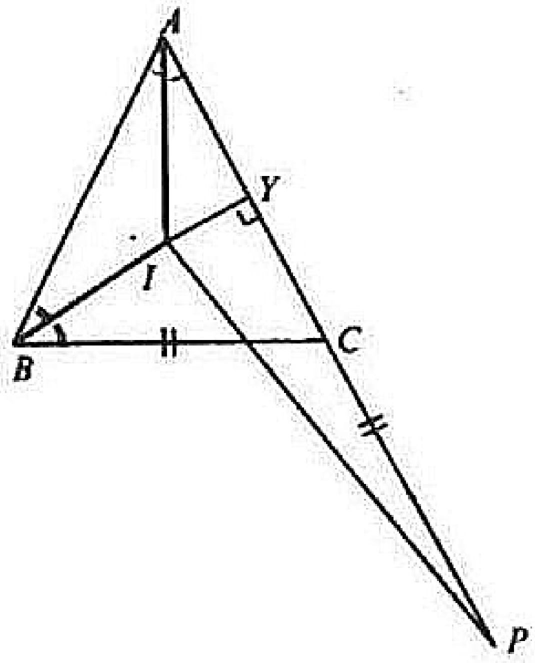


Fig. 4.105

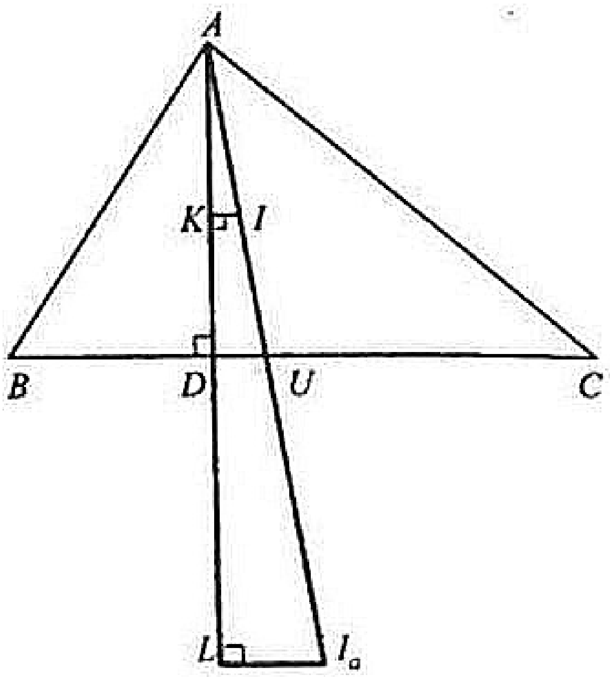


Fig. 4.106

(See Fig. 4.105). Produce AC to P such that $CP = BC$. In ΔAIP we know $AP = c + a$, $\angle IAP = A/2$ and the altitude IY through I. So, ΔAIP may be constructed. $\angle PBC = C/2$ and so $\angle PBI = C/2 + B/2 = 90° - A/2$. Therefore B lies on the arc of a circle on PI at which PI subtends $90° - A/2$. Also B lies on the tangent from A other than AP to the incircle (I, r). This fixes B. Now, C is the intersection of AP with the second tangent from B other than BA to the incircle (I, r). This completes the construction of ΔABC. ❑

Construction 43. Given a, h_a and $b + c$ construct ΔABC.

See Fig. 4.106. We have $ah_a = 2$ area of $\Delta ABC = (a + b + c)r$. This says that r is the fourth proportional to $a + b + c$, a, h_a, and hence can be constructed. I, I_a divide AU harmonically. Therefore K, L divide AD harmonically. Therefore given r, h_a the exradius r_a can be constructed. Now with a and $r_a - r$, using $2a = R - (r_a - r)$ we can construct R. Knowing c, h_a, R, the triangle is easily constructed. ❑

Construction 44. Given a triangle and a point on one of its sides, trisect the triangle by means of straight lines drawn through this point.

Let P be any given point on AC. We can construct t such that $3AP : AC = AB : t$. Let Q be on AB such that $AQ = t$. (See Fig. 4.107). Construct CT such that $3CP : CA = CB : CT$. Then we have $3AP \,.\, AQ - AB \,.\, AC$ and hence

$$\frac{1}{3} = \frac{AP \,.\, AQ}{AB \,.\, AC} = \frac{\Delta AQP}{\Delta ABC}$$

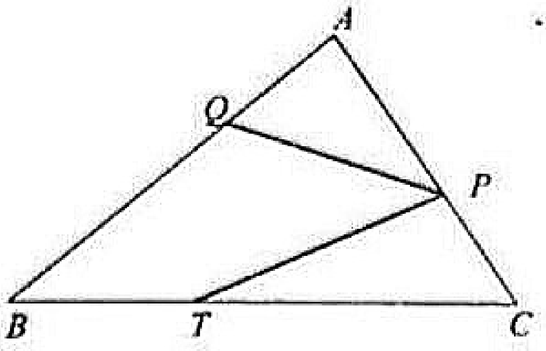


Fig. 4.107

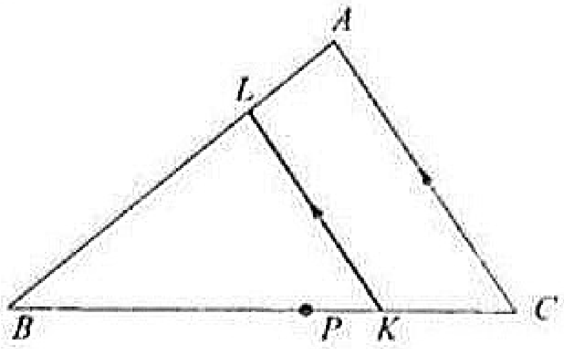


Fig. 4.108

Similarly $$\frac{1}{3} = \frac{\Delta PTC}{\Delta ABC}$$ and we see that *PQ*, *PT* are the required lines.

□

Construction 45. Draw a triangle similar to a given triangle *ABC* and equal in area to *p*/*q* times the area of ΔABC.

Let *P* be a point on *BC* such that $BP/BC = p/q$ (If $p > 1$, *p* lies on *BC* produced). Construct the mean proportional *BQ* between *BP* and *BC*. Let *K* be the point on *BC* such that $BK = BQ$. Draw $KL \parallel CA$ meeting *BA* at *L* (Fig. 4.108).

Then $$\frac{\Delta LBK}{\Delta ABC} = \frac{BK^2}{BC^2} = \frac{BP\,.\,BC}{BP^2} \quad \text{(by construction)}$$

$$= \frac{BP}{BC} = \frac{p}{q}.$$

Therefore ΔLBP is the required triangle. □

Construction 46. Construct a triangle similar to a given triangle *ABC* and equal in area to a second triangle *DEF*.

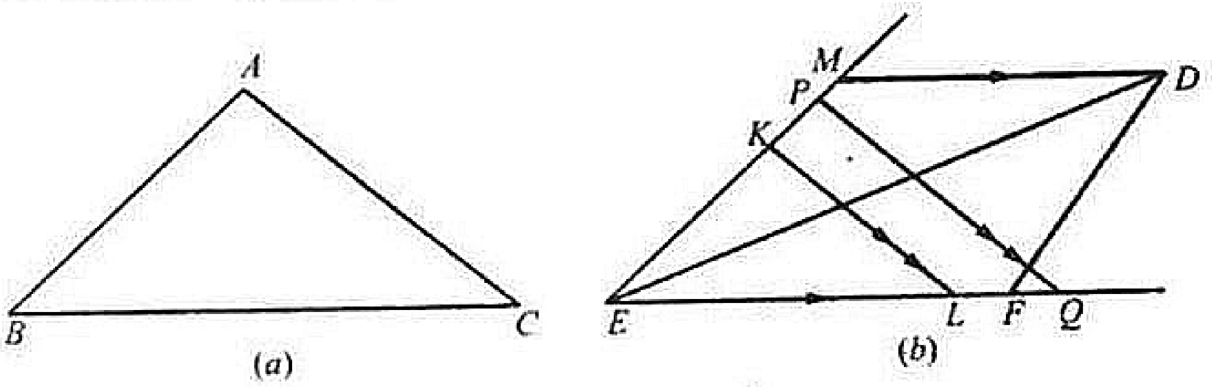


Fig. 4.109

On *EF* construct *KEF* similar to ΔABC. Draw $DM \parallel FE$ meeting *EK* (produced if necessary) at *M* (Fig. 4.109). Let *EP* be the mean proportional between *EK* and *EM*. Draw $PQ \parallel KF$ meeting *EF* at *Q*. Then

$$\frac{\Delta PEQ}{\Delta KEF} = \frac{EP^2}{EK^2} = \frac{EK\,.\,EM}{EK^2} = \frac{EM}{EK} = \frac{\Delta DEF}{\Delta KEF}$$

or ΔPEQ is the required triangle. □

Construction 47. Divide *AB* internally and externally at *X*, *Y* respectively such that $AX^2 = AB - XB$ and $AY^2 = AB \cdot YB$ (medial section).

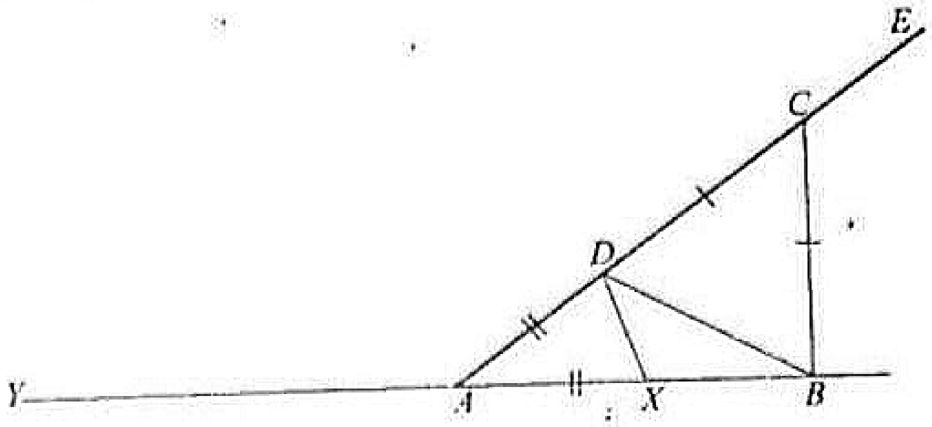


Fig. 4.110

Draw BC perpendicular to AB such that $BC = (1/2)\ AB$. Take D on CA such that $CD = CB$. Take X on AB such that $AX = AD$. We have $AB^2 + BC^2 = AC^2 = (AD + DC)^2$

$$= AD^2 + DC^2 + 2AD . DC$$

$$= AX^2 + BC^2 + 2AX . BC$$

Therefore $\quad AB^2 = AX^2 + 2AX . BC = AX^2 + AX . AB$

Therefore $\quad AX^2 = AB^2 - AXAB = AB(AB - AX) = AB . XB.$

Thus X is the required point. For external division, extend AC to E such that $CE = CB$. Take Y on BA produced (Fig. 4.110) such that $AY = AE$. Then one can easily check that Y is the required point. ❑

Construction 48. On a given base BC construct an isosceles triangle ABC such that $\angle B = \angle C = 2\angle A$.

Divide BC externally at Y in medial section, *i.e.*, $BY^2 = BC . YC$. (Fig. 4.111). Construct an isosceles triangle ABC on BC such that $AB = AC = BY$. We have $AC^2 = BY^2 = BC . YC$. Therefore CA is a tangent to the circle YBA at the point A. Hence $\angle CAB = \angle AYB$ (angle in the alternate segment) $= \angle BAY$ (by construction). Therefore $\angle ABC = \angle B = 2\angle CAB = 2\angle A$. ❑

Note. 1. $\angle B = \angle C = 72°$ and $\angle A = 36°$.

2. Given one of the equal sides AB, describe an isosceles ΔABC with $\angle B = \angle C = 2\angle A$. For this divide AB internally at X such that $BX^2 = BC . XC$. Then AX is the base for the required triangle ABC.

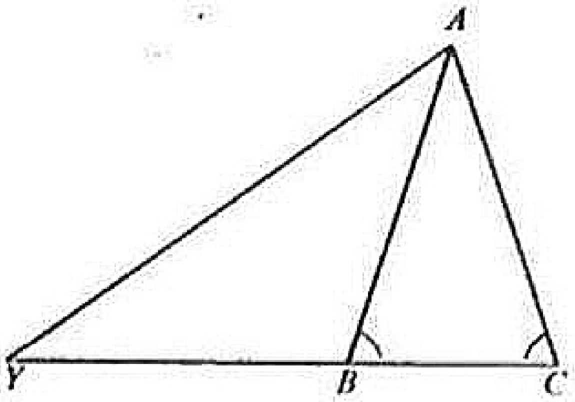


Fig. 4.111

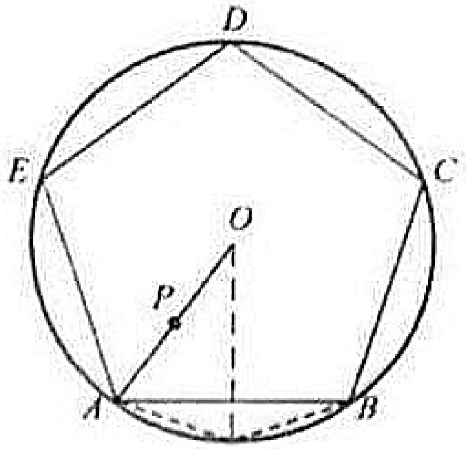


Fig. 4.112

Construction 49. Inscribe a regular pentagon in a given circle.

Let A be any point on the given circle with centre O. Find P on AO such that $AP^2 = AO . PO$. Let AQ be a chord equal in length to AP. Then by Note(2) of the previous construction, $\angle AOQ = 36°$. Take QB again equal to AP. This gives $\angle QOB = 36°$ and so $\angle AOB = 72°$. So, AB is a side of the required regular pentagon. We continue this construction to get C, D, and E (Fig. 4.112). ❑

Note. A, Q, B in Fig. 4.112 are three consecutive vertices of a regular decagon inscribed in the same circle.

Construction 50. Construct ΔABC given $2s$, A and t_a.

See Fig. 4.113. As $AZ_a = s$, from the given data we may construct ΔAI_aZ_a. Take U on AI_a such that $AU = t_a$. Draw the excircle opposite to A, namely (I_a, I_aZ_a). Now, AZ_a

the other tangent from A to this excircle and a tangent from U to the same circle are the sides of $\triangle ABC$. ❑

Note. The above problem has two, one or no solutions.

Construction 51. Construct a triangle ABC given its altitudes h_a, h_b and h_c.

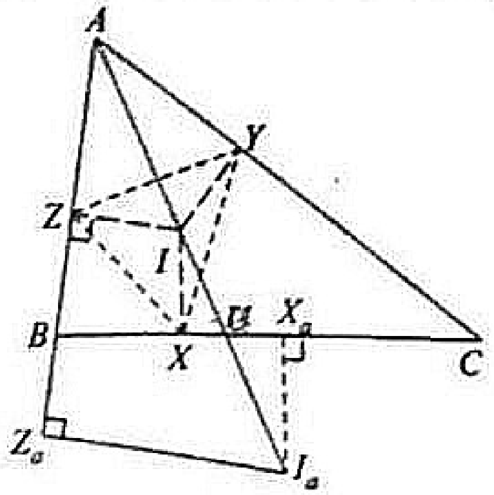


Fig. 4.113

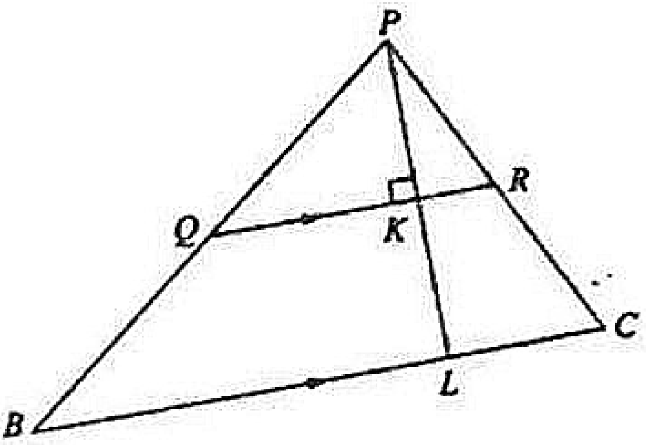


Fig. 4.114

We have $\quad ah_a = bh_b = ch_c = 2\Delta ABC.$

$$\therefore \quad \frac{a}{h_b} = \frac{b}{h_a} = \frac{c}{k} \quad \text{where} \quad k = \frac{h_a h_b}{h_c}$$

$$\therefore \quad \frac{h_c}{h_a} = \frac{h_b}{k} \quad \text{and } k \text{ can be constructed.}$$

$\triangle ABC \,|||\, \triangle PQR$ where $QR = h_b$, $RP = h_a$, $PQ = k$. (Fig. 4.114). Let PK be the altitude through P for $\triangle PQR$. Take L on PK such that $PL = h_a$. Let the parallel through L to meet PQ, PR at B, C respectively. Then if A is taken at P, we see that ABC is the required triangle. ❑

Note. For $\triangle PQR$ to exist we must have

$$h_a + h_b > k > h_a - h_b. \quad \text{In other words}$$

$$h_a + h_b > \frac{h_a h_b}{h_c} > h_a - h_b \quad \text{or equivalently}$$

$$\frac{1}{h_b} + \frac{1}{h_a} > \frac{1}{h_c} > \frac{1}{h_b} - \frac{1}{h_a}.$$

Construction 52. Construct a triangle given the points where the altitudes produced meet the circumcircle.

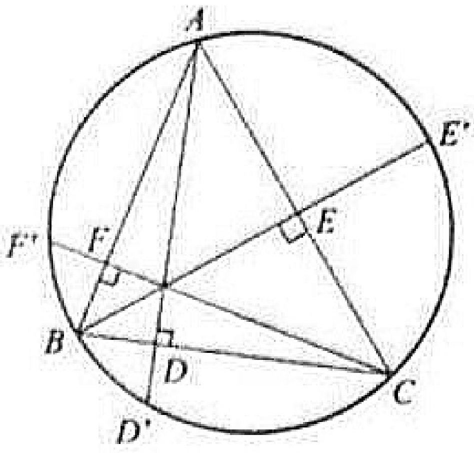


Fig. 4.115

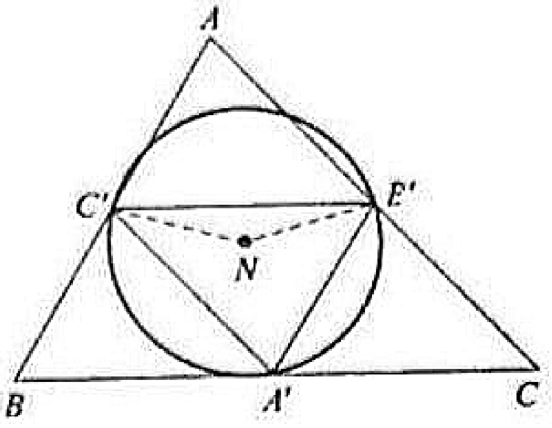


Fig. 4.116

EXERCISE 4.5

1. Draw a circle with centre O. Choose a point A on the circle; cut off chord AB, BC, CD, DE, EF each equal to the radius. Prove that $AF = AB$ and that $ABCDEF$ is a regular hexagon.
2. On a given line segment. BC, construct an equilateral triangle ABC. Bisect the angles B and C by straight lines meeting at S. Draw SD, SE parallel to AB, AC respectively to meet BC at D, E. Prove that D, E trisect BC, *i.e.*, $BD = DE = EC$.
3. Given a line segment x units long, construct one of length x^2 units.
4. Divide a straight line AB in the ratio $\sqrt{2} : \sqrt{3}$.
5. Divide a line segment internally at X and externally at Y, so that $AX^2 : XB^2 = AY^2 : YB^2 = 2 : 5$.
6. S is a given circle with centre O. Through a given point A, draw a straight line to cut S at X, Y such that $XY = BC =$ a given line segment (in length).
7. Draw a circle touching a given circle and a given straight line at a given point P.
8. Draw a circle to touch a given line AB and a given circle at a given point P.
9. A and B are given points; draw a circle S with centre A so that the tangent to S from B is of given length (less than AB).
10. In a given circle, place a chord of given length. How many such chords can be drawn?
11. Inscribe a circle in a given triangle.
12. Draw a circle through two given points A, B to touch a given straight line CD.
13. Draw a circle to touch two given straight lines OA, OB and pass through a given point C.
14. Draw a circle to touch a given circle (centre C and radius r) and also to touch two given straight lines OA, OB.
15. Draw a circle through two given points A, B to touch a given circle.
16. Draw a circle, with its centre on a given straight line, to pass through a given point and touch a given circle.
17. Draw a circle to pass through a given point A, to touch a given straight line BC and a given circle.
18. Draw a circle through a given point A to touch two given circles C_1, C_2.
19. Draw a circle to touch three given circles.
20. Let A, B be two given points on a circle S, l is a given straight line and C is a given point on it; find a point M on the circle such that if AM, BM meet l at P, Q then $CP/CQ = \lambda$, a given ratio.
21. Construct a triangle so that its sides pass through three noncollinear points and be divided by these points internally in given ratios.
22. Draw a circle tangent to two concentric circles and passing through a given point.
23. Inscribe a square in a given quadrilateral.
24. In a given triangle inscribe a parallelogram having a given angle and having its adjacent sides in a given ratio.
25. Construct ΔABC, given R, a, $(b + c)b$.
26. Construct ΔABC, given in position (*i*) I_a, I_b, I_c (*ii*) I, I_b, I_c (*iii*) S, I, I_a.
27. Construct a quadrilateral given the four sides and the sum of two opposite angles.

4.6 SOME GEOMETRIC GEMS

In this section we try to give an assorted collection of beautiful problems in the geometry of straight lines, triangles and circles in a plane.

Problem 1. Let ABC be an acute angled triangle and D any point on BC. Find points E, F on the sides CA, AB of ΔABC such that the perimeter of ΔDEF is of minimum length.

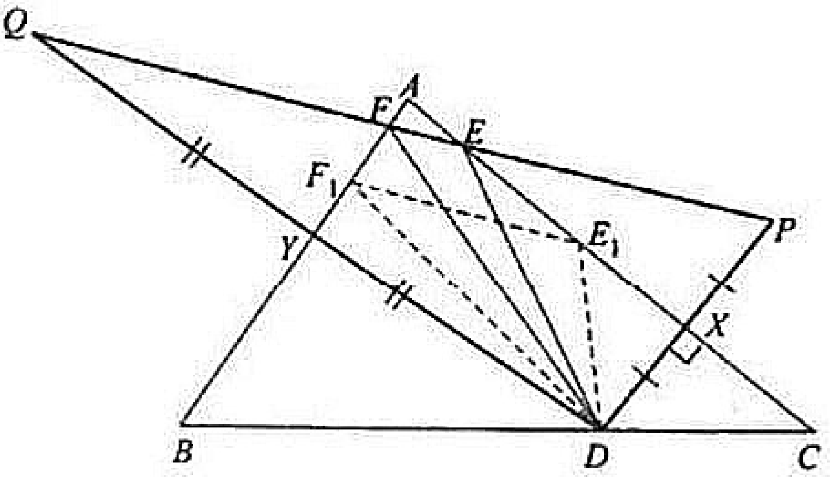


Fig. 4.121

Draw DX, DY perpendicular to CA, AB (See Fig. 4.121) and produce them to P and Q such that $DX = XP$ and $DY = YQ$. Let PQ cut the sides CA, AB at E, F-respectively. We claim that DEF is the required triangle. For, by construction $DE = EP$ and $DF = QF$. Hence the perimeter of $\Delta DEF = DE + EF + FD = PE + EF + FQ = PQ$. If E_1 and F_1 are any two points on CA, AB respectively then $DE_1 = E_1P$ and $DF_1 = F_1Q$ and hence perimeter of $\Delta DE_1F_1 = PE_1 + E_1F_1 + F_1Q \geq PQ$ (note that the distance between any two points in a plane is the minimum along the straight line joining the two points!). Hence E, F are the required points. ☐

Problem 2. Given a circle (S, R) with centre S and radius R show that an infinite number of triangles may be inscribed in it, having their centroid at a given point within the circle.

Let A be any point on the given circle and G be the given point inside the circle. Produce AG to A' such that $AG : GA' = 2 : 1$. See Fig. 4.122. Let the perpendicular to SA' cut the circle at B and C. Then ABC is a triangle inscribed in the given circle having G as its centroid. Now, if A' lies outside the circle, then we do not have any solution. We note that as A moves, the locus of A' is a circle. In fact the locus of A' is the circle $(N, R/2)$ where N is the point on SG dividing it externally in the ratio $3 : 1$. (Recall that

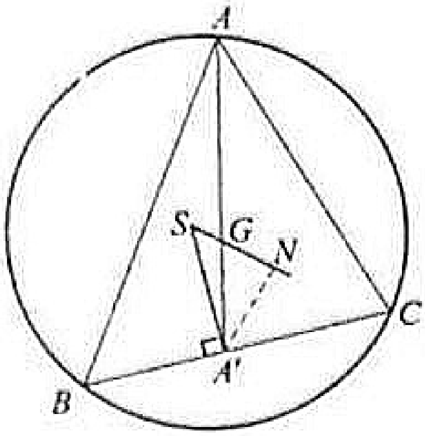


Fig. 4.122

$$\frac{RC_2}{RP} = \frac{OP}{OB} \tag{2}$$

Using (2) in $$OA + OB + OC \geq OA\left(\frac{A_1A_2}{RQ}\right) + OB\left(\frac{B_1B_2}{PR}\right) + OC\left(\frac{C_1C_2}{PQ}\right)$$

we get, $$OA + OB + OC \geq OA\left(\frac{A_1P + PA_2}{RQ}\right) + OB\left(\frac{B_1Q + QB_2}{PR}\right)$$

$$+ OC\left(\frac{C_1R + RC_2}{PQ}\right) + \frac{OC}{PQ}\left(\frac{QR\,.\,OQ}{OA} + \frac{RP\,.\,OP}{OB}\right)$$

$$= \frac{OA}{RQ}\left(\frac{PR\,.\,OR}{OB} + \frac{PQ\,.\,OQ}{OC}\right) + \frac{OB}{PR}\left(\frac{PQ\,.\,OP}{OC} + \frac{QR\,.\,OR}{OA}\right)$$

$$= OP\left(\frac{OB}{PR}\cdot\frac{PQ}{OC} + \frac{OC\,.\,PR}{PQ\,.\,OB}\right) + \text{similar terms}$$

$$\geq 2(OP + OQ + OR) \quad \{\text{since if } x > 0,\ x + 1/x \geq 2\}$$ ❑

Problem 7. Given an acute angled triangle ABC, find points D, E, F, on the sides BC, CA, AB of ΔABC such that the perimeter of ΔDEF is a minimum.

For a given point D on BC, this problem is solved in problem 1 of this section. By construction (see Fig. 4.121) we have $AP = AQ = AD$. Also, $\angle DAX = \angle XAP$ and $\angle DAY = \angle YAQ$ and $\angle DAX + \angle DAY = \angle A$.

Therefore, $\angle QAP = 2\angle A$. This means that for any choice of die point D on BC, $\angle QAP = 2\angle A$ = constant in the isosceles triangle AQP. Further QP = permeter of ΔDEF and therefore the perimeter of ΔDEF is a minimum when the side QP of the isosceles triangle AQP with constant vertical angle is a minimum. This happens when the equal sides have minimum length. But $AQ = AP = AD$ and AD is a minimum when D is the foot of the perpendicular from A on BC. Thus the required triangle is the (pedal) orthic triangle DEF. ❑

Problem 8. Let F be any point on the side AB of ΔABC. D be the intersection of BC with the straight line $AD \parallel FC$ through A. Similarly, let E be the intersection of CA with the line $BE \parallel FC$ through B. Prove that $\dfrac{1}{AD} + \dfrac{1}{BE} = \dfrac{1}{CF}$.

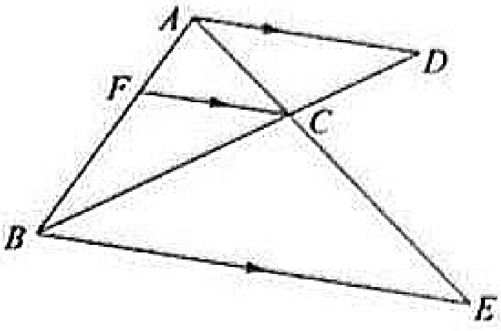


Fig. 4.127

$\Delta CAF \;|||\; \Delta EAB$ since $CF \parallel EB$ and $\Delta CBF \;|||\; \Delta DBA$. Therefore, we get

$$\frac{CF}{EB} = \frac{AF}{AB} \quad \text{and} \quad \frac{CF}{DA} = \frac{BF}{BA}$$

Adding we get $CF\left(\frac{1}{BE} + \frac{1}{AD}\right) = \frac{AF + FB}{AB} = 1$

$$\therefore \qquad \frac{1}{AD} + \frac{1}{BE} = \frac{1}{CF}$$

❑

Problem 9. If a polygon is inscribed in a circle and a second polygon is cricumscribed by drawing tangents to the circle at the vertices of the first polygon, then the product of the perpendiculars on the sides of the first, from any point on the circle, equals the product of the perpendiculars from the same point to the sides of the second.

This problem will illustrate how a degenerate special case, on repeated applications, may prove the general case. We consider the special case now. See Fig. 4.128. Suppose AB is a chord of a circle and the tangents at A and B meet at C. P is any point on the circle and PL, PM, PN are the perpendicular from P on the sides AB, BC, CA of ΔABC. Then we claim that $PM \,.\, PN = PL^2$.

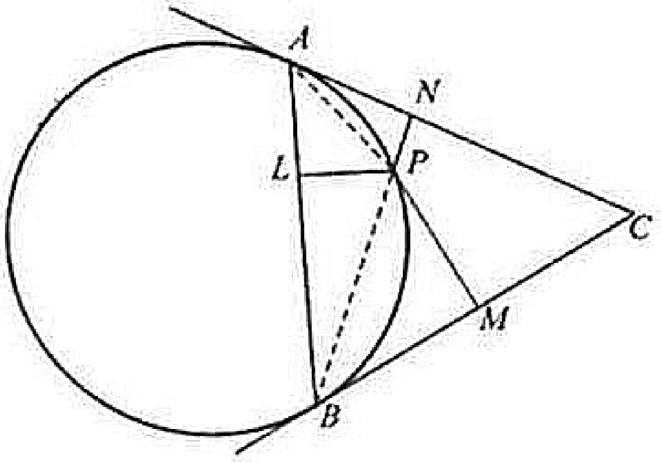


Fig. 4.128

Proof of claim: We have, $\Delta PAN \;|||\; \Delta PBL$

$$\therefore \qquad \frac{PN}{PL} = \frac{PA}{PB}$$

Again, $\Delta PBM \;|||\; \Delta PAL$ and

$$\frac{PA}{PB} = \frac{PL}{PM}. \quad \text{Thus, we have}$$

$$\frac{PA}{PB} = \frac{PL}{PM} = \frac{PN}{PL} \quad \text{or} \quad PM \cdot PN = PL^2$$

This ancillary result that we have just now proved is the problem 9, for a two sided polygon!

Now, consider an n sided polygon $A_1A_2A_3 \ldots A_n$, inscribed in a circle and let $B_1, B_2 \ldots B_n$ be the polygon circumscribing the same circle got by drawing the tangents at the vertices $A_1, A_2, \ldots A_n$ (Fig. 4.129).

Suppose Q is any point within ΔABC. Now, rotate ΔQAB about B through 60° to get ΔC_1Q_1B (Fig. 4.134).
We note that ΔAC_1B is isosceles as $AB = C_1B$ and further $\angle ABC_1$ = angle of rotation $= 60°$. Therefore, ΔAC_1B is equilateral. This means that irrespective of the position of Q, C_1 must be the third vertex of the equilateral triangle on AB, drawn outwardly, and hence C_1 coincides with C'. Again, $\angle QBQ_1 = 60°$ and $QB = Q_1B$ implies that ΔBQQ_1 is also equilateral.

This means that $QA + QB + QC = C_1Q_1 + Q_1Q + QC$
= the length of the polygonal path $CQ\,Q_1\,C_1$. (1)

This polygonal path joining C and C_1 has minimum length when Q and Q_1 lie on the line segment CC_1, in which case Q becomes the Fermat point of ΔABC (in view of (1)). When CQQ_1C_1 is a straight line, we should have $\angle BQQ_1 = 60°$ and hence $\angle BQC_1 = 60°$; this implies that the circumcircle of $\angle AC_1B$ passes through Q and this fixes the position of Q, on CC'. Therefore, the Fermat point P lies on CC' and $CC' = PA + PB + PC$. By symmetry, the Fermat point P also lies on AA' and BB'. Further $AA' = BB' = CC$ $= PA + PB + PC$. □

Note. There is another beautiful solution to Fermat's problem using Ptolemy's theorem.

Let ABC be any triangle. Let B and C be acute angles of ΔABC. Construct the equilateral triangle $BA'C$ on BC and consider its circumcircle. (Fig. 4.135). If P is on the minor arc BA' of the circle $BA'C$ then by Ptolemy's theorem,

$PB \,.\, CA' + BC \,.\, PA' = PC \,.\, BA'$ which gives $PB + PA' = PC$ and hence $PB + PC > PA'$. Similarly if P is on the minor arc CA', we get $PB + PC > PA'$. If P is on the minor arc BC, then we have

$$PB \,.\, CA' + PC \,.\, BA' = BC \,.\, PA' \text{ or}$$
$$PB + PC = PA'$$

∴ Unless P lies on the minor arc BC we have

$$PB + PC > PA' \text{ or } PA + PB + PC > PA + PA'$$

Now $PA + PA'$ is a minimum only when P lies on AA' and in which case $PA + PA' = AA'$. Thus the minimum of $PA + PB + PC$ is AA' and it occurs when P is at the intersection of AA' and the circumcircle of $\Delta BA'C$. In case $\angle BAC = 120°$, A coincides with P. If $\angle BAC > 120°$, A lies within the circle $BA'C$ and A is still the Fermat point of ΔABC.
□

Problem 13. If $C'AB$, $A'BC$ and $B'CA$ are the equilateral triangles drawn outwardly on the sides of a given triangle ABC then the centres X, Y, Z of the equilateral triangles form another equilateral triangle.

Let us draw the major arcs of the circles $C'AB$, $A'BC$ and $B'CA$. Let PQ be any line segment through A intercepted by the circles $C'AB$ and $B'CA$ at P, Q respectively (Fig. 4.136). Let PB and QC meet at R. We note that $\angle QPB = \angle PQC = 60°$ and hence $\angle BRC = 60°$. Therefore, R lies on the circle $A'BC$. Further ΔPQR is equilateral. From the centres X and Y, drop the perpendiculars XL, YK on the side QR of ΔPQR. Then L and K are the midpoints of the chords RC and CQ. Draw $XM \perp YK$ so that $XMKL$ is a rectangle. Now, K and L being the midpoints of QC and CR we see that $QR = 2KL$. Therefore QR is the largest when $LK = XM = XY$ (Fig. 4.136); *i.e.*, when M coincides with Y or $QR \parallel XY$. Further maximum $QR = 2XY$. For a similar reason PQ is a maximum

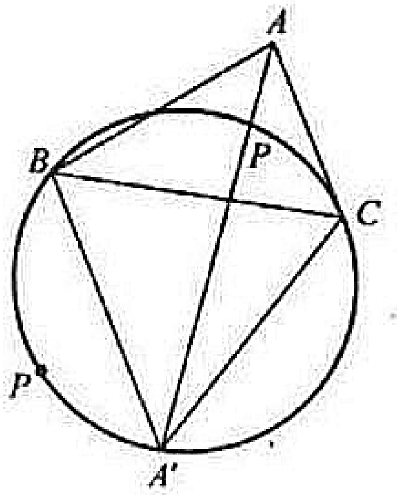


Fig. 4.135

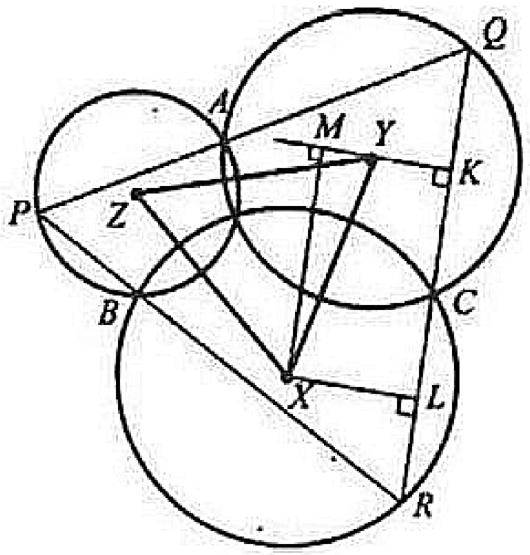


Fig. 4.136

when $PQ \parallel YZ$ and $PQ = 2YZ$ and maximum PR is twice ZX. But for any choice of P on the major arc of circle $C'AB$, we always have PQR as an equilateral triangle. In particular when ΔPQR is the largest, we have $PQ = 2YZ = QR = 2XY = RP = ZX$. Hence ΔXYZ is equilateral. □

Problem 14. Consider the two geometric transformations Rot (O_1, a) and Rot (O_2, β) where Rot (A, θ) means rotation about the point A through an angle θ. Find the sum of the two rotations Rot (O_1, α) and Rot (O_2, β).

Consider a line segment AB. Under Rot (O_1, α), AB is transformed into A_1B_1. We have $AB = A_1B_1$ in length and the angle between A_1B_1 and AB is α the angle of rotation. Under Rot (O_2, β) A_1B_1 is transformed into A_2B_2, with $A_2B_2 = A_1B_1$ in length and the angle between A_2B_2 and A_1B_1 is β (Fig. 4.137). Therefore, the angle between A_2B_2 and AB is $\alpha + \beta$ and hence the transformation Rot (O_2, β). Rot (O_1, α) preserves the length of line segments and rotates them through $(\alpha + \beta)$. This means that Rot (O_2, β) Rot (O_1, α) is again a rotation through $(\alpha + \beta)$ about some point O, unless $\alpha + \beta = 360°$ in which case it becomes a translation. We may get the centre of rotation of Rot (O_2, β) Rot (O_1, α), when $\alpha + \beta \neq 360°$, as follows.

Under the sum of the two rotations, O_1 goes to a point O_1' such that $O_2O_1, = O_2O_1'$ and $\angle O_1O_2O_1' = \beta$ (Fig. 4.138). If O_2'' is the point on the ray through O_1 making an angle α

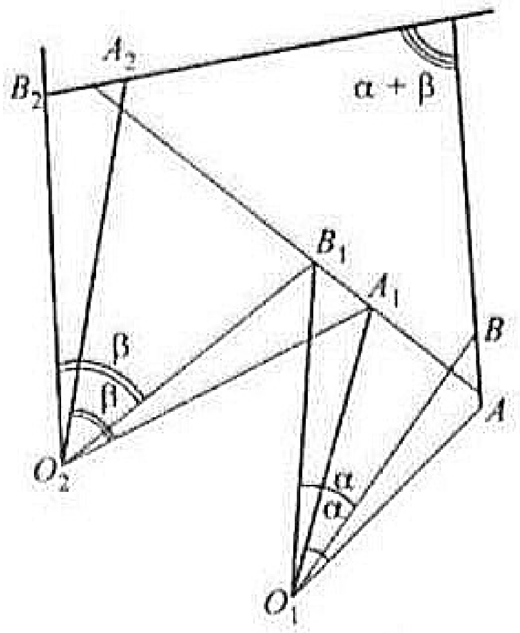


Fig. 4.137

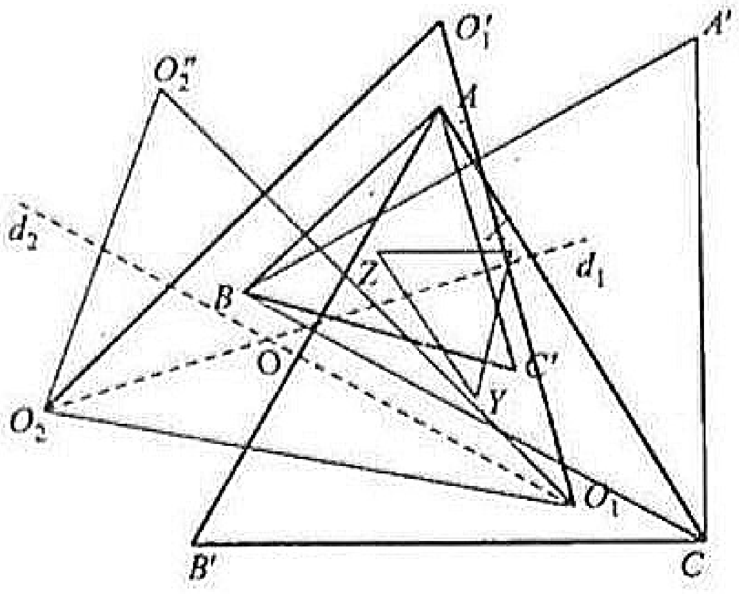


Fig. 4.138

1 with O_2O_1 as in Fig. 4.138, such that $O_1O_2'' = O_2O_1$ then O_2 is the image of O_2'' under
Rot (O_2, β) Rot (O_1, α). In fact Rot (O_1, α) takes O_2'' to O_2 and Rot (O_2, β) keeps O_2
fixed. Therefore if Rot (O_2, β) Rot (O_1, α) is Rot $(O, \alpha + \beta)$ then $OO_1 = OO_1'$ and OO_2''
$= OO_2$. This tells us that O lies on the perpendicular bisectors of O_1O_1' and $O_2''O_2$. Let
and d_1 and d_2 be the perpendicular besectors of O_1O_1' and O_2O_2'' respectively. Then d_2
makes $\alpha/2$ with O_2O_1 and d_1 makes $\beta/2$ with O_2O_1. (Fig. 4.138). If we draw d_1, d_2 ☐
satisfying the above conditions then O is the desired centre of rotation.

Problem 15. Construct equilateral triangles on the sides of a triangle ABC inwardly. Prove that the centres X, Y, Z of these triangles themselves form the vertices of an equilateral triangle.

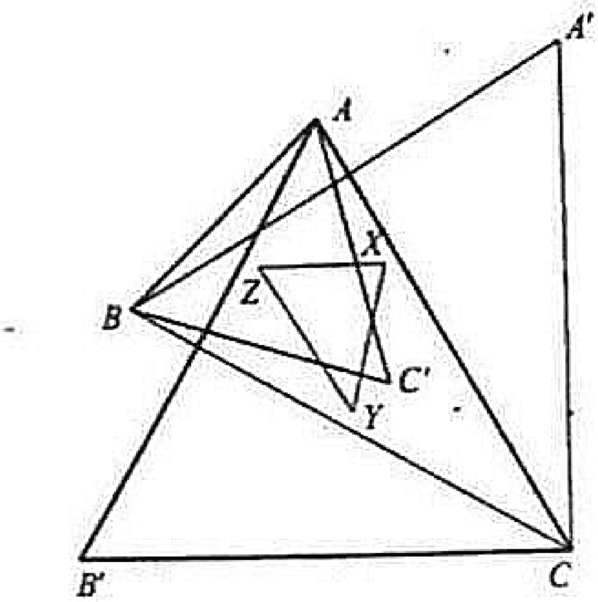


Fig. 4.139

Equilateral triangles $AC'B$, $BA'C$ and $CB'A$ are described inwardly as in Fig. 4.139. Let X, Y, Z be the centres of the triangles $BA'C$; $CB'A$ and $AC'B$ respectively. Rotate through 120° about X, then rotate through 120° about Y and then rotate through 120° about Z. The first transformation Rot $(X, 120°)$ takes B to C since $\angle BXC = 120°$; then Rot $(Y, 120°)$ takes C to A and Rot $(Z, 120°)$ takes A to B. Thus the sum of the three rotations leaves B fixed. But the sum of the three rotations Rot $(X, 120°)$, Rot $(Y, 120°)$ and Rot $(Z, 120°)$, is a translation (Problem 14); and this translation leaves B fixed implies that this translation must be the identity transformation. Now Rot $(X, 120°)$ followed by Rot $(Y, 120°)$ is Rot $(O, 240°)$ where O is the point of intersection of the lines through X and Y making angles 60° with XY (see Problem 14). This means that ΔXYO is equilateral. Now Rot $(O, 240°)$ followed by Rot $(Z, 120°)$ is the identity transformation. Hence O and Z should coincide. Thus $\angle XYZ$ is equilateral. ☐

Remark. The above proof also works for problem 13 where the equilateral triangles are drawn outwardly. The equilateral triangles XYZ formed by their centres are called the *outer Napoleon triangle* and *inner Napolean triangle* respectively.

Problem 16. If the perpendiculars from a point P of the circumcircle of ΔABC to the sides BC, CA, AB meet the circumcircle again at A_1', B_1', C_1' then AA_1', BB_1', CC_1' are all parallel to the Simson line of P with respect to ΔABC.

See Fig. 4.140. We have $\angle A_1'AC = \angle A_1'PC = \angle A_1PC = \angle A_1B_1C$ (since quadrilateral A_1CPB_1 is cyclic). Therefore $AA_1' \parallel C_1B_1A_1$. Similarly BB_1' CC_1' are also parallel to the Simson line $A_1B_1C_1$ of P. ☐

Problem 17. Find all the points on the circumcircle of a given triangle ABC whose Simson lines all have a given direction.

Draw BX parallel to the given direction meeting the circumcircle of ΔABC at X (Fig. 4.141). Draw the perpendicular from X to the side AC meeting the circle again at P. Then by the previous problem (Problem 16) P is the required point and the Simson line of P is $A_1B_1C_1$ parallel to BX through B_1 (Fig. 4.141). Again, problem 16 implies that P is the unique point whose Simson line is parallel to the given direction. ☐

Problem 18. Construct ΔABC given A, $b + c$ and h_a.

Suppose ΔABC is the required triangle. Let the internal bisector of $\angle A$ meet BC at U and the circumcircle at P. Drop the perpendiculars PC_1 and PB_1 on AB, AC respectively (Fig. 4.142). We have

$$\begin{aligned} AC_1 + AB_1 &= AB + BC_1 + AC - B_1C = AB + AC + BC_1 - B_1C \\ &= b + c + BC_1 - B_1C \end{aligned} \qquad (*)$$

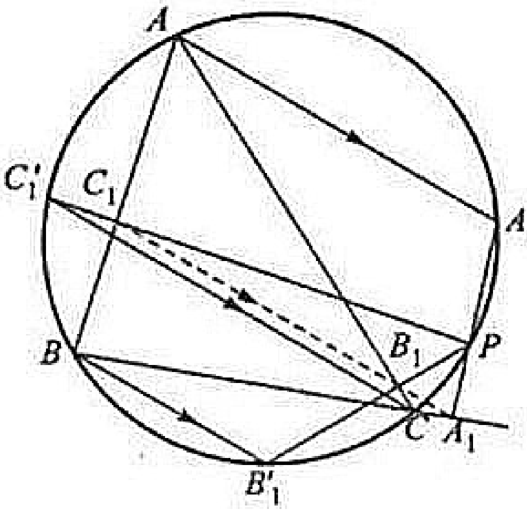


Fig. 4.140

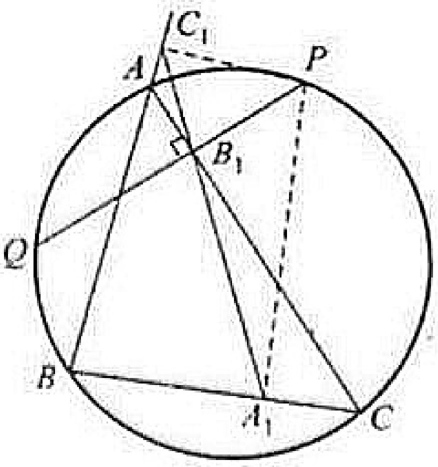


Fig. 4.141

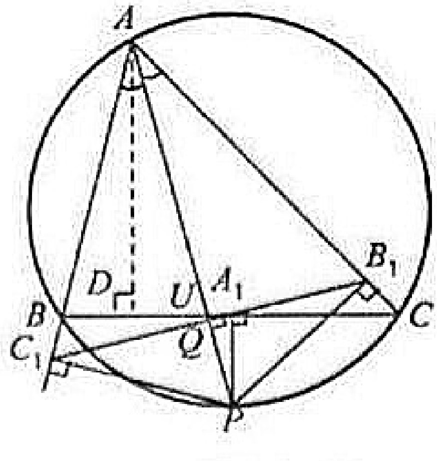


Fig. 4.142

We know that $PB = PC$ (why?) and $PC_1 = PB_1$ ($\because$ P is on the bisector of $\angle A$). Therefore $\Delta PAC_1 = \Delta PAB_1$ and $\Delta PC_1B = \Delta PB_1C$. Hence $AC_1 = AB_1$, $BC_1 = B_1C$. Substituting in (*) we get $AC_1 + AB_1 = b + c$ and therefore $AC_1 = AB_1 = (1/2)(b + c)$.

In the quadrilateral AC_1PB_1 we know $\angle B_1AC_1 = \angle A$, the sides AC_1, AB_1 and the other two angles $\angle AC_1P$ and $\angle AB_1P$ are right angles. This means that we can construct quadrilateral AC_1PB_1. Now AP is the perpendicular bisector of B_1C_1. We observe that B_1C_1 is the Simson line of P for ΔABC. Therefore B_1C_1 cuts BC at A_1 the foot of the perpendicular from P on BC. Suppose B_1C_1 meets AP at Q. Consider the triangles ADU and PQA_1.

$$\angle ADU = \angle PQA_1 = 90^\circ,\ \angle DAU = \angle QPA_1$$

(alternate angles; $AD \parallel PA_1$)

Therefore $\Delta ADU \,|||\, \Delta PQA_1$. This gives

$$\frac{PQ}{AD} = \frac{PA_1}{AU} = \frac{PA_1}{AP - UP} \tag{1}$$

In the right triangle PA_1U, A_1Q is the altitude from A_1

Therefore $PA_1^2 = PQ \,.\, PU$ (2)

Similarly from ΔPAB_1, $PB_1^2 = PQ \,.\, PA$ (3)

From (1), (2) and (3) we have $PQ \,.\, AU = PA_1 \,.\, AD = PA_1^2 = PQ \,.\, PU$

$$\therefore \quad PQ\,(AU + UP) = PA_1^2 + PA_1 \,.\, AD.$$

$$PQ\,(AP) = PA_1^2 + PA_1 \,.\, AD$$

$\therefore$ (3) gives $PA_1^2 + PA_1 \,.\, AD - PB_1^2 = 0$ (4)

In this equation (4) we know $AD = h_a$, PB_1 is known from the quadrilateral AC_1PB_1. So, PA_1 may be constructed.

(Exercise: Construct two segments given their product q and their sum p. In other words solve geometrically, the quadratic equation $x^2 - px + q = 0$).

Now draw the circle with centre P and radius PA_1 meeting B_1C_1 at A_1. The line perpendicular to PA_1 through A_1 meets AC_1, AB_1 at B, C respectively. This gives ΔABC. ❑

Note. This problem has no solution if $h_a > AQ$ (depends only on $b + c$ and A). $h_a = AQ$ implies ΔABC is isosceles and $h_a < AQ$ gives two symmetric solutions with respect to AP.

Problem 19. Let P_1, P_2 be any two points on the circumcircle of ΔABC. Then the angle between the Simson lines of P_1 and P_2 is half the angular measure of arc P_1P_2.

Let the perpendiculars from P_1, P_2 to BC meet the circumcircle again at X, Y respectively. (Fig. 4.143). Then we know that the Simson line of P_1 is parallel to AX and the Simson line of P_2 is parallel to AY (Problem 16). Therefore the angle between the Simson lines of P_1 and P_2 is $\angle XAY$.

Now, $P_1X \parallel P_2Y$ implies that arc P_1P_2 = arc XY.

The required angle $= \angle XAY = (1/2)\ \angle XSY = (1/2)\ \angle P_1SP_2$. ❑

Remark. As an immediate consequence of Problem 19 we note that the perpendiculars through P_1, P_2 to the Simson lines of P_2, P_1 respectively meet at the circumcircle of ΔABC. The same is true for the parallels through P_1, P_2.

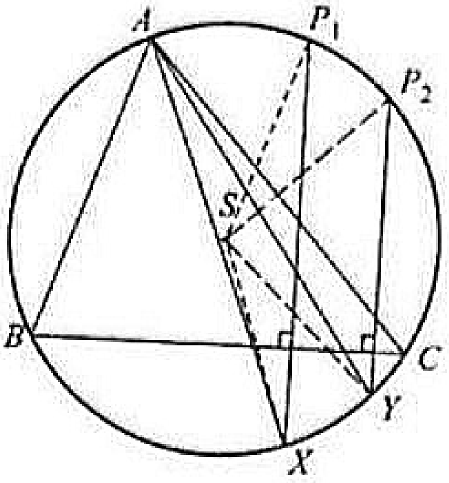


Fig. 4.143

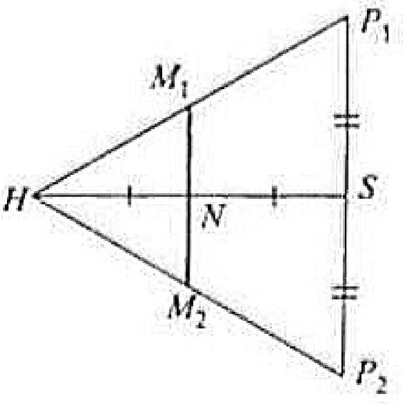


Fig. 4.144

Problem 20. If P_1 and P_2 are two diametrically opposite points on the circumcircle of ΔABC, then their Simson lines are perpendicular to each other and intersect on the nine-point circle of ΔABC.

From the previous problem, the angle between the Simson lines of two diametrically opposite points P_1, P_2 is $180°/2 = 90°$. By Theorem 63, the Simson lines of P_1 and P_2 bisect the segments HP_1 and HP_2 where H is the orthocentre of ΔABC. Let the midpoints of HP_1 and HP_2 be M_1 and M_2. (See Fig. 4.144.)

Now the nine-point centre N is the midpoint of SH and S is the midpoint of P_1P_2. Therefore N must be the midpoint of M_1M_2.

$$NM_1 = (1/2)\ SP_1 = \frac{R}{2} = NM_2 = \text{radius of the nine-point circle.}$$

M_1M_2 is a diameter of the nine-point circle of ΔABC.

Suppose the Simson lines of P_1 and P_2 meet at X. Then $\angle M_1XM_2 = 90°$ Therefore X lies on the nine-point circle of ΔABC.

Problem 21. Let $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles inscribed in the same circle. It P is a point on this circle, the angle between the Simson lines of P with respect to $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$ is a constant.

P be any point on the circumcircle of the given triangles. Draw PX, PY perpendicular to A_1C_1 and A_2C_2 respectively meeting the circle again at X, Y (Fig. 4.145). Then the Simson lines of P with respect to the two triangles are parallel to B_1X and B_2Y (Problem 16). The angle between B_1X and B_2Y is given by

$$\begin{aligned}
&(1/2)\ (\text{arc } XY - \text{arc } B_2B_1)\\
&\qquad = \angle XPY - (1/2)\ \text{arc } B_2B_1,\\
&\qquad = \angle C_1ZC_2 - (1/2)\ \text{arc } B_2B_1\\
&\qquad\qquad \text{(angle between the perpendiculars)}\\
&\qquad = \angle A_1C_2A_2 + \angle C_1A_1C_2 - (1/2)\ \text{arc } B_2B_1\\
&\qquad = (1/2)\ (\text{arc } A_1A_2 + \text{arc } C_1C_2 - \text{arc } B_2B_1)\\
&\qquad = \text{constant, independent of } P.
\end{aligned}$$

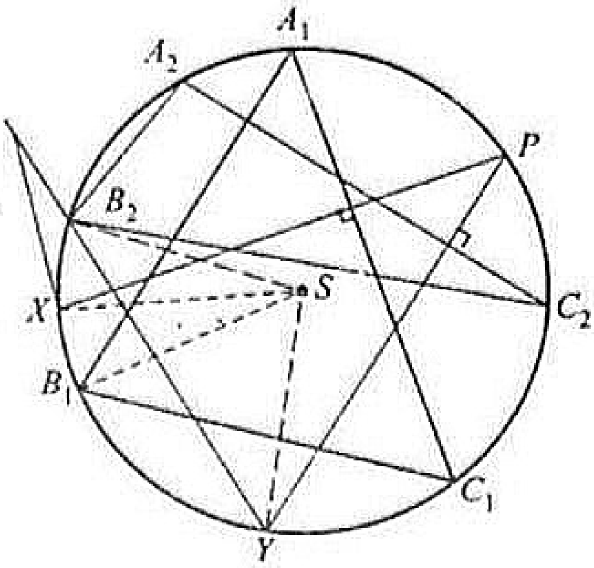


Fig. 4.145

Problem 22. Given a triangle ABC and two points P_1, P_2 on its circumcircle, there exists a third point P_3 on the circumcircle such that the Simson lines of P_1, P_2, P_3 are

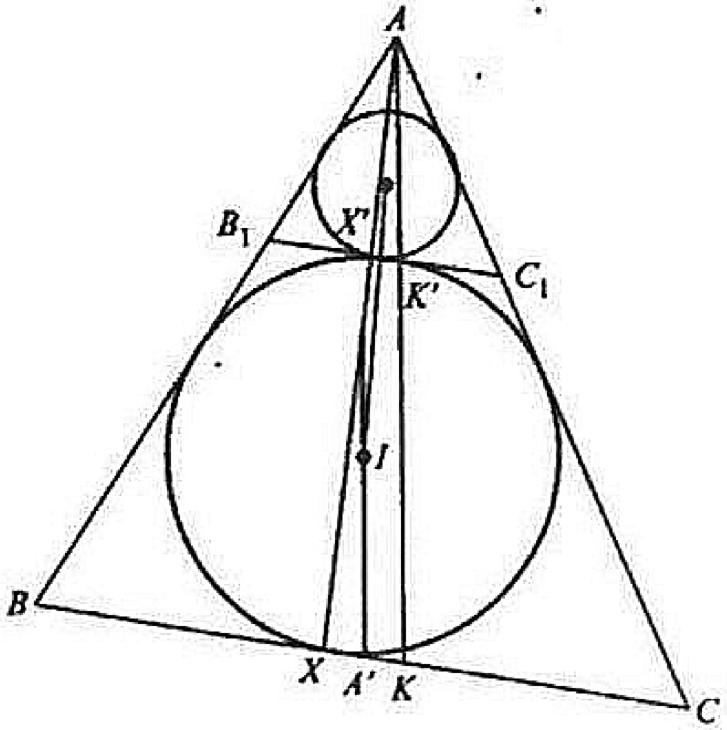


Fig. 4.148

We have $YY_1 = B_1Y_1 + B_1Y = B_1X_1 + B_1K' = 2B_1X_1 + X_1K'$

or $B_1X_1 = (1/2)\,(YY_1 - X_1K_1) = (1/2)\,(ZZ_1 - X_1K') = C_1K'.$

Similarly $BX = KC$ (Fig. 4.148).

Hence A' is the midpoint of XK as well. This means that $A'I$ should be parallel to KK' which implies that $A'I$ bisects XA. ☐

Problem 25 (Morley's theorem). The points of intersection of the adjacent trisectors of the angles of any triangle form the vertices of an equilateral triangle.

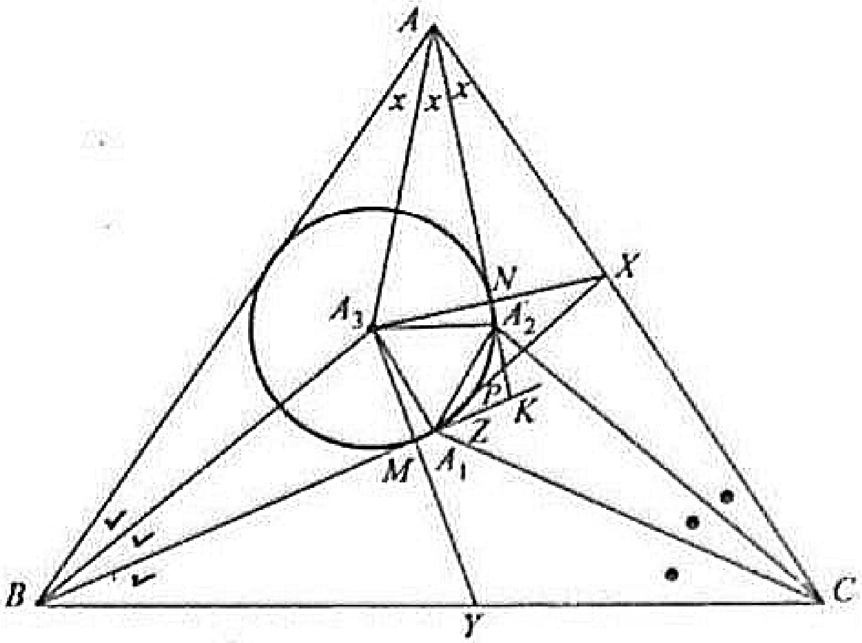


Fig. 4.149

Let the trisectors BA_1 and AA_2 meet at K (See Fig. 4.149). For the triangle ABK, the incentre is A_3. Let the incircle of ΔABK touch BA_1 and AA_2 at M, N respectively. Suppose A_3N meets AC at X and A_3M meets BC at Y. Let the tangent from X to the incircle touch the circle at P. We have $A_3N = NX$ as AN bisects $\angle A_3AX$ and $AN \perp A_3X$. So, $A_3P = A_3N = (1/2)\,A_3X$. Also $\angle A_3PX = 90°$. This means that $\angle PA_3X = 60°$ and $\angle A_3XP = 30°$ (1)

Further $\angle MA_3N = 180° - \angle MKN$ (as quadrilateral $MKNA_3$ is cyclic).

$$= 180° - (180° - (2/3)\,(\angle B + \angle A))$$

$$= (2/3)\,(\angle B + \angle A) = (2/3)\,(180° - \angle C)$$

$$= 120° - (2/3)\,(\angle C) \qquad (2)$$

Suppose the tangent XP from X to the incircle of ΔABK meets BK at Z.

Then $\angle ZYM = \angle ZA_3M$ (since ΔZA_3Y is isosceles, as $A_3M = MY$, $MZ \perp A_3Y$).

$$= (1/2)\,\angle MA_3P \text{ (since tangents at } P \text{ and } M \text{ meet at } Z)$$

$$= (1/2)\,(\angle MA_3P - \angle PA_3X) = (1/2)\,(\angle MA_3N - 60°)$$

$$= (1/2)\,(120° - (2/3)\angle C - 60°) \qquad \text{from (2)}$$

Thus $\angle ZYM = 30° - (1/3)\,\angle C$ (3)

$$\angle A_3YX = \angle A_3XY \quad (\text{as } A_3X = A_3Y = 2A_3M)$$

$$= (1/2)\,(180° - \angle XA_3Y)$$

$$= (1/2)\,\angle MKN \text{ (since quadrilateral } MKNA_3 \text{ is cyclic)}$$

$$= (1/2)\,(180° - (2/3)\,(\angle B + \angle A)$$

$$= (1/2)\,(180° - (2/3)\,(180° - \angle C)$$

$$= (1/2)\,(60° + (2/3)\,\angle C) = 30° + (1/3)\angle C.$$

$$\angle ZYX = A_3YX - \angle A_3YZ = \angle A_3YX - \angle MYZ$$

$$= \left(30° + \frac{\angle C}{3}\right) - \left(30° - \frac{\angle C}{3}\right) = \frac{2}{3}\angle C$$

and

$$\angle ZXY = \angle A_3XY - \angle A_3XZ$$

$$= \left(30° + \frac{\angle C}{3}\right) - 30° = \frac{\angle C}{3} \qquad \text{from (1)}$$

$\therefore$ $\angle XZY = 180° - \angle C$ or $ZYCK$ is cyclic.

This means that Z coincides with A_1 and the tangent from X to the incircle passes through A_1. Similarly, the tangent from Y passes through A_2. By symmetry $\angle A_1A_3P = \angle A_2A_3N$; $\angle PA_3X = 60°$ implies that $\angle A_1A_3A_2 = 60°$. Again by symmetry each angle of $\Delta A_1A_2A_3$ is equal to 60°. ❑

PROBLEMS

1. AOB is a given angle. Two circle of a radius r_1, r_2 touch OA, OB and also touch each other. Find the radius of another circle touching the sides of $\angle AOB$ and having its centre at the point of contact of the two given circles.
2. S_1 and S_2 are two non-intersecting circles with their centres at a distant 'd' apart. Prove that the four points of intersection of direct common tangents and transverse common tangents are concylic. Find the radius of this circle.
3. S is a circle of radius R and centre O. Two other circles of radii r_1, r_2 touch S internally and they intersect at A, B. Find $r_1 + r_2$.
4. Given two circles, find the locus of point P such that the ratio of the lengths of the tangents drawn from P to the given circles is a constant.
5. Given three pairwise intersecting circles, prove that the three common chords of the circles are concurrent.

6. A point A is chosen inside a circle. Find the locus of the point of intersection of the tangents drawn to the circle at the extremities of chords passing through A.
7. A straight line meets AB, BC and AC produced at D, E and F respectively. Prove that the midpoints of DC, AE and BF, are collinear.
8. Two circles touch each other internally at A. A tangent to the smaller circle meets the larger circle at B, C. Find the locus of the incentre of ΔABC.
9. Points P, Q, R are taken on BC, CA, AB such that $BP/PC = CQ/QA = AR/RB = m/n$. Prove that area $\Delta PQR = (m^3 + n^3)/(m + n)^3$ area ΔABC.
10. Construct ΔABC given m_b, m_c, h_a.
11. Construct a triangle given a, m_b/m_c, $b^2 - c^2$.
12. Prove that the feet of the four perpendiculars dropped from a vertex of a triangle on the four bisectors of the other two angles are collinear.
13. Given the six midpoints of the arcs subtended by the sides of a given ΔABC. Prove that one can construct I, I_a, I_b, I_c with ruler alone.
14. Construct ΔABC given r, r_a, m_a.
15. Given the positions of I, I_a and the length h_a as well as r_a/r, construct ΔABC.
16. Construct ΔABC given $b - c$, B, r.
17. Construct ΔABC given $b - c$, h_b, r.
18. Construct ΔABC, given A, X, X_a (notations as in the text).
19. Four points A, B, C, D such that each is the orthocentre of the triangle formed by the other three are said to form an orthocentric group of points. Prove that
    (i) the four triangles of an orthocentric group have the same orthic triangle.
    (ii) the four triangles of an orthocentric group have the same nine-point circle.
    (iii) the circumradii of the four triangles of an orthocentric group are equal.
    (iv) the circumcentres of an orthocentric group of triangles form an orthocentric quadrilateral.
    (v) An orthocentric group of triangles and the orthocentric group of their circumcentres have the same nine-point circle.
    (vi) The four vertices of a given orthocentric group of triangles may be considered as the circumcentres of a second orthocentric group of triangles.
    (vii) The four centroids of an orthocentric group of triangles form an orthocentric group.
    (viii) The nine-point centre of an orthocentric group of triangles is the same as the nine-point centre of the orthocentric group formed by the centroids of the given group of triangles.
    (ix) In any ΔABC, prove that I, I_a, I_b, I_c form an orthocentric group.
    (x) Δ is the orthic triangle of the orthocentric group I, I_a, I_b, I_c and the circumcircle of ΔABC is the nine-point-circle of the same orthocentric group.
    (xi) The circumcentres of the group I, I_a, I_b, I_c are the incentre and excentres of the symmetric of ΔABC with respect to the circumcentre S of ΔABC.
    (xii) Show that the algebraic sum of the distance of the points of an orthocentric group from a straight line passing through the nine-point centre of the group is equal to zero.
20. If h, m, t the altitude, the median, and the internal bisector from the same vertex of a triangle then prove that $4R^2h^2(t^2 - h^2) = t^4(m^2 - h^2)$ where R is the circumradius of the triangle.
21. $ABCD$ is a cyclic quadrilateral. Prove that the perpendiculars from the midpoints of AB, BC, CD, DA to CD, DA, AB, BC respectively concur at a point.

22. $ABCD$ is a cyclic quadrilateral; H_1, H_2, H_3, H_4 are the orthocentres of ΔBCD, ΔCDA, ΔDAB and ΔABC respectively. Prove that AH_1, BH_2, CH_3, DH_4 bisect each other.

23. If X is the common point to AH_1, BH_2, CH_3, DH_4 in problem 22, prove that the nine-point circles of ΔABC, ΔBCD, ΔCDA, ΔDAB pass through X.

24. With notations as in problems 22, 23 prove that $XA^2 + XB^2 + XC^2 + XD^2 = 4R^2$ where R is the radius of the circle $ABCD$.

25. Prove that the incentres of ΔABC, ΔBCD, ΔCDA, ΔDAB in a cyclic quadrilateral $ABCD$ form a rectangle.

26. For ΔABC, we call I, I_a, I_b, I_c as the tritangent centres. Prove that the sixteen tritangent centres, of the four triangles ABC, BCD, CDA, DAB of a cyclic quadrilateral lie by fours on eight straight lines; these eight lines consist of two perpendicular groups of four parallel lines.

27. If a, b, c, d are the sides of a cyclic quadrilateral $ABCD$, prove that (area $ABCD)^2 = (s-a)(s-b)(s-c)(s-d)$ where $2s = a + b + c + d$.

28. $ABCD$ is a cyclic quadrilateral in which $AC \perp BD$. Prove
    (*i*) The midpoints of the sides of the quadrilateral ABCD are concyclic and their centre is the centroid of $ABCD$.
    (*ii*) If AC, BD meet at O the perpendicular from O to AB bisects CD.
    (*iii*) If X, Y, Z, W are the feet of the perpendiculars from O on the sides of the quadrilateral, X, Y, Z, W lie on the circle passing through the midpoints of the sides.
    (*iv*) If S is the circumcentre of $ABCD$, then the distance of S from AB is equal to $CD/2$.
    (*v*) $AB^2 + CD^2 = BC^2 + AD^2 = AR^2$ where R is the circumradius of $ABCD$.

29. Prove that in any triangle ABC, $b^2 - c^2 = 2a\,A'D$ where D the foot of the altitude from A and A' is the midpoint of AB.

30. If P is a point on the side BC of ΔABC such that
    $BP/PC = m/n$, then prove that $mb^2 + nc^2 = (m+n)AP^2 + mPC^2 + nPB^2$.

31. If three equal circles have a common point then prove that the circle through the other three intersections is equal to them.

32. In problem 31, if the centres of the three equal circles are C_1, C_2, C_3 and their points of intersection are O, A, B, C prove that the figure formed by O, A, B, C is congruent to the figure formed by C, C_1, C_2, C_3 where C is the centre of circle ABC.

33. In problem 32 prove that in either of these congruent figures, the line joining any two of the vertices is perpendicular to the line joining the other two.

34. On a line segment AB a semicircle is drawn. On the other side of AB: a rectangle $ABDC$ is drawn with AC equal to the side of the square inscribed in the circle. P is any point on the semicircle: PC, PD cut AB at X, Y. Prove that $AX^2 + BF^2 = AB^2$.

35. ABC is an isosceles triangle with $AB = AC$; P, Q are points on AB : S_1 is the circle (P, PB) and S_2 is the circle (Q, QB). Similarly R, S are points on AC; S_3 is the circle (R, RC) and S_4 is the circle (S, SC). Let S_1 and S_3 meet at X, Y and S_2 and S_4 meet at Z, W. If PR and QS meet at T, prove that T is the centre of the circle passing through X, Y, Z, W. If $PR \parallel QS$, then prove that X, Y, Z, W lie on a straight line perpendicular to PR.

36. XY is a chord of a circle and P is the midpoint of XY. AB and CD are chords through P. Prove that AC and BD cut XY at equal distances from P, the same being true for AD, BC.

37. If S is a circle and AX, BY, CZ are the tangents from A, B, C to S, then prove that the circles of ΔABC touches S if and only if $AB \,.\, CS \pm AC \,.\, BY \pm BC \,.\, AX = 0$.

38. A, C, B are three colliner points in that order. Consider the semicircles on AB, BC, CA on the same sides of AB. The figure bounded by the above semicircles is called **'Shoemaker's Knife'**. Let the perpendicular through C to AB meet the semicircle on AB at D. Let TU be

the direct common tangent to the circles on AC and BC touching these circles at T, U respectively. Let TU meet CD at X. Prove that

1. arc ADC = arc ATC + arc CUB
2. $DC^2 = TU^2 = AC \,.\, BC$
3. X is the centre of the circle through C, D, T, U.
4. There area of 'Shoemaker's Knife' equals the area of the circle having CD as diameter
5. AD passes through T and BD passes through U.
6. If S_1 and S_2 are the circles inscribed in the curvilinear triangles ACD, BCD prove that S_1 and S_2 are equal circles and diameter of S_1 equals $AC \,.\, BC/AB$.
7. If the circle S_1 touches the arc AC at M then the common tangent to the two circles at M passes through B.
8. Prove that the smallest circle tangent to an circumscribing S_1 and S_2 is the circle on CD as diameter.

39. D, E, F are points on the sides BC, CA, AB of ΔABC. Prove that the circles AEF, BDF and CDE meet at a point.

40. If we call the common point of the three circles in prob. 39 as M, prove that MD, ME, MF make equal angles with the respective sides.
Further prove that $\angle BMC = \angle BAC + \angle EDC$.

41. Prove that the circumcircles of the four triangles formed by four lines have a common point.

42. Given four lines in a plane, prove that there is one and only one point from which the feet of the perpendiculars to the lines are collinear. What is that point? (Hint: problem 41).

43. ABC is a triangle and D, E, F are points on BC, CA, AB. By problem 39, the circles AEF, BFD and CDE have a common point M. If AP, BP, CP are three concurrent lines meeting the circles AEF, BFE, CDE at X, Y, Z respectively prove that X, Y, Z, M, P all lie on a circle.

44. Let F, F_a, F_b, F_c be the points of contact of the nine-point circle with the incircle and the three exscribed circles of ΔABC, prove that the point of intersection of the diagonals of the quadrilateral formed by F, F_a, F_b, F_c lies on its midline.

45. With notations as in Prob. 44, if the internal and external bisectors of angles A, B, C meet BC, AB at A_1, B_2, C_1, C_2 respectively.

prove that
$$\Delta F_aF_bF_c \;|||\; \Delta A_1B_1C_1$$
$$\Delta FF_bF_c \;|||\; \Delta A_1B_2C_2$$
$$\Delta FF_cF_a \;|||\; \Delta B_1C_2A_2$$
$$\Delta FF_aF_b \;|||\; \Delta C_1A_2B_2$$

46. $ABCD$ is a cyclic quadrilateral. Four circles $\alpha, \beta, \gamma, \delta$ touch the circle $ABCD$ at A, B, C, D respectively. Let $t_{\alpha\beta}$ be the segment of the direct common tangent to α, β if α, β touch the circle $ABCD$ in the same manner (both internally or both externally); let $t_{\alpha\beta}$ be the segment of the transverse common tangent if α, β touch the given circle in different ways. Similarly we define, $t_{\beta\gamma}$ etc. Prove that

$$t_{\alpha\beta}t_{\gamma\delta} + t_{\beta\gamma}t_{\delta\alpha} = t_{\alpha\gamma}t_{\beta\delta}.$$

47. On the circle K there are three distinct points A, B, C. Using a straight edge and a compass, construct a fourth point D on K such that a circle can be inscribed in the quadrilateral $ABCD$.

48. A circle is inscribed in ΔABC. Tangents to the circle parallel to the sides are constructed. Each of these tangents cuts off a triangle from ΔABC. In each of these triangles a circles is inscribed. Find the sum of the areas of all the four inscribed circles.

49. Let $A_0B_0C_0$ and $A_1B_1C_1$ be two acute angled triangles. Let $f = \{\Delta ABC \mid \Delta ABC \parallel\!| \Delta A_1B_1C_1, A_0$ is on BC, B_0 is on CA and C_0 is on $AB\}$. Find a triangle in f of maximum area and construct it.

50. Prove that if $n \ge 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.

51. P is any point inside ΔABC; u, v, w be the distances from P to the vertices A, B, C respectively; x, y, z be the distances from the point P to the sides BC, CA, AB respectively. Then prove that

(*i*) $au + bv + cw \ge 4\Delta$

(*ii*) $u + v + w \ge 2(x + y + z)$

(*iii*) $ux + vy + wz \ge 2(xy + yz + zx)$

(*iv*) $2(l/u + l/v + l/w) \le l/x + l/y + l/z$

(*v*) $uvw \ge R(x + y)(y + z)(z + x)/zr$

(*vi*) $uvw \ge 4Rxyz/r$

(*vii*) $uv + vw + wu \ge 2R(xy + yz + zx)/r$

(*viii*) $u + v + w \ge 6r$

52. Prove the following for a triangle ABC.

(*i*) $3(bc + ca + ab) \le (a + b + c)^2 \le 4(bc + ca + ab)$

(*ii*) $a^2 + b^2 + c^2 \ge 36(s^2 + abc/s)35$.

(*iii*) $8(s - a)(s - b)(s - c) \le abc$.

(*iv*) $abc < a^2(s - a) + b^2(s - b) + c^2(s - c) \le 3abc/2$.

(*v*) $bc(b + c) + ca(c + a) + ab(a + b) \ge 48(s - a)(s - b)(s - c)$.

(*vi*) $2s/abc \le 1/a^2 + 1/b^2 + 1/c^2$.

(*vii*) $3/2 \le a/(b + c) + b/(c + a) + c/(a + b) < 2$.

(*viii*) The perimeter of the triangle $\le s$.

53. Prove that ΔABC is acute, right or obtuse according as $a^2 + b^2 + c^2 - 8R^2$ is positive, zero or negative.

54. If in ΔABC, $a^2 + b^2 > 5c^2$, then show that c is the smallest side.

55. If p is the perimeter of the triangle whose vertices are the points of contact of the incircle with the sides of ΔABC, prove that $p \ge 6r^3\sqrt{s/4R}$.

56. ABC is a right triangle with $\angle A = 1$ radian and right angled at C. If I is the incentre and O is the midpoint of AB and N is the midpoint of OC, find whether ΔNIO is an acute, obtuse or right triangle.

57. Find a necessary and sufficient condition on a quadrilateral $ABCD$ in order that there exists a point P in the plane of $ABCD$, such that the areas of the triangles PAB, PBC, PCD, PDA are all equal.

58. ABC is a triangle such that a, b, c are all unequal; G, I, H are the centroid, incentre and orthocentre of ΔABC. Prove that $\angle GIH > 90°$.

59. Let ABC be a triangle; K, L be points on BC trisecting BC; M, N be on CA trisecting CA; P, Q be on AB trisecting AB. Construct equilateral triangles KLA; MNB', PQC' all directed inwards. Show that $A'B'C$ is an equilateral triangle having the same centroid as ΔABC.

60. In ΔABC, the internal bisector of $\angle A$ and the median through A meet BC in two distinct points L and D. The perpendicular from B upon AL meets AL and AD in P and M respectively. The perpendicular from C upon AL meets AL and AD in Q and N respectively. Show that $LM \parallel AB$ and $LN \parallel AC$.

61. Let D be a variable point on the side BC of ΔABC. Suppose the direct common tangent to the incircles of ΔABD and ΔACD, other than BC, meets AD in E. find the locus of E.

62. ABC is a mangle with $\angle A = 30°$. S is the circumcentre and I is the incentre of ΔABC. D is a point on segment AB and E is a point on segment CA such that $BD = CE = BC$. Show that $SI \perp DE$ and $SI = DE$.

63. $ABCD$ is a convex quadrilateral in which $AC = BD$; XAB, YBC, ZCD and WDA are equilateral triangles and S_1, S_2, S_3, S_4 are their centres respectively. Prove that $S_1S_3 \perp S_2S_4$.

64. S_1 and S_2 are two circles which touch externally at P; S is a circle which touches S_1 and S_2 internally. A direct common tangent to S_1, S_2 meets S at B, C. The common tangent to S_1, S_2 at P meets S at A such that P, A lie on the same side of BC. Prove that P is the incentre of ΔABC.

65. ABC is a triangle and P is any point inside ΔABC; X, Y are the feet of the perpendiculars from P to AB, AC; Z, W are the feet of the perpendiculars from A to BP, CP. Prove that the lines ZY, WX and BC are concurrent.

66. ΔABC is an acute angled triangle; A' is the midpoint of BC and P is any point on the median AA' such that $PA' = BA'$. The perpendicular from P to BC cuts BC at X. The perpendicular from X to PB cuts AB at Z and the perpendicular from X to PC cuts AC at Y. Show that the circle XYZ touches BC at X.

67. I is the incentre of ΔABC; C' is the midpoint of AB and B' is the midpoint of CA. The line $C'I$ meets AC at B_2 and the line $B'1$ meets AB at C_2. If area of ΔAB_2C_2 equals area of ΔABC, find ΔBAC.

68. AB and CD are chords of a circle cutting each other at E; M is a point on the chord AB such that $AM/AB = m/n$. The tangent at E to the circle DEM cuts BC at X and CA at Y. Prove that $YE/EX = m/(n - m)$.

69. In ΔABC, BD and CE are the bisectors of $\angle B$, $\angle C$ cutting CA, AB at D, E respectively. If $\angle BDE = 24°$ and $\angle CED = 18°$, find the angles of ΔABC.

70. $A_0B_0C_0$ is a triangle and P is inside it; A_1, B_1; C_1 are the feet of the perpendiculars from P on the sides of $\Delta A_0B_0C_0$; A_2, B_2, C_2 are the feet of the perpendiculars from P on the sides of $\Delta A_1B_1C_1$. Likewise we define $\Delta A_nB_nC_n$ for $n > 2$. Is one of $\Delta A_nB_nC_n$ for $n \geq 1$, similar to $\Delta A_0B_0C_0$?

71. C_1, C_2, ...C_n be a sequence of circles inscribed in **Shoemaker's Knife** such that C_n touches S_1, S_2 and C_{n-1}, where S_1, S_2 are seimicircles on AB, AC respectively and C_0 is the semicircle on CB (See Problem 38). Prove that the distance of the centre of C_n from AB is n times the diameter of C_n.

72. S_1 and S_2 are two circles of unit radius Touching at point P; l is a common tangent to S_1, S_2 touching them at X, Y; C_1 is the circle touching S_1, S_2 and l. C_n is the circle touching S_1, S_2 and l. C_{n-1} for $n > 1$. By computing the diameters of C_n, prove that $1/1.2 + 1/2.3 + 1/3.4 + \ldots + 1/n(n + 1) + \ldots = 1$.

73. ABC is an isosceles triangle; l is a straight line passing through a vertex of ΔABC, dividing ΔABC into isosceles triangles. Find all such isosceles triangles.

74. Find all convex polygons, for which one angle is bigger than the sum of the remaining angles.

75. Let P be a set of finitely many points in the plane, not all in a straight line. Prove that there exists a straight line in die plane containing exactly two points of P.

76. Let P be a finite set of points in a plane, no three of which are collinear and not all in a circle. Prove that there is a circle in the plane containing exactly three points of P.

5

QUADRATIC EQUATIONS AND EXPRESSIONS

5.1 INTRODUCTION

In this chapter we shall discuss equations of the form

$$y = ax^2 + bx + c \tag{1}$$

where a, b, c are real numbers. These are called *quadratic equations* in the variable x. To start with, we take a simple linear equation of the form

$$y = ax + b.$$

In particular, consider the equation

$$2x - 1 = 0$$

We know that the solution is $x = \frac{1}{2}$. Suppose on the other hand we draw the graph of the function

$$y = 2x - 1$$

on a coordinate plane. The plotting of the curve is done, as usual, by fixing values of x at reasonable intervals and working out the corresponding values of y.

x	0	1	-1	2	-2	...
$y = 2x - 1$	-1	1	-3	3	-5	...

Therefore, $(0, -1)$, $(1, 1)$, $(-1, -3)$, $(2, 3)$ and $(-2, -5)$ are all points on the graph of the function $y = 2x - 1$. A little experimentation will show that any three of these points are collinear. In fact the graph of $y = 2x - 1$ will be a straight line. That this will be so for any equation of the form $y = ax + b$ will be proved in Chapter 7. It is therefore only necessary to plot two points on the graph of $y = 2x - 1$ and join them by a straight line, on the coordinate plane.

Let us now plot the straight line $y = 2x - 1$. Two points on this line may be taken as $(0, -1)$ and $(1, 1)$. Fig. 5.1 shows the straight line joining them. Such a graphical representation of the function $y = f(x) = 2x - 1$ gives us a sure method of estimating the solution of the equation

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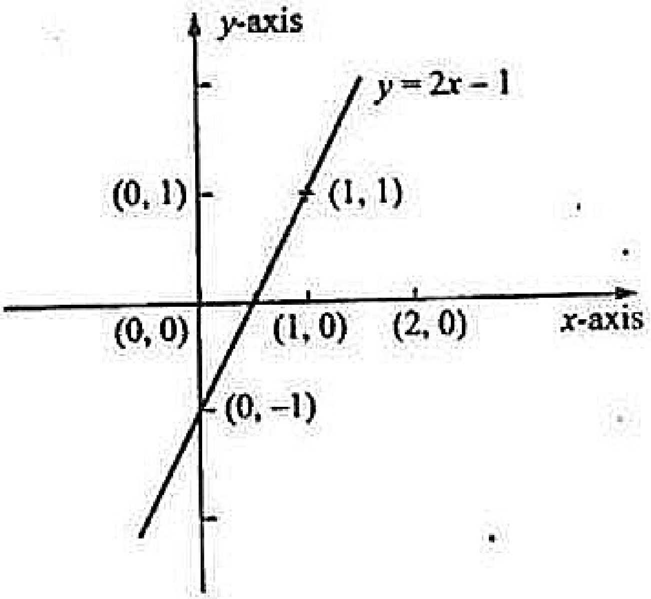


Fig. 5.1

to note that the graphical solution cannot be a precise substitute for the algebraic solution. In this case it is, because the answer is $x = 1/2$. But if the answer happens to be a more complicated number, the geometrical (that is graphical) solution may not give an accurate answer. This has nothing to do with the efficacy of the geometrical method. It is only a reflection or our equipment capabilities, like reading of measurements from a figure or a drawing.

To sum up, whenever we want to solve an equation of the form

$$ax + b = c \tag{2}$$

we may also do it geometrically by drawing the straight line graph

$$y = ax + b,$$

and looking for the x-coordinate of the point where the line $y = c$ (parallel to x-axis) cuts the line

$$y = ax + b.$$

For instance, to solve

$$2x + 3 = 5,$$

draw the graph of

$$y = 2x + 3.$$

Draw the line

$$y = 5.$$

The two meet at (1, 5). See Fig. 5.2. So $x = 1$ is the answer.

Now let us take a second degree expression, the simplest of which is $y = x^2$. Since the square of a real number is always non-negative, the graph lies above the x-axis. Moreover $x^2 = (-x)^2$ so that at both x and $-x$, y takes the same value. Thus the graph is symmetric about the y-axis. However, in the case of a first degree equation the graph being a straight line, can be completely drawn, once we plot any two points on the graph. The same cannot be done here. We can draw the graph of

$$y = x^2$$

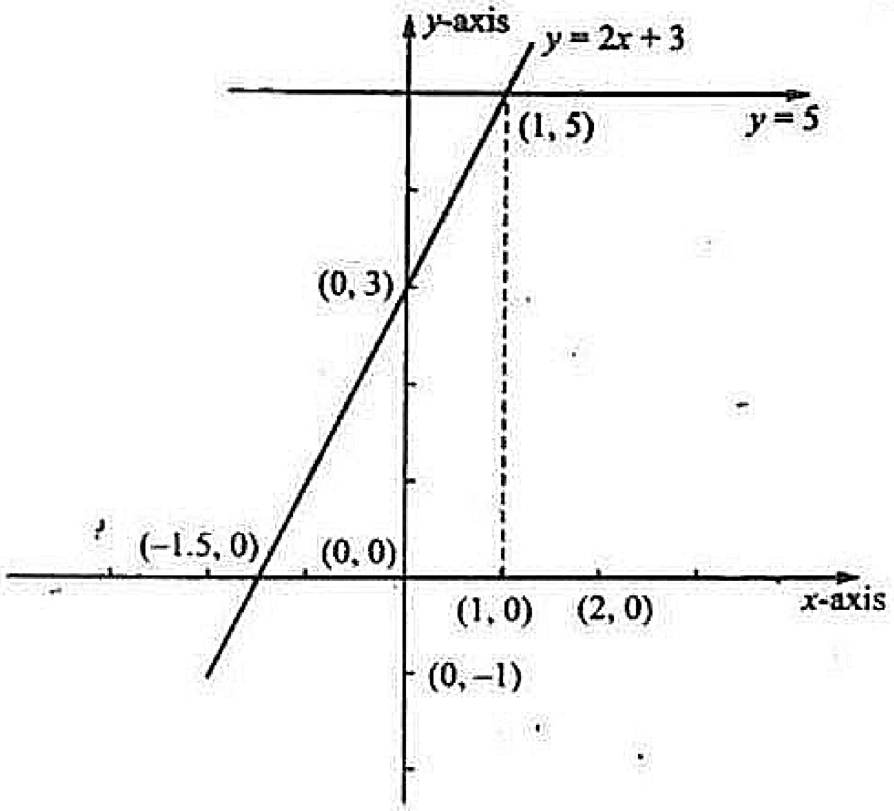


Fig. 5.2

only approximately On the coordinate plane using values of y at some chosen values of x. (See Fig. 5.3).

x	0	1	− 1	2	− 2	3	− 3	...		
$y = x^2$	0	1	1	4	4	9	9	...		

We are now ready to take the graph of the second degree expression, in the general case, viz.,

$$y = ax^2 + bx + c.$$

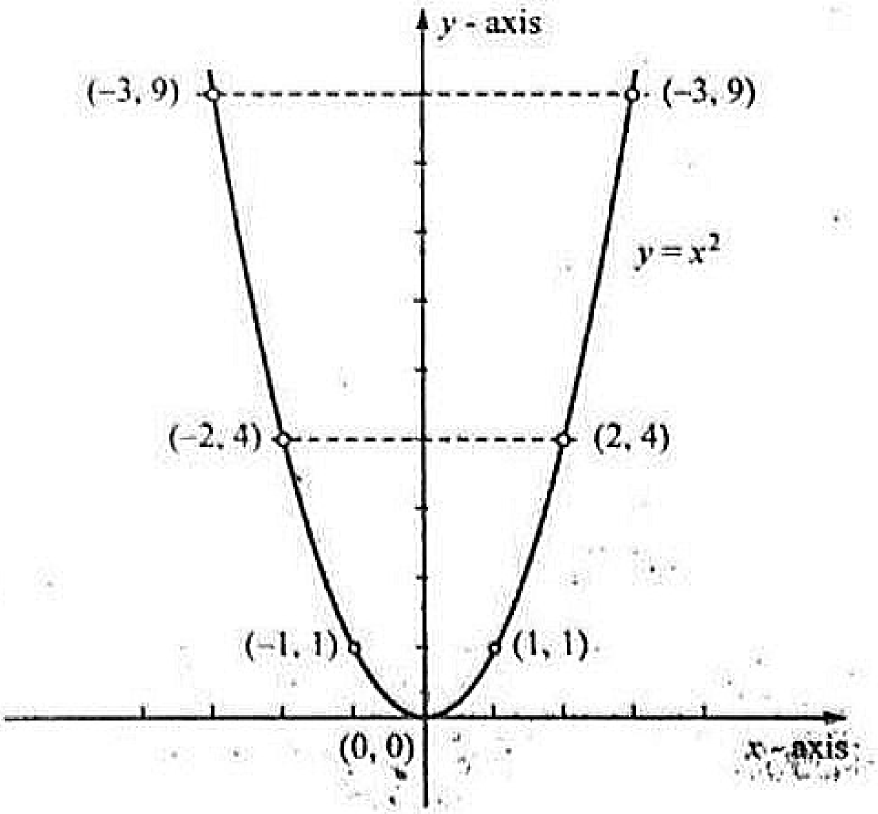


Fig. 5.3

provided the graph meets the x-axis. However, drawing the graph of such a function may not be a feasible task. We can only draw an approximate graph in general and can get only approximate solutions. An equation of the form (1) is called a quadratic equation. Because of this drawback of the graphical method, we look for algebraic methods of solving equation (1).

Suppose we can find numbers d, e, u and v such that

$$(ax^2 + bx + c) = (dx + e)\,(ux + v) \tag{2}$$

for all values of x. Then, to solve

$$ax^2 + bx + c = 0,$$

we have only to solve

$$(dx + e)\,(ux + v) = 0.$$

This would mean

either $\quad dx + e = 0 \quad$ or $\quad ux + v = 0.$

This gives

either $\quad x = -\,e/d \quad$ or $\quad x = -\,v/u.$

But note that we can write like this only if $d \neq 0$, $u \neq 0$. This is however true, since

$$du = a \qquad \text{from (2)}$$

and $a \neq 0$ by our starting assumption. Thus the quadratic equation (1) has two solutions

$$\alpha = -\,e/d,\ \beta = -\,v/u$$

whenever there is a factorization of the form (2). In other words the quadratic equation (1) can be solved if there is a factorization of $ax^2 + bx + c$ into a product of two linear factors. The solutions of equation (1) are also called the *roots of the quadratic equation* (1) or the zeros of the quadratic polynomial $ax^2 + bx + c$.

EXAMPLE 1. *Find the roots of the quadratic equation*

$$12x^2 + 25x + 12 = 0.$$

SOLUTION. Since $\quad 12 \times 12 = 144 = 16 \times 9 \quad$ and $\quad 25 = 16 + 9,$

we have

$$\begin{aligned} 12x^2 + 25x + 12 &= 12x^2 + 9x + 16x + 12 \\ &= (4x + 3)\,(3x + 4). \end{aligned}$$

Hence the roots of the given quadratic equation are

$$\alpha = -\,3/4,\ \beta = -\,4/3.$$

EXAMPLE 2. *Solve the equation*

$$2x^2 + 2\sqrt{6}x + 3 = 0.$$

SOLUTION. We can write

$$\begin{aligned} 2x^2 + 2\sqrt{6}x + 3 &= 2x^2 + \sqrt{6}x + \sqrt{6}x + 3 \\ &= (\sqrt{2}x + \sqrt{3}x)\,(\sqrt{2}x + \sqrt{3}x). \end{aligned}$$

Thus the roots of the given equation are obtained by solving two identical equations

$$\sqrt{2}x + \sqrt{3} = 0,$$

$$\sqrt{2}x + \sqrt{3} = 0.$$

In such a case we say that the given equation has two identical or coincident roots

$$\alpha = -\,\sqrt{3}/\sqrt{2},\ \beta = -\,\sqrt{3}/\sqrt{2}.$$

Example 2 also illustrates an important aspect of the roots of a quadratic equation. The roots of a quadratic equation may be identical. In other words a root of a quadratic equation may repeat itself.

EXERCISE 5.2

Find the zeros of each of the following quadratic polynomials by factoring it.

1. $3x^2 - 8x + 5$
2. $x^2 - 3x - 18$
3. $x^2 - 3x - 10$
4. $6x^2 - x - 12$
5. $18x^2 + x - 5$
6. $2\sqrt{3}x^2 + 11x + 4\sqrt{3}$
7. $-\sqrt{35}\,x^2 + 12x - \sqrt{35}$
8. $2x^2 - 3\sqrt{3}x + 3$
9. $6x^2 + 41x - 7$
10. $-2x^2 + 4x + 16$
11. $8x^2 + 2x - 3$
12. $3x^2 + 11x - 4$

5.3 METHOD OF COMPLETING THE SQUARE

In section 5.2, we have seen how to solve a given quadratic equation, whenever the corresponding quadratic expression can be written as a product of two linear factors. However, given a quadratic equation, it may be difficult to realise the corresponding linear factors. We shall also see that there are quadratic functions having no factors with real coefficients. But by mere inspection of the quadratic function, we may not be able to decide whether the function has factors with real coefficients.

Let us begin with the quadratic equation

$$x^2 + x - 1 = 0. \tag{1}$$

The quadratic function $x^2 + x - 1$ is negative at $x = 0$ and positive at $x = 1$. This means that the graph of $x^2 + x - 1$ must pass from the y-negative half-plane to the y-positive half-plane when x goes from 0 to 1. It must cut the x-axis somewhere between 0 and 1. These observations show that the equation (1) has a root between 0 and 1. Similarly, $x^2 + x - 1$ is negative at $x = -1$ and positive at $x = -2$. Hence the equation (1) also has a root between -2 and -1. The equation (1) has thus two real roots; one between 0 and 1 and the other between -2 and -1. But from the expression $x^2 + x - 1$, we cannot guess its linear factors.

On the other hand, let us also consider the equation

$$x^2 + x + 1 = 0. \tag{2}$$

Apriori, it is difficult to say whether $x^2 + x + 1$ can or cannot be written as a product of two linear factors with real coefficients. However, we observe that $x^2 + x + 1$ is positive for every real number x. In fact, if $x \geq 0$, then

$$x^2 + x + 1 \geq 1 > 0;$$

if $x \leq -1$, then

$$x^2 + x + 1 = x(x + 1) + 1 \geq 1 > 0$$

since x and $x + 1$ are both negative; if $-1 < x < 0$, then

$$x^2 + x + 1 = x^2 + (x + 1) > 0 + 0 = 0.$$

Thus the graph of $x^2 + x + 1$ does not meet the x-axis. In other words the equation has no roots in real numbers. This implies that $x^2 + x + 1$ cannot be written as a product of two linear factors with real coefficients (see also Chapter 10).

There is a general way of deciding whether a quadratic equation

$$ax^2 + bx + c = 0,\ a \neq 0 \tag{3}$$

has a real root, and a method of finding the roots of the equation (3). This method is known as the *method of completing the square.* The central idea of the method is to convert (3) to an equation of the form

$$(dx + e)^2 + f = 0.$$

where d, e and f are real numbers. This can be accomplished by considering the equation

$$4a^2x^2 + 4abx + 4ac = 0. \tag{4}$$

Since $a \neq 0$, equations (3) and (4) have the same solutions.

But (4) can be written as

$$(2ax + b)^2 + 4ac - b^2 = 0. \tag{5}$$

Thus we can take

$$d = 2a,\ e = b,\ f = 4ac - b^2.$$

Hence a number α(real or complex) is a root of (3) iff it is also a root of

$$(2ax + b)^2 = b^2 - 4ac. \tag{6}$$

We can use (6) to find the roots of (3).

The quantity $D = b^2 - 4ac$ is called the *discriminant* of the quadratic equation (3), or the discriminant of the quadratic polynomial $ax^2 + bx + c$. The nature of the roots of (3) can be completely determined by its discriminant $b^2 - 4ac$.

Theorem 1. The quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

where a, b and c are real numbers, has real roots if the discriminant D given by

$$D = b^2 - 4ac$$

is non-negative. In the case when $D \geq 0$, the roots of the given quadratic equation are

$$\alpha = \frac{-b + \sqrt{D}}{2a}, \quad \beta = \frac{-b - \sqrt{D}}{2a}$$

Proof. We have seen in the preceeding discussions that the given quadratic equation and the equation

$$(2ax + b)^2 = b^2 - 4ac = D \tag{6}$$

have the same set of solutions. Suppose the given equation has a real root x_0. Then $(2ax_0 + b)^2 \geq 0$ and x_0 is also a root of (6). This means that $D \geq 0$ is necessary for equation (3) to have a real root. Conversely, assume now that $D \geq 0$. Then D has two real square roots, $+\sqrt{D}$ and $-\sqrt{D}$. Hence taking the square root in (6), we get two linear equations

$$\begin{aligned} 2ax + b &= \sqrt{D} \\ 2ax + b &= -\sqrt{D} \end{aligned} \tag{7}$$

(compare this with our observation in 5.2). Solving these linear equations, we get two roots of the equation (6).

Since the equations (6) and (3) have the same set of roots, α and β are also the roots of the given quadratic equation (3). ❑

Remark. If $D = 0$ then the roots of the quadratic equation (3) are

$$\alpha = -b/2a,\ \beta = -b/2a$$

which are identical. On the other hand if $D \neq 0$, then $\alpha\ \alpha \neq \beta$; also $\alpha = \beta$ implies $D = 0$. Thus the equation (3) has coincident roots iff its discriminant vanishes.

EXAMPLE 1. *Find the roots of the equation*

$$x^2 + x - 1 = 0.$$

SOLUTION. The discriminant D of the quadratic polynomial is

$$D = 1^2 - 4 \times 1 \times (-1) = 5$$

Thus $D > 0$ and the given equation has two real distinct roots. They are given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} = \frac{-1 + \sqrt{5}}{2},$$

$$\beta = \frac{-b - \sqrt{D}}{2a} = \frac{-1 - \sqrt{5}}{2}$$

We also observe that $2 < \sqrt{5} < 3$. Hence the estimates

$$\frac{1}{2} < \frac{-1 + \sqrt{5}}{2} < 1,$$

$$-2 < \frac{-1 - \sqrt{5}}{2} < \frac{-3}{2}$$

are true. This conforms with our earlier observations that the given equation has a root between 0 and 1 and a root between -2 and -1.

EXAMPLE 2. *Solve the equation*

$$12x^2 + 25x + 12 = 0.$$

SOLUTION. Earlier, in example 1 of section 5.2, we have found the roots of this equation by factoring the corresponding quadratic function,

$$12x^2 + 25x + 12 = (4x + 3)(3x + 4)$$

to get the roots $\alpha = -3/4$ and $\beta = -4/3$. The same conclusion can be drawn using the method of this section. Since $a = 12$, $b = 25$ and $c = 12$, the discriminat is

$$D = b^2 - 4ac = 25^2 - 4 \times 12 \times 12 = 49.$$

Thus $D > 0$ and the equation has two real, distinct roots. These are

$$\gamma = \frac{-b + \sqrt{D}}{2a} = \frac{-25 + \sqrt{49}}{24}$$

$$= -18/24 = -3/4,$$

and $$\delta = \frac{-b - \sqrt{D}}{2a} = \frac{-25 - \sqrt{49}}{24} = \frac{-32}{24} = -4/3.$$

We see that $\alpha = \gamma$ and $\beta = \delta$.

EXAMPLE 3. *Determine whether the equation*

$$4x^2 + 4x + 1 = 0$$

has real roots and solve for them.

SOLUTION. The discriminant of the equation is

$$D = 4^2 - 4 \times 4 \times 1 = 0.$$

Hence we conclude that the given equation has two coincident roots, namely,

$$\alpha = -4/8 = -1/2,$$
$$\beta = -1/2.$$

This can also be inferred from the observation that

$$4x^2 + 4x + 1 = (2x + 1)^2.$$

EXAMPLE 4. *What are the solutions of the equation*

$$x^2 + 2x - 4 = 0$$

SOLUTION. The discriminant is given by

$$D = 2^2 + 16 = 20,$$

so that the given equation has real distinct roots. They are given by

$$\alpha = \frac{-2+\sqrt{20}}{2} = -1 + \sqrt{5}$$

$$\beta = \frac{-2-\sqrt{20}}{2} = -1 - \sqrt{5}.$$

Now we shall take a fresh look at the equation (6), namely

$$(2ax + b)^2 = b^2 - 4ac = D.$$

If $D \geq 0$, we could take square roots on both sides using the fact that every non-negative real number has two real square roots. This would given us two real solution of equation (6) and hence those of equation (3). However, if only we can give a meaning to $\sqrt{D}$ even if $D < 0$, then we will be able to solve the given equation (3) for any real values of a, b and c. If $D < 0$, then we pass to complex numbers and take square roots of D in C. In this case D has two square roots in **C**, viz., $i\sqrt{-D}$ and $-i\sqrt{-D}$. Thus we have the following theorem.

Theorem 2. If the discriminant

$$D = b^2 - 4ac$$

of the quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

is negative, then it has two complex roots which are *conjugate* to each other.

Proof. We have seen that the set of solutions of the given quadratic equation is identical with the set of solutions of

$$(2ax + b)^2 = D. \tag{*}$$

If $D < 0$, then it has two complex square roots $i\sqrt{-D}$ and $-i\sqrt{-D}$. Hence the two roots of (*) and hence of the given quadratic equation are

$$\alpha = \frac{-b+i\sqrt{-D}}{2a},$$

$$\beta = \frac{-b-i\sqrt{-D}}{2a}.$$

Since b and a are real, we see that

$$\beta = \overline{\alpha} \text{ the complex conjugate of } \alpha.$$

Thus the non-real roots of a quadratic equation with real coefficients always occur in pairs, one being the complex conjugate of the other. ☐

Theorems 1 and 2 completely resolve the question of existence of roots of a quadratic equation with real coefficients. If the discriminant $D > 0$, then the given quadratic equation has two distinct real roots. If $D = 0$, then it has two coincident roots. In the case $D < 0$, the given equation has two complex roots, one being the complex conjugate of the other. Combining all these, we can make the following statement.

Any quadratic equation

$$ax^2 + bx + c = 0$$

where a, b and c are real numbers, has exactly two roots. If α is a non-real root of the equation, then α is the other root.

EXAMPLE 5. *Solve the equation*

$$3x^2 + 2x + 1 = 0.$$

SOLUTION. The discriminant is

$$D = -8 < 0.$$

Hence the equation has no real roots. The complex roots are given by

$$\alpha = \frac{-b + i\sqrt{-D}}{2a} = \frac{-1 - i\sqrt{2}}{3},$$

$$\beta = \frac{-b + i\sqrt{-D}}{2a} = \frac{-1 + i\sqrt{2}}{3}.$$

EXAMPLE 6. *Find the roots of the equation*

$$3x^2 + 2\sqrt{3}\,x + 2 = 0.$$

SOLUTION. We have $D = (2\sqrt{3})^2 - 4 \times 3 \times 2 = -12 < 0.$

Hence the roots are

$$\alpha = \frac{-1 + i}{\sqrt{3}}, \quad \beta = \frac{-1 - i}{\sqrt{3}}.$$

EXERCISE 5.3

1. Find the roots of the following equations:
   (*a*) $2x^2 + x + 1 = 0$ (*b*) $9x^2 + 2x - 3 = 0$
   (*c*) $x^2 + 2x + 4 = 0$ (*d*) $4x^2 + 2x - 1 = 0$
   (*e*) $x^2 + 6x + 6 = 0$ (*f*) $2x^2 + 5x + 4 = 0$
   (*g*) $4x^2 + 9x + 2 = 0$ (*h*) $2\sqrt{3}\,x^2 + 4x - \sqrt{3} = 0$
   (*i*) $3x^2 + 9x - 5 = 0$ (*j*) $x^2 + 5x - 6 = 0.$
2. For each the following equations find the set of values of a for which the equation has real roots:
   (*a*) $ax^2 + 9x - 1 = 0$ (*b*) $2x^2 + ax + 2 = 0$
   (*c*) $2x^2 + 4x + a = 0$ (*d*) $2x^2 + ax - a = 0$
   (*e*) $x^2 + (a + 2)x + a = 0$ (*f*) $ax^2 + 4x + a = 0.$

3. A person goes to a vegetable market to buy vegetables. He finds that the price of carrot is the square of the price of beet-root. He buys 2 kilos of carrot and 3 kilos of beet-root, and tenders 27 rupees. What is the price of a kilo of carrot?
4. Solve for x in the equation
$$2m(1+x^2)-(1+m^2)(x+m)=0.$$
5. Find the values of a for which the equation
$$a^2x^2+2(a+1)x+4=0$$
has coincident roots.
6. Suppose p and q are real numbers which do not take simultaneously the values $p=0$, $q=1$. Suppose the equation
$$(1-q+p^2/2)\,x^2+p(1+q)x+q(q-1)+p^2/2=0$$
has two equal roots. Prove that $p^2=4q$.

5.4 RELATIONS BETWEEN ROOTS AND COEFFICIENTS

There are very useful relations between the roots of a quadratic equation
$$ax^2+bx+c=0 \tag{1}$$
and its coefficients a, b and c. These relations are true irrespective of whether (1) has only real roots or it has complex roots. If the discriminant $D \geq 0$, then the equation (1) has two real roots given by
$$\alpha=\frac{-b+\sqrt{D}}{2a},\quad \beta=\frac{-b-\sqrt{D}}{2a}. \tag{2}$$

Now we observe that
$$\alpha+\beta=\frac{(-b+\sqrt{D})}{2a}+\frac{(-b-\sqrt{D})}{2a}$$
$$=-2b/2a=-b/a.$$

Similarly,
$$\alpha\beta=\frac{(-b+\sqrt{D})}{2a}\,\frac{(-b-\sqrt{D})}{2a}$$
$$=\frac{b^2-D}{4a^2}=\frac{4ac}{4a^2}=c/a.$$

In the case $D<0$, the solutions of (1) are
$$\alpha=\frac{-b+i\sqrt{-D}}{2a},\quad \beta=\frac{-b-i\sqrt{-D}}{2a} \tag{3}$$

Now an easy computation gives again
$$\alpha+\beta=-b/a,\ \alpha\beta=c/a.$$

Thus we have the following theorem.

Theorem 3. If α and β are the roots of the quadratic equation
$$ax^2+bx+c=0$$
then,
$$\alpha+\beta=-b/a,$$
$$\alpha\beta=c/a. \tag{4}$$

EXAMPLE 1. *Find the sum and product of the roots of the equation*
$$5x^2+5x+1=0.$$

SOLUTION. If α and β are the roots of this equation, then (4) shows that

$$\alpha + \beta = -5/5 = -1,$$
$$\alpha\beta = 1/5.$$

EXAMPLE 2. *If α and β are the roots of*

$$x^2 + 4x + 6 = 0,$$

find the values of

(*i*) $1/\alpha + 1/\beta$, (*ii*) $\alpha^3 + \beta^3$, (*iii*) $\alpha/\beta + \beta/\alpha$.

SOLUTION. We have $\alpha + \beta = -4,\ \alpha\beta = 6.$

Hence
$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-4}{6} = \frac{-2}{3}.$$

We can write
$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$
$$\therefore\quad \alpha^3 + \beta^3 = (-4)^3 - (3 \times 6 \times (-4))$$
$$= -64 + 72$$
$$= 8.$$

And lastly,
$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$
$$= \frac{(\alpha + \beta) - 2\alpha\beta}{\alpha\beta}$$
$$= \frac{(-4)^2 - (2 \times 6)}{6} = 4/6 = 2/3.$$

Suppose α and β are the roots of a quadratic equation

$$ax^2 + bx + c = 0.$$

Then, we have
$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$$
$$= x^2 + \frac{b}{a}x + a = \frac{1}{a}(ax^2 + bx + c).$$

This leads to the factorization,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

Thus, if we know the roots of a quadratic equation, then we can write factorization of the corresponding quadratic function.

Conversely, if α and β are two numbers (real or complex), then the most quadratic equation having α and β as its roots is given by

$$a(x - \alpha)(x - \beta) = 0,$$

where a is a non-zero, real or complex number: If we restrict the coefficient of x^2 to be 1 then we get a unique quadratic equation

$$(x - \alpha)(x - \beta) = 0.$$

This is the same as

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

A quadratic polynomial in which the coefficient of x^2 is 1 is called a *monic quadratic polynomial* (or simply a monic quadratic).

EXAMPLE 3. *If α and β are the roots of the equation*

$$x^2 + \sqrt{2}\,x + 3 = 0,$$

find the monic quadratic having zeros

$$\alpha^2 + \beta^2 \text{ and } 2\,\alpha\,\beta.$$

SOLUTION. The monic quadratic having $\alpha^2 + \beta^2$ and $2\alpha\beta$ as its roots is given by

$$x^2 - (\alpha^2 + \beta^2 + 2\alpha\beta)x + 2\alpha\beta(\alpha^2 + \beta^2). \qquad (*)$$

However, we know that

$$\alpha + \beta = -\sqrt{2},\ \alpha\beta = 3.$$

Hence, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 2 - 6 = -4$

This gives, $\alpha^2 + \beta^2 + 2\alpha\beta = 2$

and $2\alpha\beta(\alpha^2 + \beta^2) = -24.$

Hence, from (*) the required quadratic is

$$x^2 - 2x - 24.$$

EXAMPLE 4. *Find the values of p for which the sum of the squares of the roots of*

$$x^2 + px - 2 = 0$$

is equal to 5.

SOLUTION. If α and β are the roots of the given equation, then

$$\alpha + \beta = -p,\ \alpha\beta = -2.$$

We have to find the values of p for which

$$\alpha^2 + \beta^2 = 5.$$

This gives $5 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 + 4.$

$\therefore$ $p^2 = 1.$

Solving for p, we get $p = \pm 1.$

EXAMPLE 5. *Find the values of a for which one of the roots of*

$$x^2 + (2a + 1)x + (a^2 + 2) = 0$$

is twice the other root. Find also the roots of this equation for these values of a.

SOLUTION. We may assume that the roots of the given equation are α and 2α. Using the relations between the roots of a quadratic equation and its coefficients, we have

$$\alpha + 2\alpha = -(2a + 1)$$

and $2\alpha^2 = a^2 + 2.$

The first relation gives

$$\alpha = \frac{-(2a + 1)}{3}.$$

Substituting this value of a in the second relation, we get

$$2(2a + 1)^2 = 9(a^2 + 2).$$

This is the same as

$$a^2 - 8a + 16 = 0.$$

Thus we get a quadratic equation for a and this equation has coincident roots $a = 4$. Thus there is a unique value of a for which the conditions of the problem are fulfilled. Corresponding to this value of a, the given equation reduces to

$$x^2 + 9x + 18 = 0.$$

Now $\quad x^2 + 9x + 18 = (x + 6)(x + 3)$

so that the roots are $\quad \alpha = -6, \beta = -3.$

EXERCISE 5.4

1. Find the sum and product of roots of each of the following quadratic equations

(*a*) $x^2 + 9x - 8 = 0$ (*b*) $\sqrt{2}\,x^2 - 4x + \sqrt{8} = 0$

(*c*) $3x^2 + 9x + 4 = 0$ (*d*) $4x^2 - 8x + 2 = 0$

(*e*) $6x^2 + 7x - 3 = 0$ (*f*) $28 + 31x - 5x^2 = 0$

(*g*) $x^2 - 6(x + 12) = 0$ (*h*) $x(x + 34) + 289 = 0$

(*i*) $(1/3)x^2 - 4x + 2 = 0$ (*j*) $(0.2)x^2 - (0.3)x + (0.1) = 0$

2. If α and β are the roots, compute $\alpha^3 + \beta^3$, $\alpha/\beta + \beta/\alpha$ and $\alpha^2\beta + \alpha\beta^2$ in each case.

(*a*) $4x^2 + 2x - 9 = 0$ (*b*) $x^2 - 8x + 2 = 0$

(*c*) $2x^2 + 6\sqrt{3}\,x + 3 = 0$ (*d*) $4x^2 + \sqrt{5}\,x + 6 = 0$

(*e*) $9 - 3x - (x^2/4) = 0$ (*f*) $23x - 120 - x^2 = 0$

(*g*) $x^2 + 3x - 2(x + 7) = 0$ (*h*) $8x(1 + x) + 11x - 15 = 0.$

3. Find in each case the monic quadratic having α and β as zeros where α and β are given by

(*a*) $\alpha = \sqrt{2}, \beta = \sqrt{3}$ (*b*) $\alpha = 2\sqrt{2}, \beta = \sqrt{2}$

(*c*) $\alpha = 3, \beta = \sqrt{3}$ (*d*) $\alpha = 2 + \sqrt{2}, \beta = 2 - \sqrt{2}$

(*e*) $\alpha = 0.6, \beta = 1.2$ (*f*) $\alpha = 3 + \sqrt{3}/2, \beta = 3 - \sqrt{3}/2$

(*g*) $\alpha = 2 + 3i, \beta = 2 - 3i$ (*h*) $\alpha = \pi, \beta = e$

(*i*) $\alpha = \sqrt{2} + \sqrt{5}\,i, \beta = \sqrt{2} - \sqrt{5}\,i$ (*j*) $\alpha = \sqrt{2}\,i, \beta = -\sqrt{2}\,i$

4. Suppose the sum of the roots of

$$ax^2 - 6x + c = 0$$

is -3 and their product is 2. Find the values of a and c.

5. If one of the roots of

$$2x^2 + bx + 6 = 0$$

is 3, find the other root. Find also the value of b.

6. Find the monic quadratic with roots α and β, if

$$\alpha\beta = -2, \alpha^2 + \beta^2 = 4(\alpha + \beta).$$

7. Find a necessary and sufficient condition involving only the coefficients in order that one of the roots of

$$ax^2 + bx + c = 0, \quad a \neq 0$$

is the square of the other root.

8. Find the values of x for which the roots g and h of the equation

$$t^2 - 8t + x = 0$$

satisfy the condition that

$$g^2 + h^2 = 4.$$

Find also the roots of the equations corresponding to these values of x.

9. Solve the equation

$$x^2 + px + 10 = 0$$

given that the square of the difference of the roots is 9.

10. Given that α and β are the roots of

$$6x^2 - 5x - 3 = 0$$

find the monic quadratic whose roots are $\alpha - \beta^2$ and $\beta - \alpha^2$.

11. Let α and β be the roots of an equation

$$x^2 + px + q = 0$$

and let γ and δ be the roots of

$$x^2 + Px + Q = 0.$$

Express $(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta)$

in terms of the coefficients p, q, P and Q.

12. Find all the values of a for which the equations

$$x^2 + ax + 1 = 0 \quad \text{and} \quad x^2 + x + a = 0$$

have at least one common root.

5.5 PROBLEMS LEADING TO QUADRATIC EQUATIONS

We shall consider here various examples which ultimately lead to quadratic equations. In the world of applications it is not always that quadratic equations directly appear as the problem to be solved. Ingenuity is required to recognize them below the outer veil.

EXAMPLE 1. *Solve the equation*

$$\sqrt{x} = x - 2.$$

SOLUTION. We begin by setting $y = \sqrt{x}$. Then the given equation reduces to a quadratic equation

$$y^2 - y - 2 = 0.$$

The solutions of this equation are

$$y = 2, y = -1.$$

This gives two values of x,

$$x = 4, x = 1.$$

However, as per our convention $\sqrt{x}$ is the positive square root of x whenever $x \geq 0$. Therefore

$$x - 2 = \sqrt{x} \geq 0.$$

The only value of x satisfying this is $x = 4$.

EXAMPLE 2. *Solve the equation*

$$x^4 - 20x^2 + 64 = 0.$$

SOLUTION. Again the equation is not directly quadratic. If we set $y = x^2$, we get a quadratic equation

$$y^2 - 20y + 64 = 0.$$

The solutions are $y = 16$ and $y = 4$. Hence the solutions of the given equation are precisely those of $x^2 = 16$ and $x^2 = 4$. Therefore the solutions are ± 4 and ± 2.

EXAMPLE 3. *A two digit number is four times the sum of its digits and twice the product of its digits. Find the number.*

SOLUTION. Suppose x is in ten's place and y is in unit's place of the two digit number. Then the given number is $10x + y$. The given conditions imply that

$$10x + y = 4(x + y)$$
$$10x + y = 2xy.$$

Substituting the value of y in terms of x from the first equation into the second equation and thus eliminating y we get,

$$x^2 = 3x.$$

Solving for x, we have two solutions

$$x = 0 \quad \text{and} \quad x = 3.$$

If $x = 0$, then $y = 0$ since the first relation gives $y = 2x$. Hence $10x + y = 0$ and this is not a two digit number. If $x = 3$, then $y = 6$ and $10x + y = 36$. Hence the given number is 36.

EXAMPLE 4. *Solve the equation*

$$x + (12/x) = 8.$$

SOLUTION. We observe that α is a solution of the given equation if α is a solution of the quadratic equation

$$x^2 - 8x + 12 = 0.$$

Since we can write

$$x^2 - 8x + 12 = (x - 6)(x - 2),$$

the solutions of the given equation are 6 and 2.

EXAMPLE 5. *Solve the equation*

$$x^2 + 1/x^2 - 8(x - 1/x) + 14 = 0.$$

SOLUTION. Since $\quad x^2 + 1/x^2 = (x - 1/x)^2 + 2,$

the given equation can be written in the form

$$(x - 1/x)^2 - 8(x - 1/x) + 16 = 0.$$

If we put $\quad y = x - 1/x,$

we get a quadratic equation

$$y^2 - 8y + 16 = 0.$$

It has a repeated root $y = 4$. Thus we get the equation

$$x - 1/x = 4.$$

This is the same as

$$x^2 - 4x - 1 = 0.$$

The discriminant of the equation is $D = 20 > 0$. Hence the equation has the following real roots:

$$\alpha = 2 + \sqrt{5},\ \beta = 2 - \sqrt{5}.$$

These are precisely the solutions of the given equation.

EXAMPLE 6. *A right angled triangle is such that its hypotenuse is 1 cm. longer than its base and the altitude is 1 cm. shorter than half the base. Find the base, altitude and hypotenuse.*

SOLUTION. Let us denote the lengths of the base, the altitude and the hypotenuse by x, y and h respectively. The given conditions imply that

$$y = (x/2) - 1 \text{ and } h = x + 1.$$

Using Pythagoras's theorem, we get a relation between x, y and h;

$$h^2 = x^2 + y^2.$$

This reduces to $(x+1)^2 = x^2 + ((x/2) - 1)^2$.

Simplification gives $x^2 - 12x = 0$.

Thus $x = 0$ or $x = 12$. We can reject $x = 0$, since the base of a triangle cannot be '0'. Hence $x = 12$ cm. and this leads to $h = 13$ cm., $y = 5$ cm.

EXERCISE 5.5

1. Solve the following equations

(*a*) $x^4 - 10x^2 + 9 = 0$ (*b*) $x^4 - 6x^2 + 1 = 0$

(*c*) $(1-x)(x+2)(x+3) = 9x^2 - x^3 + 4(2-7x)$

(*d*) $x^3 - x^2 + x - 1 = 0$ (*e*) $x - 7/x = 6$

(*f*) $x + 2/x = 2\sqrt{2}$ (*g*) $x^2 - 8/x^2 = 7$

(*h*) $\sqrt{x} + \sqrt{1+2x} = 1$ (*i*) $\sqrt{4-x} + \sqrt{x+9} = 5$

(*j*) $\sqrt{3x^2+10} + \sqrt{6-x^2} = 6$ (*k*) $(x^2 + 1/x^2) - 4(x + 1/x) + 6 = 0$

(*l*) $(x^2 + 1/x^2) - 8(x - 1/x) + 13 = 0$.

2. The number of square centimetres in the area of a rectangle is the same as the number of centimetres in the perimeter. If the diagonal is $3\sqrt{5}$ cm., find its sides.

3. The sum of two numbers is 6 and the sum of their reciprocals is 3/4. Find the numbers.

4. The sum of an integer and its reciprocal is 10/3. What is the integer?

5. The sum of squares of two numbers is equal to 5 times their sum. The sum of the reciprocals of the squares of these two numbers is equal to 5/18 times the sum of the reciprocals of the given numbers. Find these numbers.

6. The product of two consecutive even integers is equal to 24. Find these integers.

7. Suppose the sides of a right angled triangle are x, $x + 7$ and $x + 8$. Find the area of the triangle.

8. Solve the equation

$$2/x + 5/(x+2) = 9/(x+4).$$

9. A farmer has a rectangular garden of total area 80 sq. meters. He requires 36 meters of barbed wire for fencing it. Find the dimensions of the garden.

10. Determine the values of k for which the equation

$$\frac{x^2 + x + 2}{3x + 1} = k$$

has both roots real.

11. Find all integers a such that

$$(x - a)(x - 12) + 2$$

can be factored as $(x + b)(x + c)$ where b and c are integers.

5.6 BEHAVIOUR OF QUADRATIC FUNCTIONS

A function of the form

$$f(x) = ax + b, a \neq 0 \quad (1)$$

is called a *linear function*. As we shall observe in chapter 7, the graph of such an expression always represents a straight line on a coordinate plane. As such, its behaviour is completely determined. If this line meets the x-axis at some point x_0, and if $f(x_1) < 0$ for some $x_1 < x_0$, then we have

$$ax_0 + b = 0$$

$$ax_1 + b < 0$$

and these imply $\quad ax_0 > ax_1$.

But since $x_1 < x_0$, we must have $a > 0$. This in turn implies

$$f(x) = ax + b < ay + b = f(y)$$

for any $x < y$. Thus f is an increasing function (see Fig. 5.6).

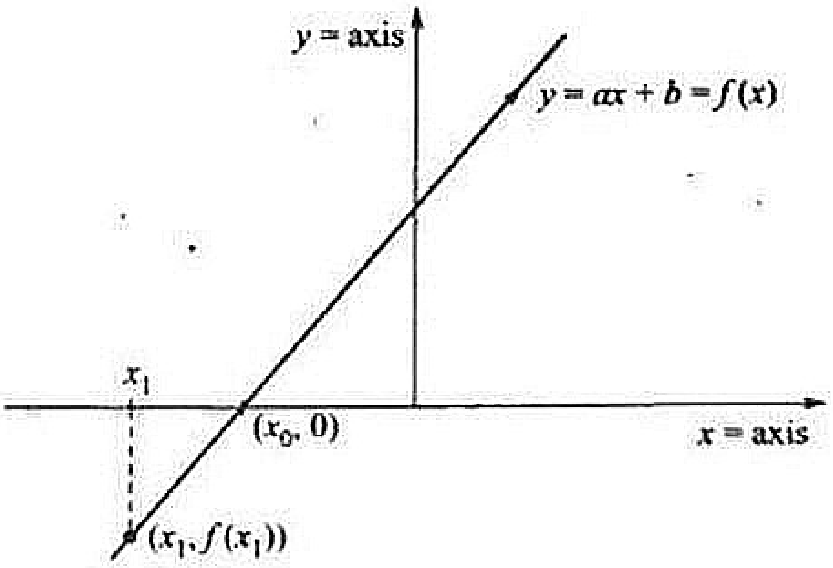


Fig. 5.6

Similarly, if $f(x_1) > 0$ for some $x_1 < x_0$ then f is a decreasing function. The student is advised to carry out the argument.

A function of the form

$$f(x) \equiv ax^2 + bx + c, \quad a \neq 0 \tag{2}$$

is called a *quadratic expression* or a quadratic polynomial. Let us consider the polynomial.

$$3x^2 + 2x + 1.$$

The discriminant of this polynomial is $D = -8$. Hence this polynomial has no real zeros. So the graph of the function

$$f(x) = 3x^2 + 2x + 1$$

does not meet the x-axis. This implies that the graph of $f(x)$ lies completely in the upper half-plane or lies completely in the lower half-plane. In turn, we conclude that $f(x)$ is positive for all values of x or negative for all values of x. But $f(0) = 1 > 0$. Therefore, $f(x) > 0$ for all values of x.

On the other hand, let us consider the polynomial

$$3x^2 + 2x - 1.$$

Its discriminant is 16 so that it has real zeros. These are

$$\alpha = -1, \beta = 1/3$$

Thus the graph of

$$g(x) = 3x^2 + 2x - 1$$

cuts the x-axis at -1 and $1/3$. Since $g(x)$ has no other zeros, the graph of $g(x)$ does not meet the x-axis at any other point (see Fig. 5.7).

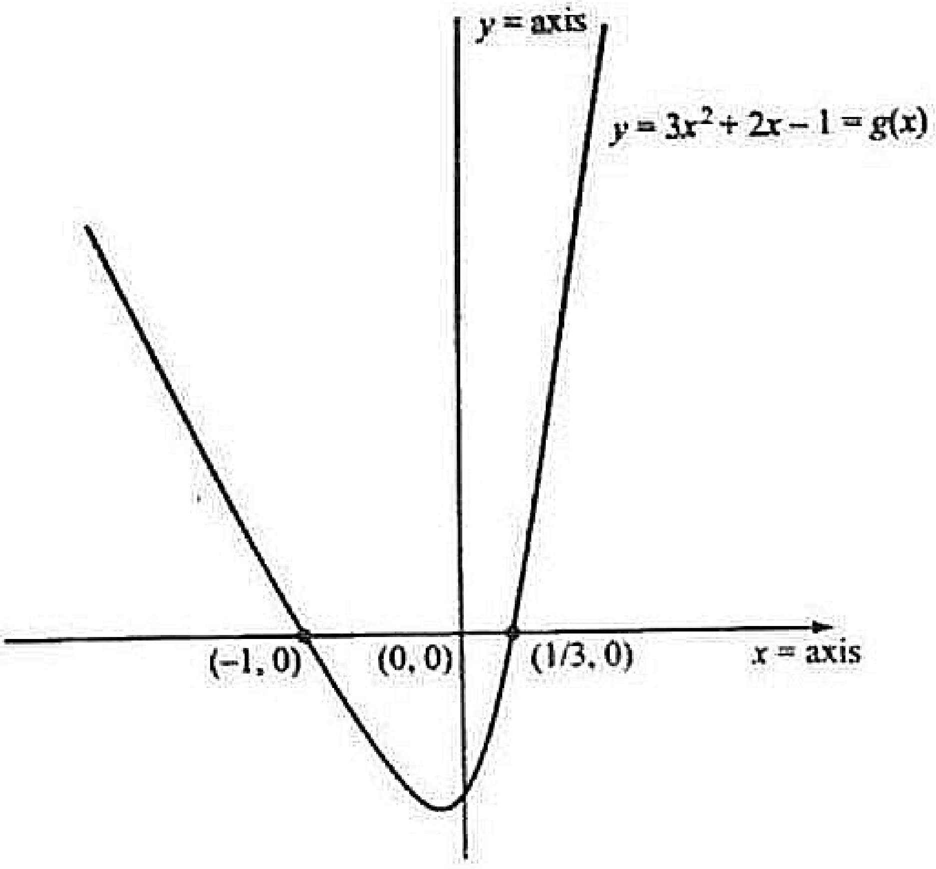


Fig. 5.7

Thus the graph of $g(x)$ passes from one y half-plane to the other y half-plane at both α and β. Since $g(x)$ has no zeros between α and β, the graph of $g(x)$ remains in the same y half-plane for $\alpha < x < \beta$. The factorization

$$3x^2 + 2x - 1 = 3(x - 1/3)(x + 1)$$

shows that

$$g(x) < 0 \quad \text{if } -1 < x < 1/3,$$
$$g(x) > 0 \quad \text{if } x < -1 \text{ or } x > 1/3.$$

Thus the graph of $g(x)$ lies in the lower half-plane for $-1 < x < 1/3$ and it lies in the upper half-plane if either $x > 1/3$ or $x < -1$. We infer that the graph of $g(x)$ passes from positive to negative at α and from negative to positive at β, as we move from left to right along the x-axis.

This type of behaviour is true of any quadratic polynomial. Let us, begin with the general quadratic polynomial

$$f(x) = ax^2 + bx + c, \; a \neq 0. \tag{3}$$

Let α and β be the roots of the equation

$$ax^2 + bx + c = 0. \tag{4}$$

Suppose the discriminant D given by

$$D = b^2 - 4ac$$

is non negative. Then the equation (4) has real roots; α and β are real in this case. Assume $\alpha < \beta$. Then the equation (4) has no roots either between α and β, or before α, or after β. Hence the graph of $f(x)$ cuts the x-axis at α and β, and at no other point. This implies that the graph of $f(x)$ lies in one of the upper and lower half-planes for $x < \alpha$ and $x > \beta$, and lies in the opposite half-plane for $\alpha < x < \beta$.

Now we can write

$$f(x) = a(x - \alpha)(x - \beta) \tag{5}$$

and therefore

$$af(x) > 0 \quad \text{if either } x < \alpha \text{ or } x > \beta,$$
$$af(x) < 0 \quad \text{if } \alpha < x < \beta.$$

This determines the sign of $f(x)$ provided we know the sign of a. If $a > 0$, then $f(x) > 0$ if $x < \alpha$ or if $x > \beta$ and $f(x) < 0$ if $\alpha < x < \beta$. Thus if $a > 0$, then the graph of $f(x)$ passes from the upper half-plane to the lower half-plane at α and passes from the lower half-plane to the upper half-plane at β. Similarly, in the case $a < 0$, the behaviour of the graph of $f(x)$ is reversed. Thus the sign of a completely determines the behaviour of the quadratic expression (3), provided the equation (4) has only real roots.

Suppose $D < 0$. Then the equation (4) has two roots α and $\overline{\alpha}$, where $\overline{\alpha}$ is; the complex conjugate of α. If α is of the form

$$\alpha = s + it$$

where s and t are real numbers, then

$$\overline{\alpha} = s - it.$$

We can write again

$$\begin{aligned} f(x) &= a(x - \alpha)(x - \overline{\alpha}) \\ &= a(x - s - it)(x - s + it) \\ &= a\{(x - s)^2 + t^2\}. \end{aligned}$$

The expression in the braces is positive for every value of x. Hence $f(x) > 0$ for all x if $a > 0$, and $f(x) < 0$ for all x if $a < 0$. This shows that the sign of $f(x)$ is completely determined by the sign of a. We record these observations.

Any quadratic expression $f(x) \equiv ax^2 + bx + c$ has the same sign between its real zeros and changes sign only when the graph of $f(x)$ passes through any of its real zeros. If $f(x)$ has no real zeros, then $f(x)$ has the same sign for all real values and the sign of $f(x)$ is determined by the sign of the coefficient a.

If the equation

$$ax^2 + bx + c = 0$$

has real roots α and β, then we have the factorization

$$ax^2 + bx + c = a(x - \alpha)(x - \beta). \tag{6}$$

Thus the quadratic polynomial $ax^2 + bx + c$ can be written as a product of linear factors, each linear factor having only real coefficients. If the given equation has no real roots, then a factorization of the form (8) with α and β real is impossible. Nevertheless, we can find a factorization of the form (8) with complex α and β (In fact $\beta = \overline{\alpha}$). If the quadratic expression $ax^2 + bx + c$ is such that it has no factorization of the form (8) with real α and β, then we say the polynomial $ax^2 + bx + c$ is, *irreducible* over **R**. Since the given equation has real roots if and only if the discriminant is non-negative, we conclude that *$ax^2 + bx + c$ is irreducible over* **R** *if and only if the discriminant $D = b^2 - 4ac$ is negative.*

All the relevant properties of the quadratic polynomial $f(x) = ax^2 + bx + c$ are given in Table 5.1.

TABLE 5.1

Sl. NO.	a	D	roots of $f(x) = 0$	Sign of $f(x)$
1.	$a > 0$	$D > 0$	Real roots $\alpha < \beta$	$f(x) > 0$ for $x < \alpha$ < 0 for $\alpha < x < \beta$ > 0 for $x > \beta$
2.	$a > 0$	$D = 0$	Real roots $\alpha = \beta$	$f(x) > 0$ for $x < \alpha$ $= 0$ for $x = \alpha$ > 0 for $x > \alpha$
3.	$a > 0$	$D < 0$	Complex roots α, $\overline{\alpha}$	$f(x) > 0$ for all x
4.	$a < 0$	$D > 0$	Real roots $\alpha < \beta$	$f(x) < 0$ for $x < \alpha$ > 0 for $\alpha < x < \beta$ < 0 for $x > \beta$
5.	$a < 0$	$D = 0$	Real roots $\alpha = \beta$	$f(x) < 0$ for $x < \alpha$ $= 0$ for $x = \alpha$ < 0 for $x > \alpha$
6.	$a < 0$	$D < 0$	Complex roots α, $\overline{\alpha}$	$f(x) < 0$ for all x

EXAMPLE 1. *Find the values of x for which the inequality*

$$x^2 - x - 2 < 0$$

is true.

SOLUTION. We observe that $x^2 - x - 2 = (x - 2)(x + 1)$.

Hence the inequality is true if the values of the two factors have opposite signs. Thus either $x - 2 < 0$, $x + 1 > 0$ or $x - 2 > 0$, $x + 1 < 0$. There is no x for which $x - 2 < 0$, $x + 1 > 0$ is true. Hence the set of values of x for which the given inequality is true is $-1 < x < 2$. We can also infer this from Table 1. The leading coefficient $a > 0$ and the discriminant $D > 0$. The zeros of $x^2 - x - 2$ are -1 and 2. Hence $x^2 - x - 2 < 0$ only between the zeros -1 and 2; *i.e.*, $-1 < x < 2$.

EXAMPLE 2. *Find all the values of x for which the inequality*

$$\frac{x^2 - 2x - 1}{x + 1} < x$$

holds.

SOLUTION. The given inequality is equivalent to

$$\frac{x^2 - 2x - 1}{x + 1} - x < 0.$$

Hence it is sufficient to find those x for which

$$\frac{-3x - 1}{x + 1} < 0$$

is true. Equivalently, it is sufficient (Why ?) to consider the inequality

$$-(3x + 1)(x + 1) < 0.$$

Hence either both $(3x + 1)$ and $(x + 1)$ must be positive or both must be negative. Hence the set of values of x for which the given inequality holds is $x > -1/3$ and $x < -1$.

EXERCISE 5.6

Determine in each of the following problems the set of values of x for which the given inequality is true.

1. $x^2 + 4x + 12 > 0$
2. $x^2 - 4x + 3 < 0$
3. $9x^2 - 6x + 1 \geq 0$
4. $(1/4)x^2 - 3(x + 5) < 0$
5. $x(x - 3) - 2 < 3x - (x^2 + 2)$
6. $x(x - 1) - 6 > 5x - x^2$
7. $21 - 7(2x - 9) > 7x^2$
8. $1/3\ (x - 2) < 3/5(x^2 + 4/3)$
9. $3x^2 - 5x - 8 \leq 0$
10. $(x^2 - 16x)^2 - 63 \geq 2(x^2 - 16x)$
11. $(x^2 - 4)(x^2 - x - 2) \leq 0$
12. $\dfrac{(x-1)(x-7)}{x+4} > 0$
13. $\dfrac{2x-1}{(x+4)(x-3)(5-x)} \geq 0$
14. $\dfrac{x^2 - 7x + 12}{2x^2 + 4x + 5} > 0$
15. $2 + \dfrac{7x-5}{8x+3} \leq 0$
16. $\dfrac{x^2 - 9}{3x - x^2 - 24} < 0$
17. $\dfrac{x-1}{x} - \dfrac{x+1}{x-1} < 2$
18. $\dfrac{1}{x-2} + \dfrac{1}{x-1} > \dfrac{1}{x}$
19. $\dfrac{3}{x+1} + \dfrac{7}{x+2} \leq \dfrac{6}{x-1}$
20. $|x^2 + 3x| \geq 2 - x^2$
21. $\dfrac{4}{|x+3| - 1} \geq |x + 2|$

PROBLEMS

1. Let $p(x)$ and $q(x)$ be two quadratic polynomials with integer coefficients. Suppose they have a non-rational zero in common. Show that

$$p(x) = rq(x)$$

for some rational number r.

2. Let a, b, c be integers, and suppose the equation

$$f(x) = ax^2 + bx + c = 0$$

has an irrational root r. Let $u = p/q$ be any rational number such that $|u - r| < 1$. Prove that

$$\left(\frac{1}{q^2} \leq |f(u)| \leq K|u - r|\right)$$

for some constant K. Deduce that there is a constant M such that

$$\left| r \frac{p}{q} \right| \geq M/q^2$$

(This is useful in approximating the non-rational zero of a polynomial by a rational number)

3. Find the value of the positive integer n for which the quadratic equation

$$\sum_{k=1}^{n} (x+k-1)(x+k) = 10n$$

has solutions α and $\alpha + 1$ for some α.

4. Let $p(x)$ be a monic quadratic polynomial over $\mathbf{Z}$. Show that, for any integer n, there exists an integer k such that

$$p(n)p(n+1) = p(k).$$

5. If the equations

$$x^2 + bx + c = 0, \quad bx^2 + cx + 1 = 0$$

have a common root, prove that either

$$b + c + 1 = 0$$

(or) $$b^2 + c^2 + 1 = bc + b + c.$$

6. If the roots of the equation $x^2 + bx + c = 0$ are real, show that the roots of the equation

$$x^2 + bx + c(x+a)(2x+b) = 0$$

are again real for every real number a.

7. Find all positive integers n for which the quadratic equation

$$a_{n+1}x^2 - 2x\sqrt{\sum_{k=1}^{n+1} a_k^2} + \sum_{k=1}^{n} a_k = 0$$

has real roots for every choice of real numbers $a_1, a_2, \ldots, a_{n+1}$.

8. Let the polynomial $p(z) = z^2 + az + b$ be such that a and b are complex numbers and $|p(z)| = 1$ whenever $|z| = 1$. Prove that $a = 0$ and $b = 0$.

9. Find necessary and sufficient conditions on the coefficients a, b, w so that the roots of the equations

$$z^2 + 2az + b = 0, \quad z - w = 0$$

are collinear in the plane.

10. Find necessary and sufficient conditions an a, b, c, d so that the equations.

$$z^2 + az + b = 0, \quad z^2 + cz + d = 0$$

are collinear in the plane.

11. Determine all quadratic polynomials $p(x)$ with complex coefficients such that the $\alpha + i\beta$ is a zero of $p(x)$ iff $-\alpha + i\beta$ is a zero of $p(x)$ (Here α and β are real numbers).

12. Let $ax^2 + bx + c$ be a quadratic polynomial with real coefficients such that

$$|ax^2 + bx + c| \le 1 \quad \text{for} \quad 0 \le x \le 1.$$

Prove that

$$|a| + |b| + |c| \le 17. \tag{*}$$

Show that the equality can hold in (*) by constructing an example.

13. Let $p(x)$ be a quadratic polynomial such that for distinct reals α and β,

$$p(\alpha) = \alpha, \; p(\beta) = \beta$$

show that α and β are roots of

$$p[p(x)] - x = 0$$

and find the remaining roots.

14. Solve the equation $\sqrt{a - \sqrt{a + x}} = x$

15. Find the real roots of

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$$

16. Let $p(x)$ be a quadratic polynomial

$$p(x)=ax^2+bx+c$$

such that $|p(x)|\le 1$ for $|x|\le 1$. Prove that

$$|cx^2+bx+a|\le 2$$

for $|x|\le 1$.

17. Suppose α is a real root of the equation

$$ax^2+bx+c=0$$

and β is a real root of the equation

$$-ax^2+bx+c=0$$

show that the equation

$$(a/2)x^2+bx+c=0$$

has a root lying between α and β.

18. Let $p(x)=ax^2+bx+c$ be such that $p(x)$ takes real values for real values of x and nonreal values for nonreal values of x. Prove that $a=0$.

19. Consider the number

$$\alpha=\left(\frac{n+\sqrt{n^2-4}}{2}\right)^m$$

where $n\ge 2$ and m are natural numbers. Prove that

$$\alpha=\frac{k+\sqrt{k^2-4}}{2}$$

for some natural number k.

20. Let $p(x)=ax^2+bx+c$ be a polynomial in $\mathbf{R}(x)$ such that $|p(\alpha)|\le 1$ for $|\alpha|\le 1$.
Prove that $\qquad |2a\alpha+b|\le 4$ for $|\alpha|<1$.

21. If a and b are positive reals, prove that

$$\frac{1}{x}+\frac{1}{x-a}+\frac{1}{x-b}=0$$

has two real roots, one between $a/3$ and $2a/3$, and another between $-2b/3$ and $-b/3$

22. If a, b, c, p, q, r are real numbers such that

$$ax^2+bx+c\ge 0,$$
$$px^2+qx+r\ge 0$$

for all real numbers prove that

$$apx^2+bqx+cr\ge 0$$

for all real x.

CHAPTER

6

TRIGONOMETRY

6.1 INTRODUCTION

Trigonometry means 'measurement of Triangles'. It is a word derived from Gonia, a Greek word, meaning, an angle. This science was nurtured on Indian soil for a number of centuries by Hindu scholars like Aryabhata (5th Cen. A.D.), Varahamihira (6th Cen. A.D.), Brahmagupta (7th Cen. A.D.) and Bhaskara (12th Cen. A.D.); but later it passed on to the west through the Arabs. After the 16th Century modern Trigonometry took shape out of the works of European mathematicians like Vieta (16th Cen. A.D.), Euler (18th Cen. A.D.) and others. Though originally it was confined to a study of the relations between the sides and angles of a triangle, in modern times, it has given rise to mathematical functions of angular magnitudes, through the medium of which many kinds of geometrical and algebraic investigations are carried out in every branch of Mathematics and its application. We shall begin the study of Trigonometry by a comment on one of the most fascinating numbers in Mathematics, namely, the irrational number, π.

Take a circle of any size. Measure, if you can, the length l of the circumference of the circle. Measure also the diameter d of the circle. The ratio l/d is always the same whatever be the circle. This is a mathematical fact, but the proof of this is beyond the scope of this book. This constant number l/d or

$$\frac{\text{Circumference of a circle}}{\text{Diameter of the same circle}}$$

is denoted by π. Its approximate values are

$$\frac{22}{7}, \frac{355}{113}, 3.1416, \textit{etc.}$$

Consider now a circle, centre O, radius r. Let A be a fixed point on the circle and P a variable point. Suppose P is initially at A and moves along the circle, say in the anticlockwise direction. By the symmetry of the circle, we may assume that the angle AOP subtended by arc AP at O is proportinal to the length of the arc itself (we cannot assume such a thing, for example, for an ellipse, in which equal arcs generally subtend unequal angles at the centre).

Definition. In a circle, the angle subtended at the centre by an arc of length equal to the radius of the circle is called a *radian*.

Consider a circle centre O and radius r (see Figure 6.1). The circumference of the circle itself can be considered as an arc of the circle and since its length is π × diameter, that

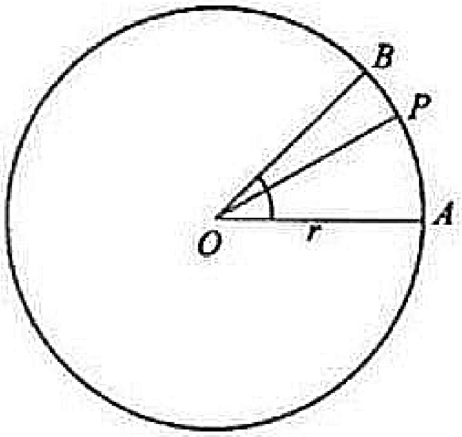


Fig. 6.1

is, $2\pi r$, it subtends an angle of 2π radians at the centre. But this angle is also equal to 360 degrees. Hence

$$2\pi \text{ radians} = 360 \text{ degrees},$$

or

$$\pi \text{ radians} = 180 \text{ decrees}.$$

Notation x radians is denoted by the symbol x^c.

Thus $1^c = \dfrac{180^\circ}{\pi}$ and $1^\circ = \dfrac{\pi^c}{180}$. Here '$c$' stands for circular measure:

From the first of these relations, it follows that one radian is $57^\circ\ 17'\ 45''$ approximately. [1 degree = 60 minutes = $60'$ and 1 minute = 60 seconds = $60''$].

Recall the representation of angles in the xy-plane, of various sizes, and of both positive and negative directions, described in the beginning of Chapter 3. In particular, also note that if two or more angles have the same final position in such a representation (*e.g.*, 40°, 400°, and -320°), they are said to be *coterminal.*

CONVENTION : In connection with angles, if no unit of measurement is mentioned, the radian is to be understood,

EXERCISE 6.1

1. Let in a circle, radius r, centre O, an arc AB of length l subtend an angle θ^c at the centre. Show that (a) $l = r\theta$ and (b) area of sector $OAB = (1/2)\, r^2\theta$. What are the corresponding formulae if angle $AOB = \theta^\circ$?
2. Two cities lying on the equator are separated by a distance of 120 miles. What is the longitudinal difference between them. (Radius of the equator may be taken as 4000 miles.)?
3. What is the angle in radians between the hands of a clock when the time is (a) 3.20 (b) 4.20?
4. Assume that the moon's radius is 1800 km. If it subtends an angle of $32'$ at the eye, what is its distance from the observer? What assumptions are you making?

6.2 TRIGONOMETRIC FUNCTIONS OR RATIOS

Let θ be any angle positive, negative or zero represented in the xy-plane by angle AOB.

Choose a point P on this final position $\overrightarrow{OB}$, $P \neq O$ and let $P = (x, y)$, $OP = r$. We shall take r to be positive always. The *Trigonometric (or circular) functions* of θ are defined as follows:

$$\text{sine } \theta = \frac{y}{r}, \text{ cosine } \theta = \frac{x}{r},$$

$$\text{tangent } \theta = \frac{y}{x}, \text{ cotangent } \theta = \frac{x}{y},$$

$$\text{secant } \theta = \frac{r}{x}, \text{ cosecant } \theta = \frac{r}{y}.$$

These functions are abbreviated to sin θ, cos θ, tan θ, cot θ, sec θ and cosec θ. When $x = 0$, tan θ and sec θ are undefined. When $y = 0$, cot θ and cosec θ are undefined.

When θ is an acute angle we can define these functions in terms of the sides of a right triangle (figure 6.2):

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PQ}{OP} = \frac{y}{r},$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{OQ}{OP} = \frac{x}{r},$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PQ}{OQ} = \frac{y}{x},$$

$$\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{OQ}{PQ} = \frac{x}{y},$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{OP}{OQ} = \frac{r}{x},$$

$$\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{OP}{PQ} = \frac{r}{y}.$$

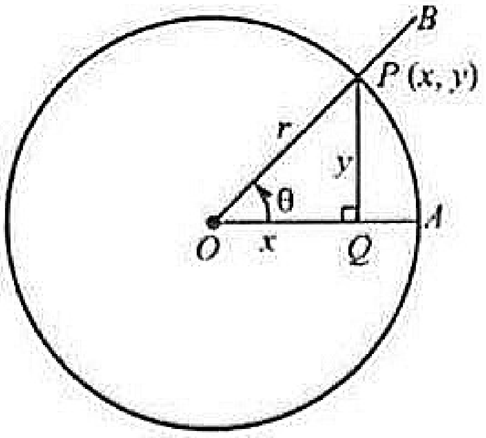


Fig. 6.2

Notation. The powers $(\sin\theta)^n$, $(\cos\theta)^n$ ... are usually written as $\sin^n\theta$, $\cos^n\theta$,...

Basic Identities satisfied by the circular functions:

If θ is any angle,

I. (*a*) $\sin\theta \operatorname{cosec}\theta = 1$, (*b*) $\cos\theta \sec\theta = 1$,

(*c*) $\tan\theta \cot\theta = 1$, (*d*) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

(*e*) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

II. (*a*) $\sin^2\theta + \cos^2\theta = 1$, (*b*) $1 + \tan^2\theta = \sec^2\theta$,
(*c*) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

These relations make sense only for those values of θ for which the functions involved are defined.

Proof. I (*a*) – (*e*) follow from the definitions.

II follows from the relation $x^2 + y^2 = r^2$, which is obtained from Pythagoras's Theorem. (See Fig. 6.2).

EXERCISE 6.2

1. Show that for any angle θ, $|\sin\theta| \le 1$, $|\cos\theta| \le 1$, $|\sec\theta| \ge 1$, $|\operatorname{cosec}\theta| \ge 1$.
2. Find the set of values of θ for which the following ratios are not defined, (*a*) $\tan\theta$ (*b*) $\cot\theta$ (*c*) $\sec\theta$ (*d*) $\operatorname{cosec}\theta$.
3. Show that the trigonometrical ratios of coterminal angles are equal.
4. Find the ratios of 0, 90°, 180°, 270°, 360° and verify that your values agree with the following table:

TABLE 6.1

angles / *ratios*	*0*	*90°*	*180°*	*270°*	*360°*
sine	0	1	0	– 1	0
cosine	1	0	– 1	0	1
tangent	0	not defined	0	not defined	0
cotangent	not defined	0	not defined	0	not defined
secant	1	not defined	– 1	not defined	1
cosecant	not defined	1	not defined	– 1	not defined

Prove the identities (5) – (12)

5. $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \,.\, \operatorname{cosec}^2\theta = (\tan\theta + \cot\theta)^2$.
6. $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 5 + (\tan\theta + \cot\theta)^2$.
7. $\dfrac{1-\cos A}{1+\cos A} = (\operatorname{cosec} A - \cot A)^2$.
8. $\dfrac{\tan A - \sec A + 1}{\tan A + \sec A - 1} = \dfrac{1-\sin A}{\cos A}$.
9. $(1 + \sin x + \cos x)^2 = 2(1 + \sin x)(1 + \cos x)$.
10. $\dfrac{\sin^2\theta}{1-\cot\theta} + \dfrac{\cos^2\theta}{1-\tan\theta} = 1 + \sin\theta\cos\theta$.
11. $\dfrac{\sin^4\theta}{1-\cot\theta} + \dfrac{\cos^4\theta}{1-\tan\theta} = 1 + \sin\theta\cos\theta - \sin^2\theta\cos^2\theta$
12. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$.
13. Consider an isosceles right triangle *OAB*, in which $\angle AOB = 90°$, $OA = OB = a$. Use Pythanoras's theorem to evaluate *AB* and hence find the rations of 45°. The values are given in Table 6.2.

14. Consider an equilateral triangle *ABC* in which each side is $2a$. Draw *AD* perpendicular to *BC*. Use Pythagoras's theorem to find *AD* and hence find the ratios of 30° and 60°. The values are given in Table 6.2.

TABLE 6.2

Ratio *Angle*	*sine*	*cosine*	*tangent*	*cotangent*	*secant*	*cosecant*
30°	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
45°	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$

15. Show that the definitions of the ratios are independent of the choice of the point *P* on $\vec{OB}$ (figure 6.2).

16. Find the signs of the six trigonometric ratios of angles in different quadrants and verify that the signs agree with the following table.

TABLE 6.3

Quadrant	*Sine*	*Cosine*	*Tangent*	*Cotangent*	*Secant*	*Cosecant*
First quadrant	+	+	+	+	+	+
Second quadrant	+	−	−	−	−	+
Third quadrant	−	−	+	+	−	−
Fourth quadrant	−	+	−	−	+	−

17. If $\cos\theta = k$ and θ is an angle in the second quadrant, determine the remaining ratios of θ.

18. If $\tan\theta = \dfrac{-3}{4}$ and $3\pi/2 < \theta < 2\pi$, evaluate $\dfrac{16 - 2\sin\theta + \cos\theta}{13 + 4\sec\theta + 6\operatorname{cosec}\theta}$.

6.3 TRIGONOMETRICAL RATIOS OF 90° ± θ, 180° ± θ, 270° ± θ, 360° ± θ, − θ

In figures 6.3, 6.4, 6.5 and 6.6, the position of $90° - \theta$ is shown for various positions of θ.

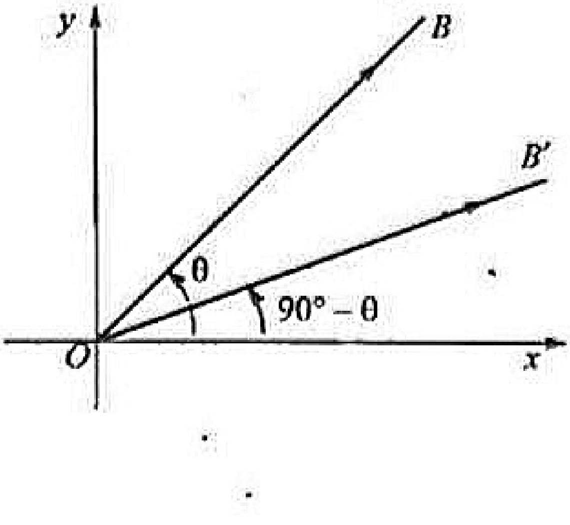


Fig. 6.3

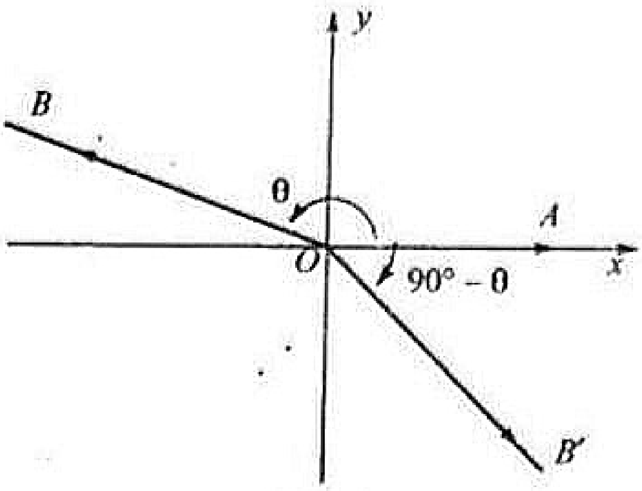


Fig. 6.4

The reader is advised to draw similar figures showing positions of 90° + θ, 180° ± θ, 270° ± θ, 360° ± θ, – θ for various positions of θ, *i.e.*, for positions of θ in each of the four quadrants. Now we wish to express the ratios of 90° – θ in terms of those of θ themselves.

Let us take the case when $90° < \theta < 180°$, that is, when θ is in the second quadrant. If angle AOB represents θ, then angle $AOB' = 90° - \theta$ is in the fourth quadrant as shown in Figure 6.7. If P and Q are points on $\overrightarrow{OB}$ and $\overrightarrow{OB'}$ respectively such that $OP = OQ = r$, and P' and Q' are the feet of perpendiculars from P and Q respectively to the y-axis and x-axis, then it is easy to see that $\Delta P'OP$ is congruent to $\Delta Q'OQ$. So $|OP'| = |OQ'|$ and $|PP'| = |QQ'|$. Consequently if $P = (x, y)$, then $Q = (y, x)$. Hence

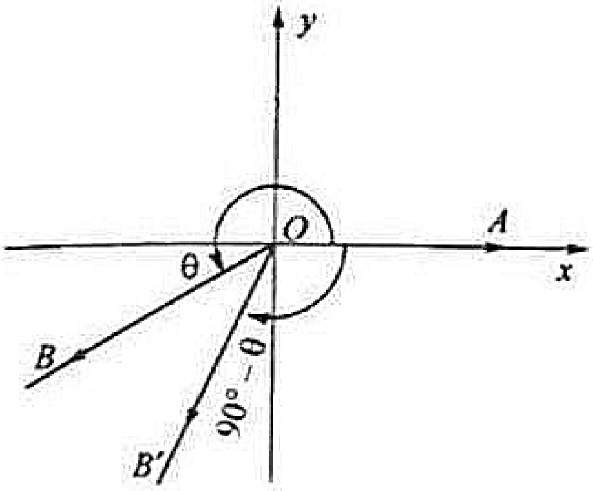


Fig. 6.5

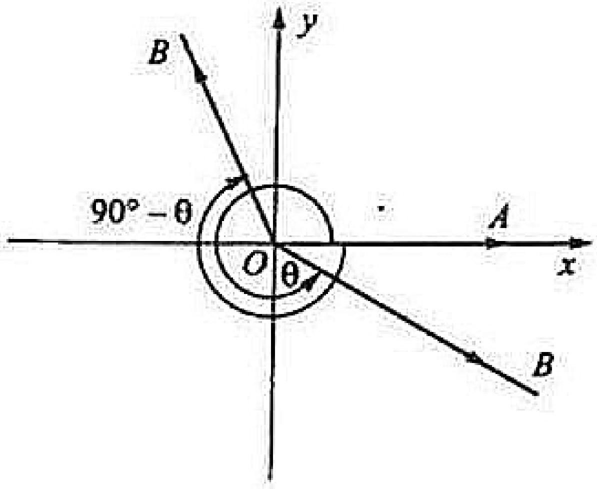


Fig. 6.6

$$\sin(90° - \theta) = x/r = \cos\theta,$$
$$\cos(90° - \theta) = y/r = \sin\theta,$$
$$\tan(90° - \theta) = x/y = \cos\theta,$$
$$\cot(90° - \theta) = y/x = \tan\theta,$$
$$\sec(90° - \theta) = r/y = \operatorname{cosec}\theta,$$
$$\operatorname{cosec}(90° - \theta) = r/x = \sec\theta.$$

Just using the relation sin (90° – θ) = cos θ, one can deduce the other five relations (how?). The same relations can be proved to be true, when θ is in any other quadrant. Similarly the ratios of 90° + θ, 180° ± θ, 270° ± θ, 360° ± θ, (– θ) can be obtained in terms of those of θ. Table 4 helps one to read off these ratios.

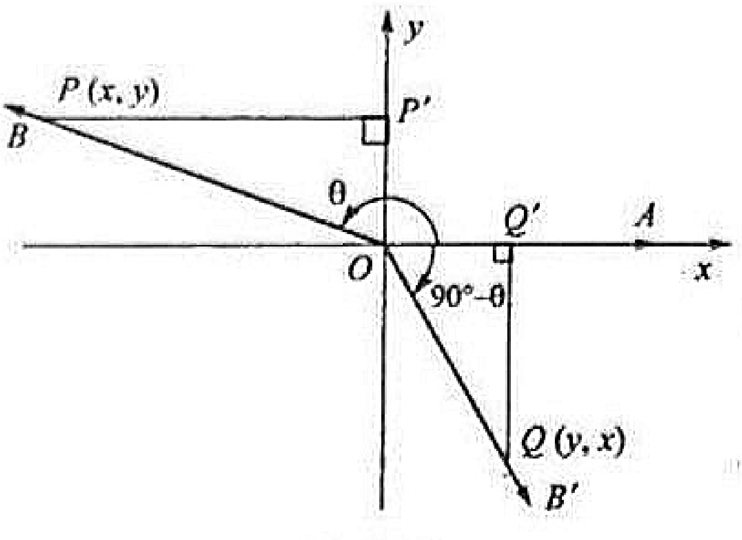


Fig. 6.7

The ratios of $360° + \theta$ and θ are the same since these two angles are co-terminal and so are the ratios of $360° - \theta$ and $-\theta$ for the same reason. The reader is advised to prove the validity of all these relations for at least one position of θ. If we look at the table, we see that ***the ratio itself changes in the presence of 90° and 270°*** and does not in the other cases. Further the sign depends on the angle we are considering.

The table also helps us to reduce the ratios of any angle to those of an acute angle; in fact to those of an angle θ where $0 \le \theta \le 45°$ (how?).

EXAMPLE 1. (*a*) $\sin 135° = \sin (180° - 45°) = \sin 45° = \frac{1}{\sqrt{2}}$.

(*b*) $\sec 240° = \sec (270° - 30°) = -\operatorname{cosec} 30° = -2$.

(*c*) $\tan (\theta - 180°) = -\tan (180° - \theta) = \tan \theta$.

(*d*) If A, B, C are the angles of a triangle, then $\cos (A + B) = \cos (\pi - C)$ $= -\cos C$ and

$$\sin\left(\frac{B+C}{2}\right) = \sin (\pi/2 - A/2) = \cos, A/2.$$

(*e*) $\sin (-1230°) = -\sin 1230° = -\sin 150°$.

$$= -\sin (180° - 30°) = -\sin 30° = -\frac{1}{2}.$$

TABLE 6.4

Ratio *Angle*	*sine*	*cosine*	*tangent*	*cotangent*	*secant*	*cosecant*
$90° - \theta$	$\cos\theta$	$\sin\theta$	$\cot\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$
$90° + \theta$	$\cos\theta$	$-\sin\theta$	$-\cot\theta$	$-\tan\theta$	$-\operatorname{cosec}\theta$	$\sec\theta$
$180° - \theta$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$	$-\cot\theta$	$-\sec\theta$	$\operatorname{cosec}\theta$
$180° + \theta$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$	$\cot\theta$	$-\sec\theta$	$-\operatorname{cosec}\theta$
$270° - \theta$	$-\cos\theta$	$-\sin\theta$	$\cot\theta$	$\tan\theta$	$-\operatorname{cosec}\theta$	$-\sec\theta$
$270° + \theta$	$-\cos\theta$	$\sin\theta$	$-\cot\theta$	$-\tan\theta$	$\operatorname{cosec}\theta$	$-\sec\theta$
$360° - \theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\operatorname{cosec}\theta$
$360° + \theta$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\operatorname{cosec}\theta$

EXERCISE 6.3

1. Simplify

(*a*) $\dfrac{\sin(180° + \theta)\cos(270° - \theta)\cot(\theta - 360°)}{\cos(\theta - 90°)\sin(360° - \theta)\tan(270° + \theta)}$.

(*b*) $\dfrac{\sin^2(90° - \theta) + \sin^2(90° + \theta) - 1}{1 - \cos^2(270° - \theta) - \cos^2(270° + \theta)}$.

(*c*) $\dfrac{\sin(540° - A)\cos(-90° + A)\tan(270° - A)}{\operatorname{cosec}(1170° + A)\sec(540° + A)\cot(-90° - A)}$.

2. Express as trigonometric ratios of acute angles and determine their values.
   (*a*) sin 1410° (*b*) cos (– 2040°) (*c*) tan (– 510°).
   (*d*) sec $56\pi/3$ (*e*) cot $(-11\pi/6)$ (*f*) cosec $23\pi/3$.
3. If $ABCD$ is a quadrilateral, then show that
   (*a*) $\sin(A+B)+\sin(C+D)=0$, (*b*) $\cos\dfrac{B+C}{2}+\cos\dfrac{A+D}{2}=0$.
   (*c*) $\tan\dfrac{A+C}{4}=\cot\dfrac{B+D}{4}$.
4. Show that if n is any integer, then
   (*a*) $\sin(n\pi+(-1)^n A)=\sin A$. (*b*) $\cos(2n\pi\pm A)=\cos A$.
   (*c*) $\tan(n\pi+A)=\tan A$. (*d*) $\sin n\pi=\tan n\pi=0$, $\cos n\pi=(-1)^n$.
   (*e*) $\sin(2n+1)\pi/2=(-1)^n$, $\cos(2n+1)\pi/2=0$.
   (*f*) $\sin(n\pi+A)=(-1)^n\sin A$, $\cos(n\pi+A)=(-1)^n\cos A$.
5. Show that
   $\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots \tan 89^\circ = 1$.

6.4 FUNCTIONS AND THEIR GRAPHS

Recall that a function f from a set A to a set B, denoted by $f: A \to B$, associates to each element of A a unique element of B. The set A is called the *domain* of f and the set B the *co-domain* of f. If x is an element of A which is associated by f with (or mapped to) the element y of B, we say y is the *image of* x under f and write $f(x) = y$.

Here are a few examples.

EXAMPLE 1. Let $A = \{a, b, c, d, e\}$ and $B = \{p, q, r, s, t, u\}$. Let $f: A \to B$ be given by $f(a) = r$, $f(b) = p$, $f(c) = r$, $f(d) = q$, $f(e) = u$. We may represent f by a Venn diagram as in Fig. 6.8.

The set of elements in B which are images of elements in A is called the *range of* f and is denoted by Ran f, Im f or $f(A)$. That is, range of $f = \{f(x) \mid x \in A\} \subset B$. In the foregoing example, $Imf = \{p, q, r, u\}$.

EXAMPLE 2. Let $f: \mathbf{N} \to \mathbf{Z}$ be defined by $f(n) = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ +1, & \text{if } n \text{ is even.} \end{cases}$

Here $Imf = \{-1, 1\}$.

We are generally interested in functions defined on **R** or its subsets such as intervals (open, closed or semiopen) or their union and taking values in **R**.

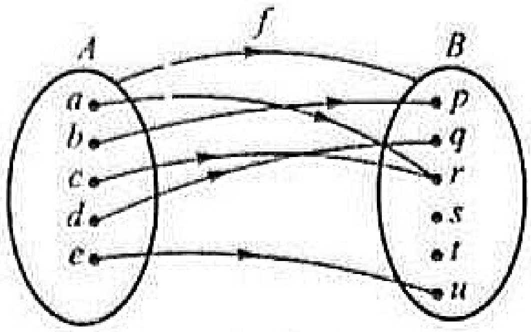


Fig. 6.8

EXAMPLE 3. Let $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2$ for x in $\mathbf{R}$. We observe that $Im f = [0, \infty)$.

We say a function $f : A \to B$ is *one-one* iff different elements in A have different images, that is, $x_1, x_2 \in A, x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$; or equivalently, $x_1, x_2 \in A, f(x_1) = f(x_2)$ implies $x_1 = x_2$. The function f in Example 3 is not one-one, because, for instance, $f(2) = f(-2) = 4$.

Question: Which of the functions in Examples 1 and 2 are one-one ?

We say that a function $f : A \to B$ is *onto* iff every element in B is the image of some element in A; or equivalently, $Im f = B$.

The function in Example 3 is not onto either, because no negative real number in the co-domain is an image under f. Are the functions in Examples 1 and 2 onto ?

A function $f : A \to B$ is said to be *bijective* iff f is both one-one and onto.

EXAMPLE 4. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 2x - 1$ for x in $\mathbf{R}$. This is both one-one and onto (why?) and so f is bijective.

EXAMPLE 5. Let $f : \mathbf{N} \to \mathbf{N}$ be given by $f(x) = x^2$ for all x in $\mathbf{N}$.

This is one-one but not onto. So f is not bijective. Note that in Example 3 we had the same defining relation for f, namely $f(x) = x^2$, but, there f was not one-one. So the defining relation alone does not determine one-one-ness of f.

EXAMPLE 6. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = \sin x$ for all x in $\mathbf{R}$.

This is called the sine function. Here x may be taken to be in degrees or better in radians. Thus the relations $\sin 0 = 0$, $\sin \pi/6 = 1/2$, $\sin(-\pi/3) = -\sqrt{3}/2$ describe the images of 0, $\pi/6$ and $-\pi/3$ under the sine function. Here $Im f = [-1, 1]$ and so f is not onto. Also f is not one-one either as $f(0) = f(\pi) = f(2\pi) = \ldots = f(n\pi) = \ldots = 0$.

Similarly we have the cosine function, tangent function and so on. Note that for the tangent function, the domain is not all of $\mathbf{R}$. In fact, the function describes the tangent

$$f : \mathbf{R} - \{(2n + 1) \frac{\pi}{2} \mid n \in Z\} \to \mathbf{R}, f(x) = \tan x$$

function. The tangent function is not defined at odd multiples of $\pi/2$ and so these are excluded from the domain. Also $Im f = \mathbf{R}$ and so the function is onto. Is it one-one ?

As stated before, our main interest is in functions defined on $\mathbf{R}$ or intervals and taking values in $\mathbf{R}$. Such functions are called *real-valued functions of a real variable*. They are of the form $f : X \to \mathbf{R}$, $X \subset \mathbf{R}$. These functions can be represented in the xy – plane by curves or graphs and usually just by looking at the graph of a real function, we can obtain a lot of information. The graph of a function $f : X \to \mathbf{R}$, $X \subset \mathbf{R}$ is the set of points $\{(x, f(x)) \mid x \in X\}$ in the xy – plane. The relation $y = f(x)$ is the equation of the graph.

EXAMPLE 7. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2$, for all x in $\mathbf{R}$.

The equation to the graph of this function is $y - x^2$ and the graph has already been shown in Figure 5.3 of Chapter 5.

Suppose we draw a line parallel to x-axis and it cuts the graph of $y = f(x)$ in more than one point. What do we conclude? We conclude that f is not one-one. Thus f in Example 7 is not one-one, since every line parallel to the x-axis and lying above it cuts the graph in two distinct points (x_1, x_1^2) and $(-x_1, x_1^2)$.

How do we get an idea of the range of a function from its graph? Let us take the projection of the curve in Example 7 on the y-axis and examine which part of the y-axis is covered by this projection. We see that the whole of the positive y-axis and the origin are covered by this projection and so we conclude that $Im\, f = [0, \infty)$. So f is not onto.

EXAMPLE 8. Let us draw the graph of the sine function given in Example 6. The equation of the graph is $y = \sin x$ and the graph itself is given by Fig. 6.9.

The sine curve looks like a wave extending in either direction of the x-axis to infinity and does not go above $y = 1$ or below $y = -1$. This is because the range of the function is $[-1, 1]$. The function is neither one-one nor onto.

EXAMPLE 9. Let us draw the graph of the tangent function given by $f : \mathbf{R} - \{(2n-1)\pi/2 \mid n \in \mathbf{Z}\} \to \mathbf{R}, f(x) = \tan x$.

The equation of the graph is $y = \tan x$ and the graph is as given in Fig. 6.10.

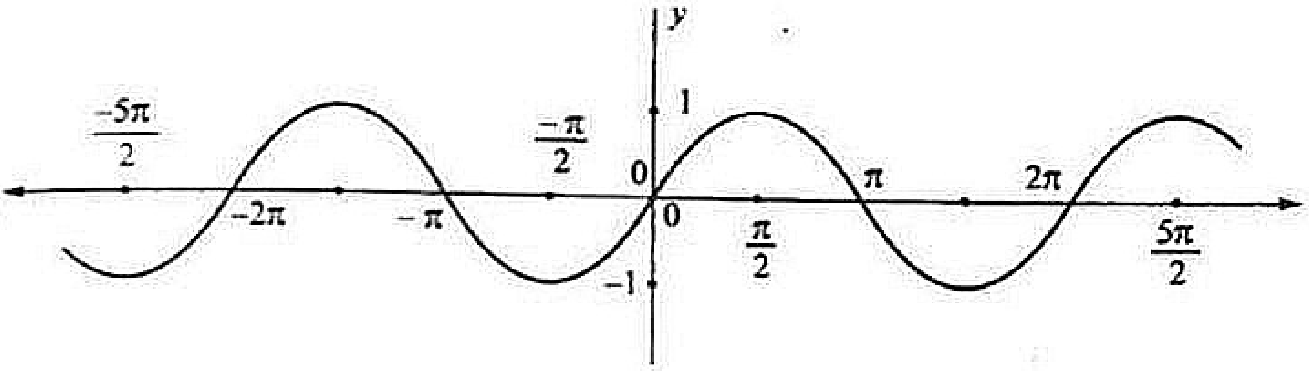


Fig. 6.9

The graph consists of infinitely many disjoint branches each lying between the two lines

$$x = \frac{(2n-1)\pi}{2}, x = \frac{(2n+1)\pi}{2}, n = 0, \pm 1, \pm 2, \ldots..$$

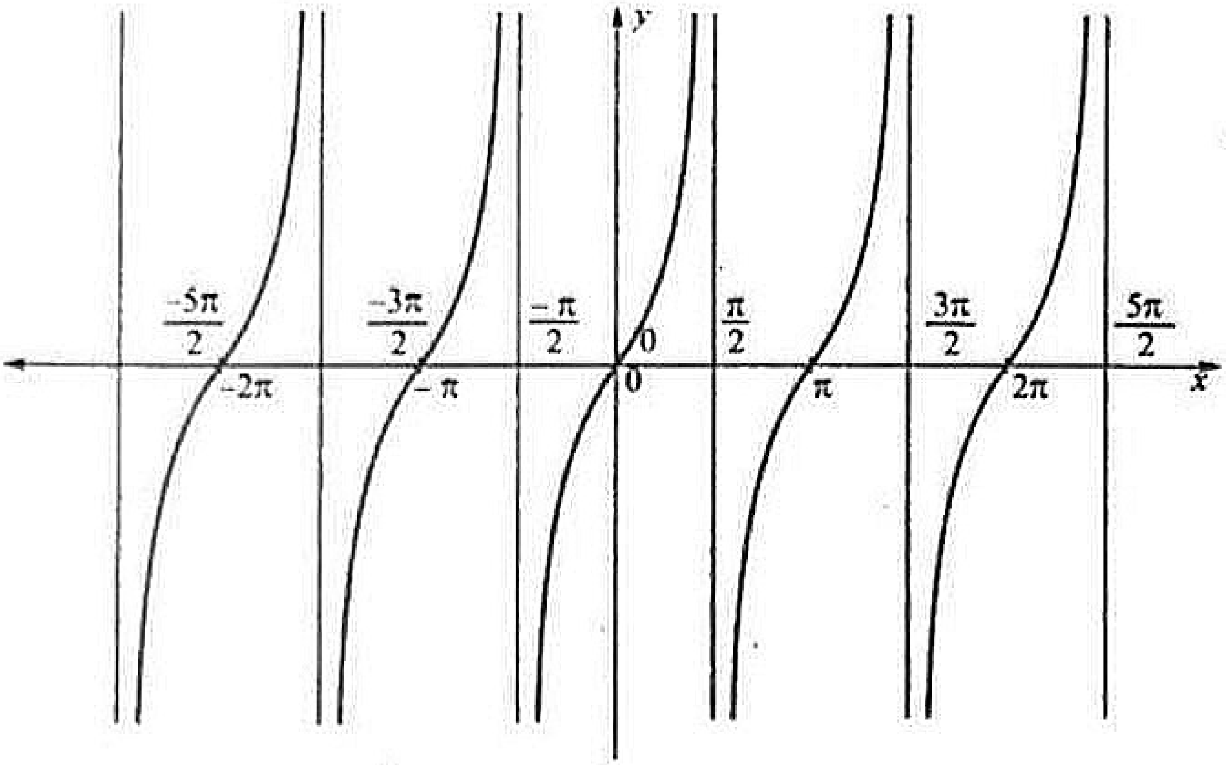


Fig. 6.10

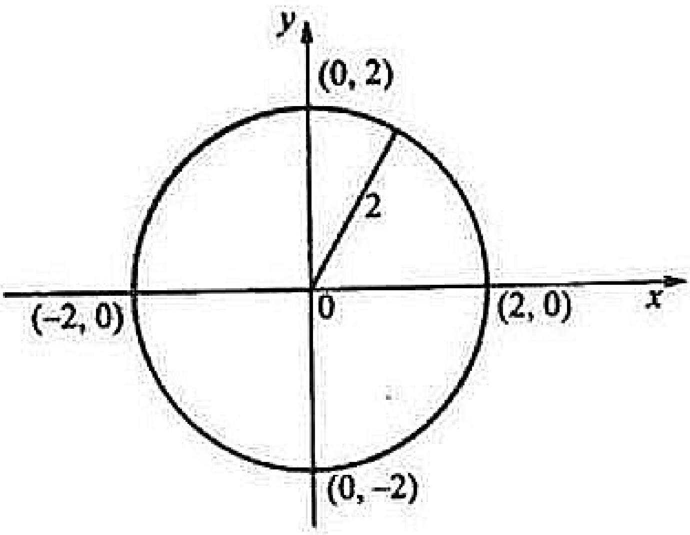


Fig. 6.11

$$y = \pm\sqrt{(4 - x^2)}.$$

Thus y has two values for each x in $(-2, 2)$.

Nevertheless, we have a 'function-like' structure. In fact, the graph is the union of the two graphs given by the functions

$$f : [-2, 2] \to \mathbf{R}, f(x) = \sqrt{4 - x^2}$$

and $$g : [-2, 2] \to \mathbf{R}, g(x) = -\sqrt{4 - x^2}.$$

Here f corresponds to the upper semicircle and g to the lower semicircle.

Observe that a **graph is symmetric with respect to the y-axis if its equation is unaltered by replacing x by $-x$** (e.g., $y = x^2$, $y = x^4$). **Similarly a graph is symmetric with respect to the x-axis if its equation is unaltered by replacing y by $-y$.** The graph of the circle above in Example 13 is symmetric with respect to both the axes.

EXERCISE 6.4

1. (*a*) Determine which of the figures from Fig. 6.12 to Fig. 6.15 represent functions from A to B?

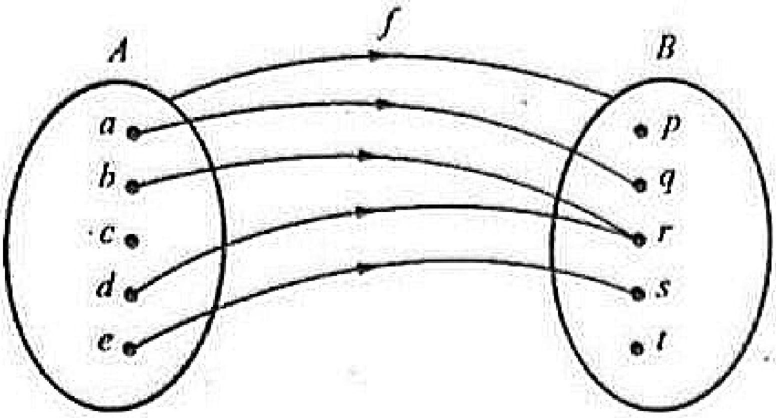


Fig. 6.12

(*b*) Which f is a function ?

(*i*) $f : \mathbf{Z} \to \mathbf{N}, f(x) = x^2$, for all x in $\mathbf{Z}$.

(*ii*) $f : \mathbf{Z} \to \mathbf{Z}, f(x) = x^2$, for all x in $\mathbf{Z}$.

(*iii*) $f: \mathbf{N} \to \mathbf{R}$, $f(x) = \dfrac{1}{2x^2 - 7x - 15}$, for all x in **N**.

(*iv*) $f: \mathbf{R} \to \mathbf{R}$, $f(x) = \dfrac{1}{x^2 + x + 1}$, for all x in **R**.

2. Find the inverses of the following functions whenever they exist.

(*a*) $f: \mathbf{R} \to \mathbf{R}, f(x) = x^3$, for all x in **R**.

(*b*) $f: \mathbf{R} \to \mathbf{R}, f(x) = \begin{cases} 2x, & \text{if } x \geq 0, \\ x, & \text{if } x < 0. \end{cases}$

(*c*) $f: \mathbf{R} \to \mathbf{R}, f(x) = |x|$, for all x in **R**.

(*d*) $f: \mathbf{R} \to \mathbf{R}, f(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$

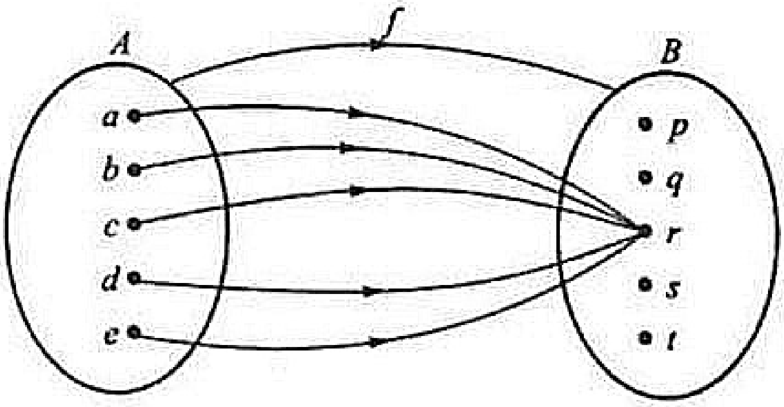


Fig. 6.13

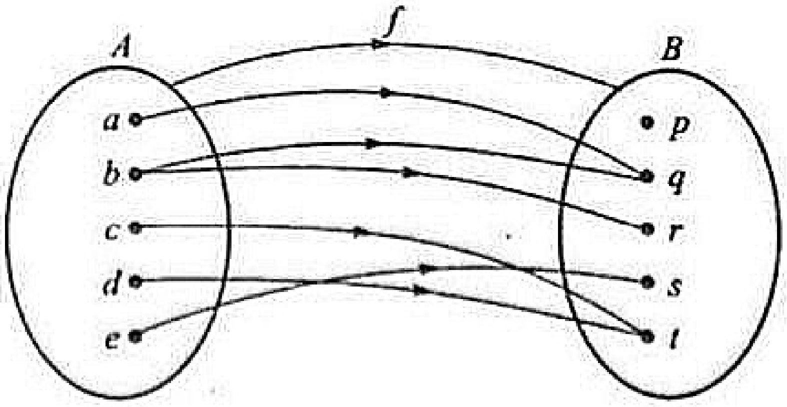


Fig. 6.14

(*e*) $f: (-\pi/2, \pi/2) \to \mathbf{R}, f(x) = \tan x$, for all x in $(-\pi/2, \pi/2)$

(*f*) $f: \mathbf{N} \to \mathbf{N}, f(n) = n^2 + 9n + 14$, for all n in IN.

(*g*) $f: \mathbf{N} \to \mathbf{N}, f(n) = \begin{cases} n+1, & \text{if } n \text{ is odd}, \\ n-1, & \text{if } n \text{ is even}. \end{cases}$

(*h*) $f: \mathbf{R} - \{0\} \to \mathbf{R}\ f(x) = 1/x$

3. Draw the graphs of the following

(*a*) $y = \cos x$. (*b*) $y = \cot x$. (*c*) $y = \operatorname{cosec} x$.

(*d*) $y = \sec x$. (*e*) $y = \sin 2x$, (*f*) $y = \tan^2 x$.

(*g*) $y = |x|$.

$$= \frac{PT}{PR}\cdot\frac{PR}{OP} + \frac{RS}{OR}\cdot\frac{OR}{OP} = \cos A \sin B + \sin A \cos B.$$

(*b*) Again, $\cos(A+B) = \frac{OQ}{OP} = \frac{OS - QS}{OP} = \frac{OS}{OP} - \frac{TR}{OP} = \frac{OS}{OR}\cdot\frac{OR}{OP} - \frac{TR}{PR}\cdot\frac{PR}{OP}$

$$= \cos A \cos B - \sin A \sin B.$$

(*c*) Further, to prove the last relation, we observe from (*a*) and (*b*) that

$$\sin (A+B) = \cos A \cos B\,(\tan A + \tan B);\text{ and}$$

$$\cos (A+B) = \cos A \cos B\,(1 - \tan A \tan B).$$

Dividing the first relation by the second we obtain the result. ☐

Theorem 2. If A and B are any two angles then

(*a*) $\sin (A-B) = \sin A \cos B - \cos A \sin B$;

(*b*) $\cos (A-B) = \cos A \cos B + \sin A \sin B$;

(*c*) $\tan (A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

Proof. One has only to replace B by $-B$ in Theorem 1 and observe that $\sin(-B) = -\sin B$, $\cos(-B) = \cos B$ and $\tan(-B)$ $-\tan B$. ☐

EXAMPLE 1. $\sin(\theta + \pi/4) = \sin\theta\cos\pi/4 + \cos\theta\sin\pi/4$

$$= (1/\sqrt{2})\,(\sin\theta + \cos\theta).$$

EXAMPLE 2. $\cos(2A + B/3) = \cos 2A \cos B/3 - \sin 2A \sin B/3$.

EXAMPLE 3. $\tan(\pi/4 + \theta) = \dfrac{1+\tan\theta}{1-\tan\theta}$.

EXAMPLE 4. $\sin(\pi/4 - \theta) = \sin(\pi/4)\cos\theta - \cos(\pi/4)\sin\theta$

EXAMPLE 5. $\tan 15^\circ = \tan(60^\circ - 45^\circ) = \dfrac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}.$$

EXAMPLE 6. *Show that* $\sin(A+B)\,.\,\sin(A-B) = \sin^2 A - \sin^2 B$.

SOLUTION. L.H.S. $= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$

$$= \sin^2 A\,(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$$

$$= \sin^2 A - \sin^2 B = \text{R.H.S.}$$

EXAMPLE 7. Prove that $\cos(A+B).\cos(A-B) = \cos^2 A - \sin^2 B$.

6.5 B : CONVERSION FORMULAE (PRODUCTS INTO SUMS)

Using the formulae for $\sin(A+B)$, $\sin(A-B)$, $\cos(A+B)$ and $\cos(A-B)$, one easily deduces the following important conversion formulae:

Theorem 3. If A and B are two angles,

(*a*) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$,

(*b*) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$, (*c*) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$,

(*d*) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$.

To prove these we start from the R.H.S. of each equation and obtain the corresponding L.H.S.

Note. Formulae (*a*) and (*b*) are essentially the same; for, we have only to interchange A and B.

EXAMPLE 8. $2\sin 80^\circ\cos 20^\circ = \sin(80^\circ + 20^\circ) + \sin(80^\circ - 20^\circ)$

$$= \sin 100^\circ + (\sqrt{3}/2) = \sin 80^\circ + (\sqrt{3}/2)$$

EXAMPLE 9. $2\sin 10^\circ\sin 50^\circ = \cos(50^\circ - 10^\circ) - \cos(50^\circ + 10^\circ)$

$$= \cos 40^\circ - (1/2).$$

EXAMPLE 10. $2\cos\left(\dfrac{A+3B}{2}\right)\cos\left(\dfrac{3A-B}{2}\right) = \cos(2A+B) + \cos(A-2B)$

EXAMPLE 11. $\sin(A+B)\,.\,\sin(A-B) = \dfrac{1}{2}\,[2\sin(A+B).\,\sin(A-B)]$

$$= \frac{1}{2}\,(\cos 2B - \cos 2A).$$

6.5 C : CONVERSION FORMULAE (SUMS INTO PRODUCTS)

We have another set of four more important formulae which express sums (and differences) as products. They are given in the following Theorem.

Theorem 4. If C and D are two angles,

(*a*) $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$;

(*b*) $\sin C - \sin D = 2\sin\dfrac{C-D}{2}\cos\dfrac{C+D}{2}$;

(*c*) $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$;

(*d*) $\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$;

Note that the last angle is $(D-C)/2$ and not $(C-D)/2$. These are proved by writing $A = (C+D)/2$ and $B = (C-D)/2$ in Theorem 3. □

EXAMPLE 12. $\sin 70^\circ + \sin 10^\circ = 2\sin\dfrac{70^\circ + 10^\circ}{2}\cos\dfrac{70^\circ - 10^\circ}{2}$

$$= 2\sin 40^\circ\cos 30^\circ = \sqrt{3}\,\sin 40^\circ.$$

EXAMPLE 13. $\cos 20^\circ - \sin 20^\circ = \cos 20^\circ - \cos 70^\circ$

$$= 2\sin\frac{20^\circ + 70^\circ}{2}\sin\frac{70^\circ - 20^\circ}{2}$$

$$= 2\sin 45^\circ\cos 25^\circ = \sqrt{2}\,\sin 25^\circ.$$

EXAMPLE 14. $\sin 7\theta - \sin 3\theta = 2\sin\dfrac{7\theta - 3\theta}{2}\cos\dfrac{7\theta + 3\theta}{2}$

$$= 2\sin 2\theta\cos 5\theta.$$

6.5 E : RATIOS OF 18° AND 36° :

Let $\theta = 18°$, so that $5\theta = 90°$ and $2\theta = 90° - 3\theta$.
This gives $\sin 2\theta = \cos 3\theta$; that is $2 \sin\theta \cos\theta = 4 \cos^3\theta - 3 \cos\theta$
Dividing by $\cos\theta$ $(\neq 0)$, we get

$$2 \sin\theta = 4 \cos^2\theta - 3 = 1 - 4 \sin^2\theta.$$

Therefore, $4 \sin^2\theta + 2 \sin\theta - 1 = 0$. Solving this for $\sin\theta$, we obtain

$$\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

Since $\theta = 18°$, $\sin\theta$ is positive.

Therefore have,

$$\sin 18° = \frac{\sqrt{5}-1}{4}.$$

Now the other ratios of 18° can be found out, as also those of 36°, 72° and 54°. For example, $\cos 72° = \sin 18° = (\sqrt{5} - 1)/4$; and so on.

Now we give geometrical proofs of the expressions for sin 18° and cos 36°. Consider an isosceles triangle ABC in which $AB = AC$ and $\angle A = 36°$, so that $\angle B = \angle C = 72°$. Let the internal biector of $\angle C$ meet AB in D. Since $\angle BCD = \angle ACD = 36°$, we have $\angle BDC = 72°$. So BCD is an isosceles triangle and $BC = DC$. Again $\angle DAC = \angle DCA = 36°$ and so triangle ACD is isosceles in which $AD = DC$. Draw AE and DF perpendicular to BC and AC respectively. Let $AB = AC = a$ and $BC = DC = DA = x$. Now triangle ABC is similar to triangle CDB. Hence

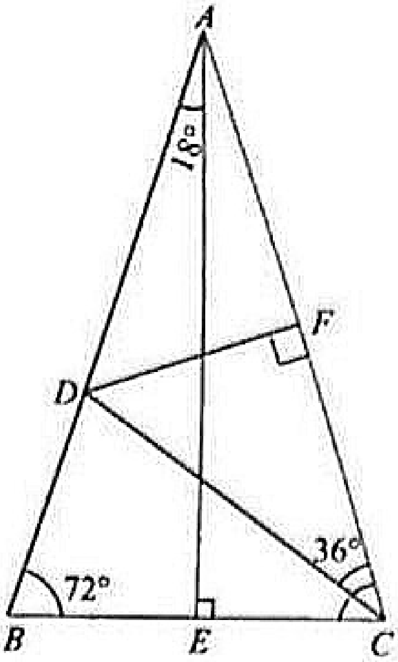


Fig. 6.17

$$AB/CD = BC/DB.$$

That is, $a/x = x/(a-x)$. Simplifying we get $x^2 + ax - a^2 = 0$.

Solving we have $x/a = (-1 + \sqrt{5})/2$, as negative sign is ruled out.

In triangle ABE, $\angle BAE = 18°$, $\therefore$ $\sin \angle BAE = BE/AB$.

That is, $$\sin 18° = \frac{x/2}{a} = \frac{1}{2}(x/a) = (\sqrt{5} - 1)/4.$$

Also from triangle DAF, $\cos \angle DAF = AF/AD$.

That is, $$\cos 36° = \frac{a/2}{x} = \frac{1}{2}(a/x) = \frac{1}{2}\frac{2}{(\sqrt{5}-1)} = \frac{\sqrt{5}+1}{4}.$$

EXAMPLE 23. $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°.$

EXAMPLE 24. $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°.$

EXAMPLE 25. *Show that 2 sin 48° sin 12° = sin 18°.*

SOLUTION. L.H.S $= \cos 36° - \cos 60°$

$$= \frac{\sqrt{5}+1}{4} - \frac{1}{2} = \frac{\sqrt{5}-1}{4} = \sin 18° = \text{R.H.S.}$$

EXERCISE 6.5

1. Show that $\tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40° = \sqrt{3}$.
2. Show that $\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$.
3. Show that

   (*a*) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$.

   (*b*) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$.

   (*c*) $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$.

4. Expand $\sin(A+B+C)$, $\cos(A+B+C)$, $\tan(A+B+C)$ in terms of the ratios of A, B, C.
5. (*a*) Show that

   $$\tan(A_1 + A_2 + \dots + A_n) = \frac{s_1 - s_3 + s_5 - s_7 + \dots}{1 - s_2 + s_4 - s_6 + \dots}.$$

   where s_r = sum of products of tangents of the angles $A_1, A_2, \dots, A_n$ taken r at a time. (Use induction).

Prove the following identities [(6) – (12)]:

6. $\cos 3\theta = 4\cos\theta \cos(\theta - \pi/3)\cos(\theta + \pi/3)$.
7. $\tan 3\theta = \tan\theta \tan(\pi/3 - \theta)\tan(\pi/3 + \theta)$.
8. $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.
9. $(\sin 8\theta)/\sin\theta = 8(16\cos^7\theta - 24\cos^5\theta + 10\cos^3\theta - \cos\theta)$.
10. $\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$.
11. $1 + \tan A \tan(A/2) = \tan A \cot(A/2) - 1 = \sec A$.

12. (*a*) $\cot A - \tan A = 2 \cot 2A$.

(*b*) Hence express $\tan A + 2 \tan 2A + 2^2 \tan 2^2 A + \ldots + 2^{n-1} \tan 2^{n-1} A$ in the form $a \cot A + b \cot 2^n A$, where a and b are integers.

Prove the following relations [illegible]

13. $\cos 20° \cos 40° \cos 60° \cos 80° = 1/16$.

14. $\sin 20° \sin 40° \sin 60° \sin 80° = 3/16$.

15. $\cos^2 74° + \cos^2 14° - \cos 74° \cos 14° = 3/4$.

16. (*a*) $\tan \left(7\frac{1}{2}\right)° = \sqrt{6} - \sqrt{4} - \sqrt{3} + \sqrt{2}$.

(*b*) $\cot 15° = 2 + \sqrt{3}$.

$\cot 20° - \cot 40° + \cot 80° = \sqrt{3}$.

$\tan 6° \tan 42° \tan 66° \tan 78° = 1$.

$$\cos \frac{\pi}{15} \cos \frac{2\pi}{15} . \cos \frac{3\pi}{15} \ldots \cos \frac{7\pi}{15} = 1/2^7.$$

$\cos\theta + \cos 5\theta + \cos 7\theta = \cos 2\theta + \cos 4\theta$, where $\theta = 12°$.

Show that $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$.

Show that $4 \sin^3 \theta \cos 3\theta + 4 \cos^3 \theta \sin 3\theta = 3 \sin 4\theta$.

If $\cos \alpha + \cos \beta = p$ and $\sin \alpha + \sin \beta = q$, evaluate $\tan^2 (\alpha - \beta)/2$ and $\tan^2 (\alpha - \beta)/2$ in terms of p and q.

If $\cos \theta = \dfrac{\cos u - e}{1 - e \cos u}$, then show that

$$\tan (\theta/2) = \pm \sqrt{\frac{1+e}{1-e}} \tan (u/2).$$

If $\operatorname{cosec} (\theta + \alpha) \operatorname{cosec} (\theta - \alpha) = 2 \operatorname{cosec} \theta$ then show that either $\alpha = 2n\pi$, where n is an integer or $\sin \theta = \pm \sqrt{2} \cos (\alpha/2)$.

For what values of A in $[0, 2\pi]$ does the following relation hold good?

(*a*) $2 \sin(A/2) = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}$;

(*b*) $2 \cos(A/2) = \sqrt{1 + \sin A} + \sqrt{1 - \sin A}$.

Use the idea given in exercise 26 to

show that $$\sin 9° = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4}$$

and $$\cos 9° = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4}.$$

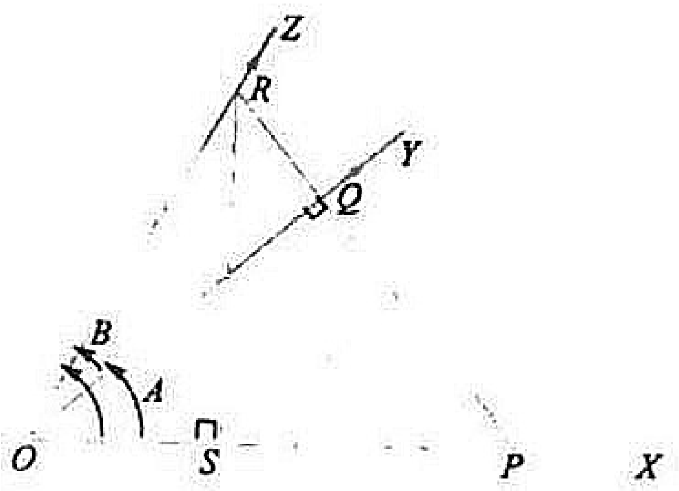
Evaluate $\sin 27°$ and $\cos 27°$.

Find the ratios of 15° and 75°.

(*a*) In the adjoining figure (Fig. 18), $\angle XOY = A$, $\angle YOZ = B$.

The line PQR is perpendicular to $\overrightarrow{OY}$ and RS is perpendicular to $\overrightarrow{OX}$.

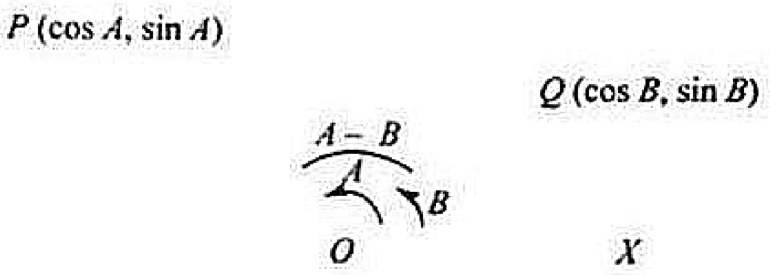
Use the fact that $2\Delta OPR = OP.RS - PR.OQ$ to prove that sin $(A+B) = \sin A \cos B + \cos A \sin B$.



(*b*) Use Table 6.3, Section 6.3 to obtain expressions for $\sin(A-B)$, $\cos(A+B)$ and $\cos(A-B)$.

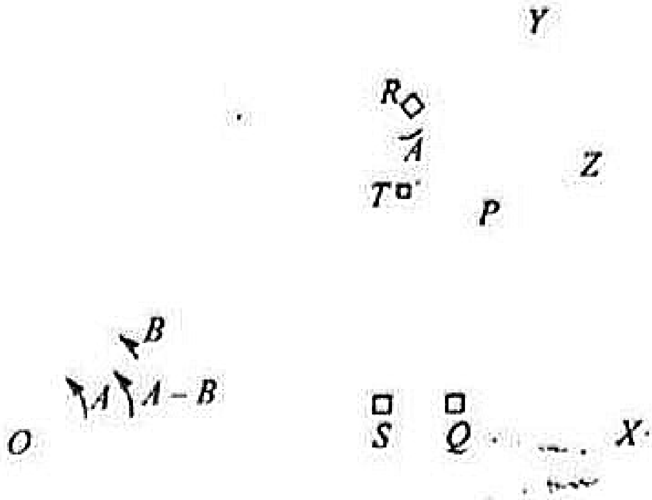
In the *xy*-plane consider the unit circle (centre 0, radius 1) and take two points $P = (\cos A, \sin A)$ and $Q = (\cos B, \sin B)$ on the circle as shown in Fig. 6.19. Use the distance formula for PQ in two ways to obtain

$\cos(A-B) = \cos A \cos B + \sin A \sin B$.



Give a purely geometrical proof of the formula for $\tan(A+B)$, using figure 6.16. (Use the similarity of triangles PTR and OSR).

In figure 6.20, $\angle XOY = A$, $\angle ZOY = B$ and PQ, PR, RS, PT are respectively perpendicular to OX, OY, OX, RS. Give geometrical proofs of the formulae in Theorem 2, that is, the expressions for the ratios of $A - B$. (Imitate the proof of Theorem 1).



34. In figure 6.21, $\angle XOY = C$, $\angle ZOY = D$, OW bisects $\angle YOZ$, the line PRQ is perpendicular to OW. The lines PS, QT, RL are all perpendicular to OX and the line QNM is perpendicular to PS. Observe that $\angle XOW = (C + D)/2$ and $\angle YOW = \angle ZOW = (C - D)/2$. Deduce the results of Theorem 4.

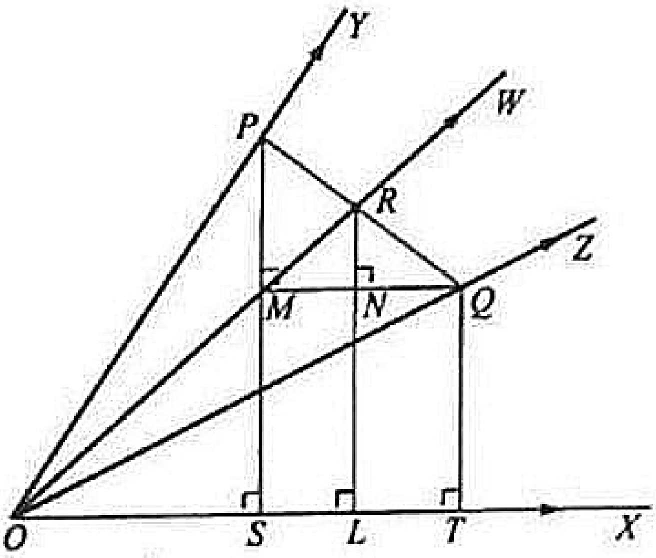


Fig. 6.21

35. Let BC be a diameter of a circle centre O and P a point on the circle such that $\angle CBP = A$. PQ perpendicular to BC as in figure 6.22. Observe that $\angle COP = 2A$ and $\angle CPQ = A$. Obtain the expressions for sin 2A, cos 2A and tan 2A.

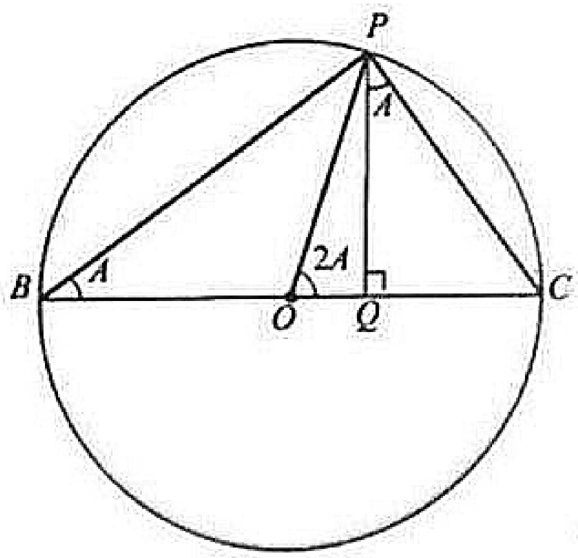


Fig. 6.22

6.6 TRIGONOMETRICAL IDENTITIES

Using the fact that the sum of the angles of a triangle ABC is 180 degrees, we can derive several identities, which will be useful later. Generally, in these identities we express sums into products. If the sums are symmetric functions of A, B, C, so are the products. The conversion rules (sums into products) derived in Section 6.5 are extensively used.

First we mention a few simple relations governed by the condition $A + B + C = 180°$:

(*i*) $\sin(A + B) = \sin(180° - C) = \sin C$.

(*ii*) $\cos(B + C) = \cos(180° - A) = -\cos A$.

(*iii*). $\tan(C + A) = \tan(180° - B) = -\tan B$.

(*iv*) $\sin((A+B)/2) = \sin(90° - C/2) = \cos(C/2)$.

(*v*) $\cos((B+C)/2) = \cos(90° - A/2) = \sin(A/2)$.

(*vi*) $\tan((C+A)/2) = \tan(90° - B/2) = \cot(B/2)$.

(*vii*) $\sin((A+B)/4) = \sin((\pi - C)/4) = \sin\left(\frac{\pi}{2} - \frac{\pi + C}{4}\right)$

$$= \cos((\pi + C)/4).$$

EXAMPLE 1. *If $A + B + C = 180°$ show that*

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C.$$

SOLUTION. We have

$$\begin{aligned}\cos 2A + \cos 2B + \cos 2C &= 2\cos(A+B)\cos(A-B) + \cos 2C\\ &= -2\cos C\cos(A-B) + 2\cos^2 C - 1\\ &= -1 - 2\cos C[\cos(A-B) - \cos C]\\ &= -1 - 2\cos C[\cos(A-B) + \cos(A+B)]\\ &= -1 - 2\cos C.\, 2\cos A\cos B\\ &= -1 - 4\cos A\cos B\cos C.\end{aligned}$$

EXAMPLE 2. *If $A + B + C = \pi$, show that*

$$\sin A + \sin B - \sin C = 4\sin(A/2)\sin(B/2)\cos(C/2).$$

SOLUTION. We have

$$\begin{aligned}\sin A + \sin B - \sin C &= 2\sin(A+B)/2\cos(A-B)/2 - \sin C\\ &= 2\cos(C/2)\cos((A-B)/2) - 2\sin(C/2)\cos(C/2)\\ &= 2\cos(C/2)[\cos((A-B)/2) - \sin(C/2)]\\ &= 2\cos(C/2)[\cos((A-B)/2) - \cos((A+B)/2)]\\ &= 2\cos(C/2)\, 2\sin(A/2)\sin(B/2)\\ &= 4\sin(A/2)\sin(B/2)\cos(C/2).\end{aligned}$$

EXAMPLE 3. *If the sum of the three angles A, B, C is 2 right angles, show that*

$$\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2) = 1 - 2\sin(A/2)\sin(B/2)\sin(C/2).$$

SOLUTION. We have

$$\begin{aligned}&\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2)\\ &= 1 - (\cos^2(A/2) - \sin^2(B/2)) + \sin^2(C/2)\\ &= 1 - \cos((A+B)/2)\cos((A-B)/2) + \sin^2(C/2)\\ &= 1 - \sin(C/2)\cos((A-B)/2) + \sin^2(C/2)\\ &= 1 - \sin(C/2)[\cos((A-B)/2) - \sin(C/2)]\\ &= 1 - \sin(C/2)[\cos((A-B)/2) - \cos((A+B)/2)]\\ &= 1 - \sin(C/2)\, 2\sin(A/2)\sin(B/2)\\ &= 1 - 2\sin(A/2)\sin(B/2)\sin(C/2).\end{aligned}$$

Alternatively, one may use the formula

$$\sin^2\theta/2 = (1/2)(1 - \cos\theta)$$

and write the left hand expression in the form

$$3/2 - 1/2\,(\cos A + \cos B + \cos C),$$

and proceed with $\cos A + \cos B + \cos C$ as in Example 2.

EXAMPLE 4. *If $A + B + C$ is a multiple of π, then prove that*

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

SOLUTION. Let $A + B + C = n\pi$, where n is any integer. Then

$$A + B = n\pi - C.$$

Therefore $\tan (A + B) = \tan (n\pi - C)$.

So $$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C.$$

$\therefore$ $\tan A + \tan B = -\tan C + \tan A \tan B \tan C.$

That is, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. As a consequence, we see that if $A + B + C = \pi$, then $\tan nA + \tan nB + \tan nC = \tan nA.\tan nB.\tan nC$, for any integer n.

EXERCISE 6.6

If $A + B + C = 180°$, show that

1. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
2. $\sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C$.
3. $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$, and hence that
   $\cos 2A - \cos 2B - \cos 2C = -1 + 4 \cos A \sin B \sin C$.
4. $\sin A + \sin B + \sin C = 4 \cos (A/2) \cos (B/2) \cos (C/2)$.
5. $\cos A + \cos B + \cos C = 1 + 4 \sin (A/2) \sin (B/2) \sin (C/2)$.
6. $-\cos A + \cos B + \cos C = -1 + 4 \sin (A/2) \cos (B/2) \cos (C/2)$.
7. $\sin^2(A/2) - \sin^2(B/2) + \sin^2(C/2) = 1 - 2 \cos (A/2) \sin (B/2) \cos (C/2)$.
8. $\cos^2(A/2) + \cos^2(B/2) + \cos^2(C/2) = 2 + 2 \sin (A/2) \sin (B/2) \sin (C/2)$.
9. $\cos^2(A/2) + \cos^2(B/2) - \cos^2(C/2) = 2 + 2 \cos (A/2) \cos (B/2) \sin (C/2)$.
10. $\cos^2A + \cos^2B - \cos^2C = 1 - 2 \sin A \sin B \cos C$.
11. $\sin^2A + \sin^2B + \sin^2C = 2 + 2 \cos A \cos B \cos C$.
12. $\cot (A/2) + \cot (B/2) + \cot (C/2) = \cot (A/2) \cot (B/2) \cot (C/2)$.
13. $\tan(B/2) \tan (C/2) + \tan (C/2) \tan (A/2) + \tan (A/2) \tan (B/2) = 1$.
14. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.
15. $\sin (A/2) + \sin (B/2) + \sin (C/2)$
    $= 1 + 4 \sin ((\pi - A)/4) \sin ((\pi - B)/4) \sin ((\pi - C)/4)$
    $= 1 + 4 \cos ((\pi + A)/4) \cos ((\pi + B)/4) \cos ((\pi + C)/4)$.
16. $\cos (A/2) - \cos (B/2) + \cos (C/2)$
    $= 4 \cos ((\pi + A)/4) \cos ((\pi - B)/4) \cos ((\pi + C)/4)$.
17. $\sin (A/2) + \sin (B/2) - \sin (C/2) = -1 + 4 \sin \dfrac{\pi + A}{4} \sin \dfrac{\pi + B}{4} \sin \dfrac{\pi - C}{4}$.
18. $\cos (A/2) + \cos (B/2) + \cos (C/2) = 4 \cos \dfrac{\pi - A}{4} \cos \dfrac{\pi - B}{4} \cos \dfrac{\pi - C}{4}$.
19. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin (A/2) \sin (B/2) \sin (C/2)$.

If $A + B + C = 2W$, show that

20. $\sin (W - A) + \sin (W - B) + \sin (W - C) - \sin W$
    $= 4 \sin (A/2) \sin (B/2) \sin (C/2)$.

21. $\cos^2 W + \cos^2(W - A) + \cos^2(W - B) + \cos^2(W - C)$
$$= 2 + 2 \cos A \cos B \cos C.$$

If $A + B + C = 0$, then show that

22. $\sin A + \sin B + \sin C = -4\sin (A/2) \sin (B/2) \sin (C/2)$.
23. $\cos^2 A + \cos^2 B - \cos^2 C = 1 + 2 \sin A \sin B \cos C$.
24. $\sin 2A + \sin 2B + \sin 2C$
$$= 2(\sin A + \sin B + \sin C) \times (1 + \cos A + \cos B + \cos C).$$
25. If p and q are respectively the product of sines and cosines of the angles of a triangle then the tangets of the angles are the roots of
$$q x^3 - p x^2 + (1 + q) x - p = 0.$$
26. If $A + B + C = 180°$, then

$\cos mA + \cos mB + \cos mC = 1 \pm \sin (mA/2) \sin (mB/2) \sin (mC/2)$, according as m is of the form $4n + 1$ or $4n + 3$; and

$\sin mA + \sin mB + \sin mC = \pm 4 \sin mA \sin mB \sin mC$, according as m is of the form $4n$ or $4n + 2$. Here the sign ± in the two results can be replaced by $(-1)^{(m-1)/2}$ and $(-1)^{m/2}$ respectively.

6.7 INVERSE CIRCULAR FUNCTIONS

In section 6.4, Example 8 it was observed that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin x$ was neither one-one nor onto. If we wish to define inverse of the sine function we must cut to size the domain as well as the co-domain (see Remark 2, Section 6.4). Since the range of the sine function is $[-1, 1]$ and it takes all these values once and only once in the interval $[-\pi/2, \pi/2]$, we have the following definition.

Definition. If $f : [-\pi/2, \pi/2] \to [-1, 1]$ is defined by $f(x) = \sin x$ for all x in the domain of f, then f being a bijective function has an inverse f^{-1} which is called the *inverse sine function*. We write $\sin^{-1} x$ (or arc sin x) for $f^{-1}(x)$.

In other words, if $-1 \le x \le 1$, then the numerically smallest angle θ whose sine is x is defined as $\sin^{-1} x$.

EXAMPLE 1. (*i*) $\sin^{-1}(1/\sqrt{2}) = 45°$; (*ii*) $\sin^{-1} 1 = \pi/2$;
(*iii*) $\sin^{-1} 0 = 0$; (*iv*) $\sin^{-1} (-1) = -\pi/2$;
(*v*) $\sin^{-1} (-1/2) = -\pi/6$.

Remark 1. If $0 < x \le 1$ and $\sin^{-1} x = \theta$, then $\sin^{-1} (-x) = -\theta$.

Similarly we proceed to define $\cos^{-1} x$, $\tan^{-1} x$ etc. For the inverse of the cosine function, we observe that the range of the cosine function is again $[-1, 1]$ and it takes all these values exactly once in $[0, \pi]$.

Definition. If $f : [0, \pi] \to [-1, 1]$ is defined by $f(x) = \cos x$, for all x in $[0, \pi]$, then f is a bijective function and its inverse f^{-1} is called the *inverse cosine function*. We write $\cos^{-1} x$ (or arc cos x) for $f^{-1}(x)$.

That is, if $-1 \le x \le 1$, the smallest non-negative angle θ whose cosine is x is defined as $\cos^{-1} x$.

EXAMPLE 2. (*i*) $\cos^{-1}(1/2) = \pi/3$; (*ii*) $\cos^{-1} 1 = 0$;
(*iii*) $\cos^{-1} 0 = \pi/2$; (*iv*) $\cos^{-1} (-1) = \pi$;
(*v*) $\cos^{-1} (-1/2) = 2\pi/3$.

Remark 2. If $0 < x \le 1$ and $\cos^{-1} x = \theta$, then $\cos^{-1}(-x) = \pi - \theta$.

For the inverse tangent function, we see that the range of the tangent function is the whole real line **R** and it takes every real value just once in $(-\pi/2, \pi/2)$.

Definition. If $f: (-\pi/2, \pi/2) \to \mathbf{R}$ is defined by $f(x) = \tan x$ for all x in $(-\pi/2, \pi/2)$, then f is a bijective function and so has an inverse. The inverse function f^{-1} is called the *inverse tangent function*. Also $f^{-1}(x)$ is denoted by $\tan^{-1} x$ (or arc tan x).

In other words, if x is a real number, then the numerically smallest angle θ whose tangent is x is written $\tan^{-1} x$.

EXAMPLE 3. (*i*) $\tan^{-1}\sqrt{3} = \pi/3$; (*ii*) $\tan^{-1} 0 = 0$

(*iii*) $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$.

Remark 3. If $x > 0$ and $\tan^{-1} x = \theta$, then $\tan^{-1}(-x) = -\theta$.

Definition. The function $f: (0, \pi) \to \mathbf{R}$ defined by $f(x) = \cot x$ for all x in $(0, \pi)$ is a bijective function and its inverse f^{-1} is called the *inverse cotangent function*. We write $\cot^{-1} x$ (or arc cot x) for $f^{-1}(x)$.

EXAMPLE 4. (*i*) $\cot^{-1} 1 = \pi/4$; (*ii*) $\cot^{-1}(-1) = 3\pi/4$;

(*iii*) $\cot^{-1} 0 = \pi/2$; (*iv*) $\cot^{-1}(\sqrt{3} - 2) = 7\pi/12$.

Definition. The function $[0, \pi/2) \cup (\pi/2, \pi] \to \mathbf{R}\backslash(-1, 1)$ given by $f(x) = \sec x$ for all x in the domain of f is a bijective function and its inverse f^{-1} is called the *inverse secant function*. We write $\sec^{-1} x$ (arc sec x) for $f^{-1}(x)$.

EXAMPLE 5. (*i*) $\sec^{-1} 1 = 0$; (*ii*) $\sec^{-1} 2 = \pi/3$;

(*iii*) $\sec^{-1}(-2/\sqrt{3}) = 5\pi/6$; (*iv*) $\sec^{-1}(-1) = \pi$.

Definition. The function $f: [-\pi/2, 0) \cup (0, \pi/2] \to \mathbf{R}\backslash(-1, 1)$ given by $f(x) = \operatorname{cosec} x$ for all x in the domain of f being a bijection has an inverse f^{-1} which is called the *inverse cosecant function*. We write $\operatorname{cosec}^{-1} x$ (or arc cosec x) for $f^{-1}(x)$.

EXAMPLE 6. (*i*) $\operatorname{cosec}^{-1} 1 = \pi/2$; (*ii*) $\operatorname{cosec}^{-1} 2 = \pi/6$;

(*iii*) $\operatorname{cosec}^{-1}(-1) = -\pi/2$; (*iv*) $\operatorname{cosec}^{-1}(-2/\sqrt{3}) = -\pi/3$.

The six functions defined above are called the *inverse circular functions* or *inverse trigonometrical* ratios.

We list some simple properties of these inverse functions.

A. (*a*) $\sin(\sin^{-1} x) = x$ for $x \in [-1, 1]$, and if $-\pi/2 \le \theta \le \pi/2$ then $\sin^{-1}(\sin\theta) = \theta$.

(*b*) $\cos(\cos^{-1} x) = x$ for $x \in [-1, 1]$, and if $0 \le \theta \le \pi$, then $\cos^{-1}(\cos\theta) = \theta$.

(*c*) $\tan(\tan^{-1} x) = x$ for any real x, and if $-\pi/2 < \theta < \pi/2$, then $\tan^{-1}(\tan\theta) = \theta$.

B. (*a*) If $-1 \le x \le 1$, then $\sin^{-1} x + \cos^{-1} x = \pi/2$;

(*b*) If x is any real number, then $\tan^{-1} x + \cot^{-1} x = \pi/2$;

(*c*) If $x \ge 1$ or $x \le -1$, then $\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$.

C. (*a*) If $x \ge 1$ or $x \le -1$, then $\operatorname{cosec}^{-1} x = \sin^{-1}(1/x)$;

(*b*) If $x \ge 1$ or $x \le -1$, then $\sec^{-1} x = \cos^{-1}(1/x)$;

(*c*) If $x > 0$, then $\cot^{-1} x = \tan^{-1} (1/x)$ and
if $x < 0$, then $\cot^{-1} x = \pi - \tan^{-1} (1/x)$.

D. (*a*) If $-1 \le x \le 1$, then $\sin^{-1} x + \sin^{-1} (-x) = 0$;
(*b*) If $-1 \le x \le 1$, then $\cos^{-1} x + \cos^{-1} (-x) = \pi$;
(*c*) If x is any real number, then $\tan^{-1} x + \tan^{-1} (-x) = 0$.

The graphs of the inverse circular functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$ are given in Figures 6.23 to 6.28 respectively. The student is advised to note the nuances of these graphs carefully.

EXAMPLE 7. *Show that*

$$\sin^{-1} (3/5) + \sin^{-1} (8/17) = \sin^{-1} (77/85).$$

SOLUTION. Assume that $\sin^{-1} (3/5) = \alpha$, $\sin^{-1} (8/17) = \beta$ and $\sin^{-1} (77/85) = \gamma$, so that $\sin \alpha = 3/5$, $\sin \beta = 8/17$ and $\sin \gamma = 77/85$. Then we have to show that $\alpha + \beta = \gamma$.

Now $\sin (\alpha + \beta)$

$$\begin{aligned} &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= (3/5)(15/17) + (4/5)(8/17) \\ &= 77/85 = \sin \gamma. \end{aligned}$$

Hence $\alpha + \beta = \gamma$, which proves the result, since we have $\alpha + \beta < \pi/2$.

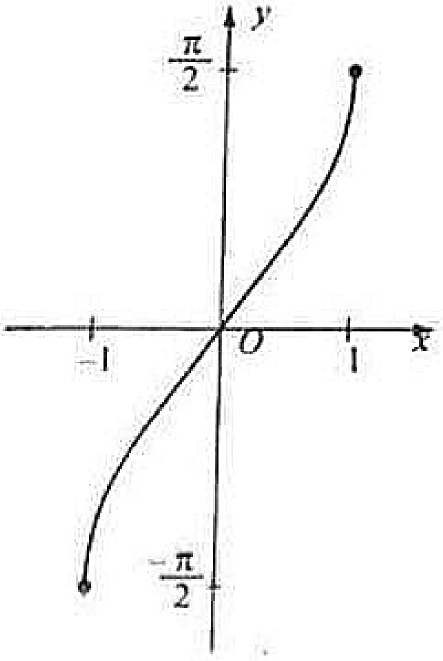


Fig. 6.23

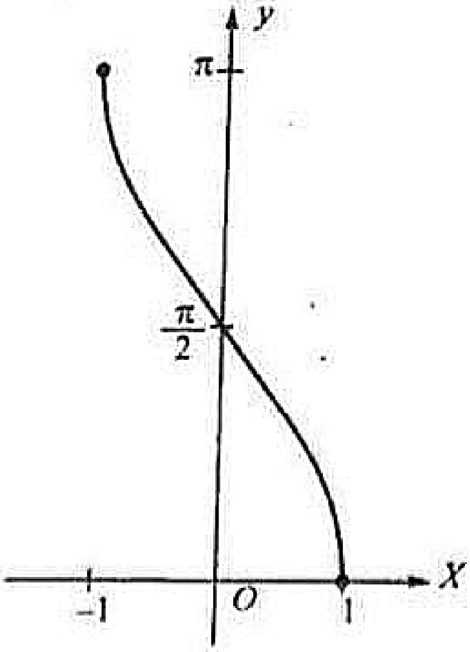


Fig. 6.24

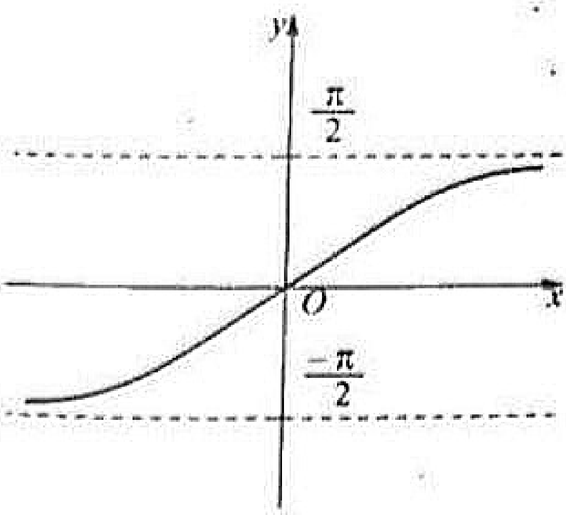


Fig. 6.25

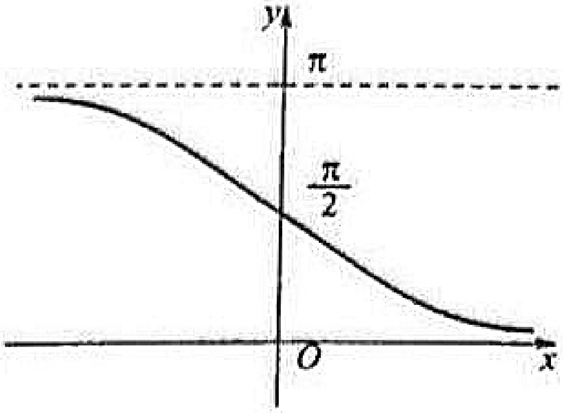


Fig. 6.26

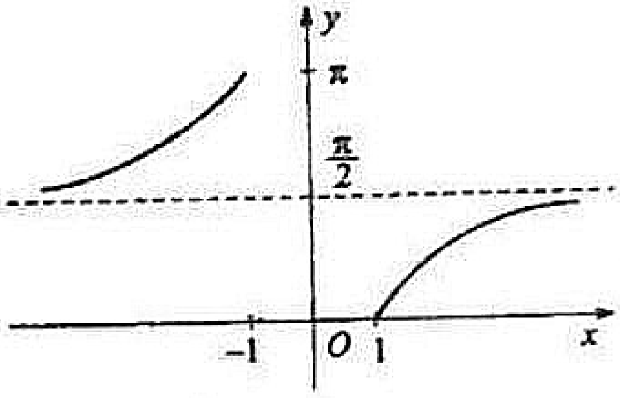


Fig. 6.27

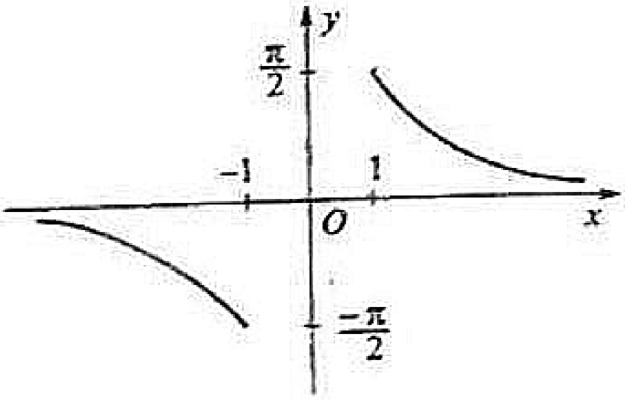


Fig. 6.28

EXAMPLE 8. *Prove that*

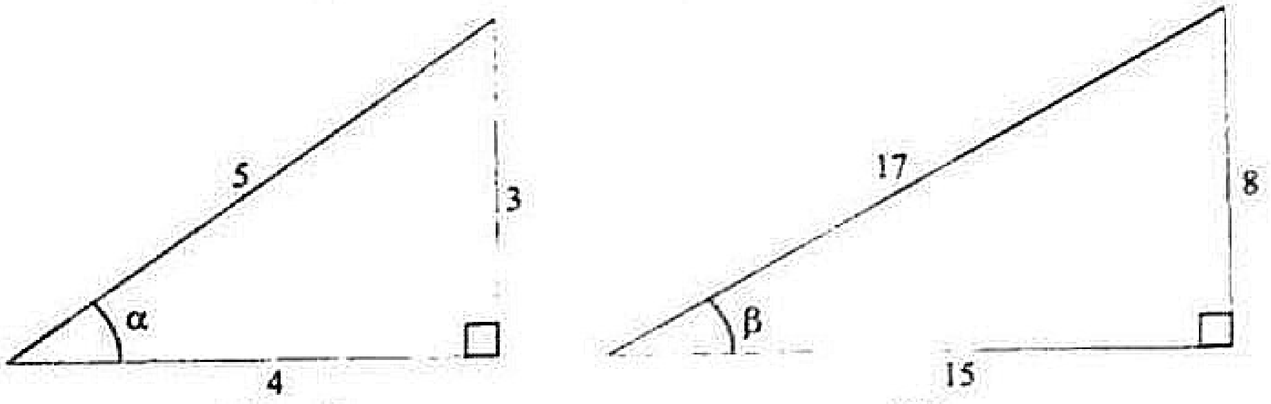
$$4\tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4.$$

SOLUTION. Let $\tan^{-1}(1/5) = \alpha$ and $\tan^{-1}(1/239) = \beta$,

so that $\tan\alpha = 1/5$ and $\tan\beta = 1/239$.

We need to show that

$$4\alpha - \beta = \pi/4 \text{ or } 4\alpha = \pi/4 + \beta.$$



(a) (b)

Fig. 6.29

Now $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2(1/5)}{1-(1/5)^2} = 5/12.$$

Therefore, $$\tan 4\alpha = \frac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \frac{2(5/12)}{1-(5/12)^2} = 120/119.$$

Also $$\tan\left(\frac{\pi}{4}+\beta\right) = \frac{1+\tan\beta}{1-\tan\beta} = \frac{1+(1/239)}{1-(1/239)} = \frac{120}{119}.$$

Hence $\tan 4\alpha = \tan((\pi/4)+\beta)$, giving $4\alpha = (\pi/4)+\beta$ as desired.

If $ab < 1$, show that

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab}.$$

Let. $\tan^{-1} a = \alpha$, $\tan^{-1} b = \beta$, $\tan^{-1}\dfrac{a+b}{1-ab} = \gamma$.

Then we have to prove that $\alpha + \beta = \gamma$.

Now $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{a+b}{1-ab} = \tan\gamma$.

Hence $\qquad \alpha + \beta = \gamma$, which is what is wanted.

Question 1. Where have we used the fact that $ab < 1$?

2. What happens if $ab > 1$, that is, how should we change the given relation if $ab > 1$?

EXAMPLE 10. *Solve the equation* $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}(8/31)$.

SOLUTION. Let $\tan^{-1}(x+1) = \alpha$, $\tan^{-1}(x-1) = \beta$ so that $\tan\alpha = x+1$, $\tan\beta = x-1$. The given relation becomes

$$\alpha + \beta = \tan^{-1}(8/31).$$

This in turn gives $\qquad \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{8}{31}$.

Therefore $\qquad \dfrac{(x+1)+(x-1)}{1-(x+1)(x-1)} = \dfrac{8}{31}$.

Simplifying, we obtain $4x^2 + 31x - 8 = 0$.

Solving this quadratic equation, we have $x = -8$, $1/4$.

Here $x = -8$ is inadmissible, because for this value of x, L.H.S. of the given equation is a negative angle, whereas, its R.H.S. is a positive angle. Thus we have only one solution, namely, $x = 1/4$.

EXERCISE 6.7

Prove the following relations: [(1) – (15)].

1. $\cos^{-1}(4/5) + \cos^{-1}(12/13) = \cos^{-1}(33/65)$.

2. $2\cos^{-1}(3/\sqrt{13}) + \cot^{-1}(16/63) + \dfrac{1}{2}\cos^{-1}(7/25) = \pi$.

3. $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$.

4. $\tan^{-1}(m/n) + \tan^{-1}((n-m)/(n+m)) = \pi/4$ or $-3\pi/4$, according as $m/n > -1$ or $m/n < -1$.

5. $\tan^{-1}(5/12) + \sin^{-1}(7/25) = \cos^{-1}(253/325)$.

6. $3\tan^{-1}(1/4) + \tan^{-1}(1/20) + \tan^{-1}(1/1985) = \pi/4$.

7 $\tan^{-1}(1/3) + \tan^{-1}(1/5) + \tan^{-1}(1/7) + \tan^{-1}(1/8) = \pi/4$.

8 $2\tan^{-1}(1/5) + \tan^{-1}(1/7) + 2\tan^{-1}(1/8) = \pi/4$.

9 $\tan^{-1}\dfrac{a-b}{1+ab} + \tan^{-1}\dfrac{b-c}{1+bc} + \tan^{-1}\dfrac{c-a}{1+ca} = 0$.

10 $\cos(2\tan^{-1}(1/7)) = \sin(4\tan^{-1}(1/3))$.

11 $\tan^{-1}\dfrac{a(a+b+c)}{bc} + \tan^{-1}\dfrac{b(a+b+c)}{ca} + \tan^{-1}\dfrac{c(a+b+c)}{ab} = \pi$.

12 $\tan^{-1}\dfrac{\sqrt{x^2+a^2} - x + b}{\sqrt{a^2-b^2}} + \tan^{-1}\dfrac{x\sqrt{a^2-b^2}}{b\sqrt{x^2+a^2+b^2}} + \tan^{-1}\dfrac{\sqrt{a^2-b^2}}{b} = n\pi$.

In the first case, $\frac{x-\alpha}{2} = n\pi, n \in \mathbf{Z}$.

That is, $x = 2n\pi + \alpha, n \in \mathbf{Z}$.

In the second case, $\frac{x+\alpha}{2} = (2n+1)\pi/2, n \in \mathbf{Z}$.

That is, $x = (2n+1)\pi - \alpha, n \in \mathbf{Z}$.

Combining the two solutions we have

$$x = n\pi + (-1)^n \alpha, n \in \mathbf{Z}. \qquad \square$$

Remark 1. The one solution that is mentioned in the theorem has to be found either from the knowledge of standard values or from tables.

Remark 2. Note that in the proof we have used the solutions of $\sin x = 0$ and $\cos x = 0$ obtained in an earlier section (see problem 11, Exercise 6.4).

EXAMPLE 2. Now let us look at an equation of the form $\cos x = k$, $-1 \le k \le 1$. Let us take, for instance, $k = 1/\sqrt{2}$, which gives $\cos x = 1/\sqrt{2}$.

The number $1/\sqrt{2}$ is positive and the cosine function is positive in the first quadrant and the fourth quadrant. The (only) solution in the first quadrant is $x = 45^\circ = \pi/4$ and the solution set that this generates is $\{2n\pi + \pi/4 \mid n \in \mathbf{Z}\}$. The (only) solution in the fourth quadrant is $x = -45^\circ = -\pi/4$. The solution set corresponding to this solution is $\{2n\pi - \pi/4 \mid n \in \mathbf{Z}\}$. Hence the general solution is given by $x = 2n\pi \pm \pi/4$, $n \in \mathbf{Z}$. Similarly we can look at equations like $\cos x = -1/\sqrt{2}$. In general, we have the following theorem.

Theorem 8. If $-1 \le k \le 1$ and α is one solution of $\cos x = k$, then the general solution is given by $x = 2n\pi \pm \alpha$, n being any integer.

Proof. We have $\cos x = k = \cos\alpha$; i.e., $\cos x - \cos\alpha = 0$.

Hence $$-2\sin\frac{x-\alpha}{2}\sin\frac{x+\alpha}{2} = 0.$$

If $\sin\frac{x+\alpha}{2} = 0$, we have $\frac{x+\alpha}{2} = n\pi, n \in \mathbf{Z}$

giving $x = 2n\pi - \alpha, n \in \mathbf{Z}$.

If $\sin\frac{x-\alpha}{2} = 0$, we have $\frac{x-\alpha}{2} = n\pi, n \in \mathbf{Z}$

giving $x = 2n\pi + \alpha, n \in \mathbf{Z}$.

Thus the general solution is given by

$$x = 2n\pi \pm \alpha, n \in \mathbf{Z}.$$

(*a*) The solution of $\cos x = 1$ is $x = 2n\pi \pm 0 = 2n\pi$, $n \in \mathbf{Z}$.

(*b*) The solution of $\cos x = 0$ is $x = 2n\pi \pm \pi/2$, $n \in \mathbf{Z}$. That is $x = (4n \pm 1)\pi/2$, $n \in \mathbf{Z}$. Since numbers of the form $4n + 1$, $n \in \mathbf{Z}$ and those of the form $4n - 1$, $n \in \mathbf{Z}$ together exhaust all odd integers, we may write the solutions more compactly in the form $x = (2n+1)\pi/2$, $n \in \mathbf{Z}$.

(*c*) The solution of $\cos x = -1$ is given by $x = 2n\pi \pm \pi$, $n \in \mathbf{Z}$. That is $x = (2n \pm 1)\pi$, $n \in \mathbf{Z}$. Now numbers of the form $2n + 1$, $n \in \mathbf{Z}$ are the same as those of the form $2n - 1$, $n \in \mathbf{Z}$ because both are collections of all odd integers. Hence avoiding duplicity, we can write the solution in the simpler form $x = (2n + 1)\pi$, $n \in \mathbf{Z}$.

Summarizing, we can state that the cosine function (*a*) takes the value 1 at an even multiple of π; (*b*) vanishes at an odd multiple of $\pi/2$; and (*c*) takes the value -1 at an odd multiple of π.

EXAMPLE 3. Now consider an equation of the form $\tan x = k$. Let us take $k = 2 - \sqrt{3}$. Here k is a positive number and the tangent function is known to be positive in the first and third quadrants. The solutions are $x = 15° = \pi/12$ and $x = 180° + 15° = \pi + \pi/12$ respectively in these quadrants. These give rise to the solution sets $\{2n\pi + \pi 12 \mid n \in \mathbf{Z}\}$ and $\{2n\pi + \pi + \pi/12 \mid n \in \mathbf{Z}\}$. Their union is $\{2n\pi + \pi/12 \mid n \in \mathbf{Z}\}$ U $\{(2n + 1)\pi + \pi/12 \mid n \in \mathbf{Z}\}$ which can be compactly written in the form $\{n\pi + \pi\backslash 12 \mid n \in \mathbf{Z}\}$. We can similarly deal with equations such as $\tan x = -(2 - \sqrt{3})$.

Theorem 9. Let k be any real number and α be a particular solution of the equation $\tan x = k$. Then the general solution is $x = n\pi + \alpha$, where $n \in \mathbf{Z}$.

Proof. We have $\tan x = k = \tan\alpha$. So $\tan x - \tan\alpha = 0$. Simplifying, we have $\sin(x - \alpha) = 0$, and hence $x - \alpha = n\pi$, $n \in \mathbf{Z}$ giving $x = n\pi + \alpha$, $n \in \mathbf{Z}$. ❑

Remark 4. Equation of the form $\cot x = k$, $\sec x = k$, $\operatorname{cosec} x = k$ can be respectively (and equivalently) transformed into the forms $\tan x = 1/k$ $(k \neq 0)$, $\cos x - 1/k$, $\sin x = 1/k$, which can be solved as above.

EXAMPLE 4. *Solve:* $\cos^2\theta - \sin\theta = -1$.

SOLUTION. We use the relation $\cos^2\theta = 1 - \sin^2\theta$ and rewrite the given equation as a quadratic equation in $\sin\theta$. Accordingly, we have

$$\sin^2\theta + \sin\theta - 2 = 0.$$

Factorising, we get

$$(\sin\theta - 1)(\sin\theta + 2) = 0.$$

Thus $\sin\theta = 1$ or -2; $\sin\theta = -2$ has no solutions. If $\sin\theta = 1$, we have $\theta = n\pi + (-1)^n \pi/2$, $n \in \mathbf{Z}$, which is the same as $\theta = (4n + 1)\pi/2$, $n \in \mathbf{Z}$.

EXAMPLE 5. *Solve:* $\sin 5x + \sin 3x = 2\cos x$.

SOLUTION. The given equation can be written as

$$2\sin 4x\cos x = 2\cos x.$$

That is, $2\cos x(\sin 4x - 1) = 0$. So, either $\cos x = 0$ or $\sin 4x = 1$.

The equation $\cos x = 0$ gives $x = (2n + 1)\pi/2$, $n \in \mathbf{Z}$.

The equation $\sin 4x = 1$ gives $4x = (4n + 1)\pi/2$, $n \in \mathbf{Z}$.

That is, $x = (4n + 1)\dfrac{\pi}{8}$, $n \in \mathbf{Z}$.

Hence the solution set is

$$\{(4n + 1)\pi/8 \mid n \in \mathbf{Z}\} \cup \{(2n + 1)\pi/2 \mid n \in \mathbf{Z}\}.$$

[illegible] Suppose we are asked to solve the equation in Example 5 in $[0, 2\pi]$. Then $\cos x = 0$ has two solutions, $x = \pi/2$ and $x = 3\pi/2$, while $\sin 4x = 1$ has four solutions, $x = \pi/8$, $x = 5\pi/8$, $x = 9\pi/8$, $x = 13\pi/8$. In all, we have six solutions.

EXAMPLE 6. *Solve the equation*

$$a \cos x + b \sin x = c.$$

What is the condition to be satisfied by a, b, c for the existence of a solution? Hence solve $\sqrt{3}\cos x - \sin x = 1$.

We know that for any ordered pair (a, b) of real numebrs (not simultaneously zero), there exists a corresponding pair of real numbers r and θ such that $r > 0$;

$$a = r\cos\theta \text{ and } b = r\sin\theta.$$

To see this, plot the point $P = (a, b)$ in the xy – plane (P may lie in any quadrant or an any axis). Join OP. Let $OP = r$ and $\angle xOP = \theta$, where θ is measured in the positive direction. From the very definitions of $\sin\theta$ and $\cos\theta$, we have $\cos\theta = a/r$ and $\sin\theta = b/r$, where $a = r\cos\theta$ and $b = r\sin\theta$.

Here $\quad r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$ or $\pi + \tan^{-1}(b/a)$.

The angle θ should satisfy $r\cos\theta = a$ and $r\sin\theta = b$ and so lies in the same quadrant as $P = (a, b)$.

The given equation then becomes $r\cos\theta\cos x + r\sin\theta\sin x = c$.

Hence $\quad r\cos(x - \theta) = c$; *i.e.*, $\cos(x - \theta) = c/r$.

For this equation to have a solution, we must have

$$-1 \le c/r \le 1,$$

which is the same as

$$-\sqrt{a^2+b^2} \le c \le \sqrt{a^2+b^2}$$

If $\cos\alpha = c/r$, then the general solution is given by

$$x - \theta = 2n\pi \pm \alpha,\ n \in Z.$$

giving

$$x = 2n\pi + \theta \pm \alpha,\ n \in Z.$$

Thus a criterion for the solvability of the equation is

$$|c| \le \sqrt{a^2+b^2}.$$

In simple cases, it is advisable to divide the given equation directly by $\sqrt{a^2+b^2}$ and recognise $\frac{a}{a^2+b^2}$ and $\frac{b}{a^2+b^2}$ as $\cos\theta$ and $\sin\theta$ for some suitable angle θ. We apply this method to the given numerical problem. Dividing the equation by $\sqrt{(\sqrt{3})^2 + (-1)^2} = 2$, we obtain $(\sqrt{3}/2)\cos x - (1/2)\sin x = 1/2$.

Observing that $\sqrt{3}/2 = \cos\pi/6$ and $1/2 = \sin\pi/6$, we get $\cos\pi/6\cos x - \sin\pi/6\sin x = 1/2$. That is, $\cos(x + \pi/6) = 1/2 = \cos\pi/3$, which gives $x + \pi/6 = 2n\pi \pm \pi/3$, $n \in Z$. Hence the solution is $x = 2n\pi - \pi/6 \pm \pi/3$, $n \in Z$.

EXAMPLE 7. *Solve the equation*

$$\sin(x + \pi/4) = \sin 2x.$$

SOLUTION. At the first glance, we may be tempted to use the formula for $\sin C - \sin D$ which anyway gives the correct solution. But we may also argue in the following manner:

The value $2x$ may be considered as a particular solution of $x + \pi/4$ and hence the general solution is given by

$$x + \pi/4 = n\pi + (-1)^n 2x, n \in Z,$$

since the sine relation is involved.

Solving this for x, we get

$$x = \frac{n\pi - \pi/4}{1-(-1)^n 2}, n \in Z.$$

Compare this with the solution obtained by the first method, namely,

$$x = \frac{\pi}{4} + 2n\pi, n \in Z \text{ or } \frac{(2n+1)\pi - \pi/4}{3}, n \in Z.$$

In fact these two solution sets are the same !

EXAMPLE 8. *Solve:* $\sec x + \operatorname{cosec} x = 2\sqrt{2}$.

SOLUTION. The given equation is the same as

$$\sin x + \cos x = 2\sqrt{2} \sin x \cos x.$$

Dividing by $\sqrt{2}$ and using the fact that $\sin \pi/4 = 1/\sqrt{2} = \cos \pi/4$, we get

$$\sin (x + \pi/4) = \sin 2x,$$

which is the same as the equation in Example 7.

EXERCISE 6.8

Solve the equations (1) – (12) for θ.

1. $2 \cos 2\theta - 7 \cos \theta = 0$.
2. $\sin 3\theta + 5 \sin \theta = 0$.
3. $\sin (m+n)\theta + \sin (m-n)\theta = \sin m\theta$.
4. $\tan 5\theta + \cot 2\theta = 0$.
5. $\tan \theta + \tan 2\theta + \tan 3\theta = 0$.
6. $\cos 3\theta = \cos^3 \theta$.
7. $(\sqrt{2} - 1) \cos \theta + \sin \theta = 1$.
8. $2 \cos \theta + 3 \sin \theta = 3$.
9. $\sqrt{3} \sin \theta + \cos \theta = \cos (\pi/5) \sec \theta$.
10. $\tan \theta + \tan (\theta + \pi/3) + \tan (\theta - \pi/3) = 3$.
11. $\cot \theta + \cot 2\theta + \cot 3\theta = 0$.
12. $\sin \theta + \sin 2\theta + \sin 3\theta = 0$.
13. Explain how to solve $a \cos^2 \theta + 2b \cos \theta \sin \theta + c \sin^2 \theta = d$. What is the condition governing a, b, c, d for the existence of a solution?
14. Solve the equation $a \cos \theta + b \sin \theta = c$, by transforming this into a quadratic equation in $\tan \theta/2$. Hence solve $(\sqrt{2} - 1) \cos \theta + \sin \theta = 1$.
15. Solve the equation $\cos \theta + \sin 2\theta = 0$ by the following two methods:
    (*a*) Write the equation as $\cos \theta (1 + 2 \sin \theta) = 0$ and solve $\cos \theta = 0$, $2 \sin \theta + 1 = 0$ separately.

(*b*) Write the equation as $\sin 2\theta = \sin(\theta - \pi/2)$ and solve as in. Example 7. Compare the two solution sets obtained in (*a*) and (*b*) and show that they are the same.

16. Solve the equation for θ:
$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$, A, B, C being the angles of a triangle.

17. Find the fallacy in the following argument: Let us solve the equation

$$2\cos^2\theta = \sin\theta + \sin 3\theta - 5\cos\theta.$$

This is the same as

$$2\cos^2\theta = 2\sin 2\theta\cos\theta - 5\cos\theta.$$

Hence, $\quad 2\cos\theta = 2\sin 2\theta - 5.$

That is, $\quad 2\sin 2\theta - 2\cos\theta = 5.$

Now $|2\sin 2\theta - 2\cos\theta| \le 2 + 2 = 4$ and so the given equation has no solution. But $\theta = \pi/2$ is a solution of the given equation, as can easily be checked.

18. Prove the following relations:

(*a*) $\{2n\pi + \alpha \mid n \in Z\} \cup \{(2n+1)\pi - \alpha \mid n \in Z\}$
$$= \{n\pi + (-1)^n\alpha \mid n \in Z\}.$$

(*b*) $\{2n+1 \mid n \in Z\} = (2n-1 \mid n \in Z\}.$

(*c*) $\{4n+1 \mid n \in Z\} \cup \{4n-1 \mid n \in Z\} = \{2n+1 \mid n \in Z\}.$

(*d*) $$\left\{\frac{n\pi - \pi/4}{1-(-1)^n 2} \,\middle|\, n \in Z\right\} = \{\pi/4 - 2n\pi \mid n \in Z\} \cup \left\{\frac{(2n+1)\pi - \pi/4}{3} \,\middle|\, n \in Z\right\}$$

(*see Example 7*).

19. Find the minimum and maximum values of the following expressions in θ:

(*a*) $\cos\theta + \sin\theta$.

(*b*) $\sqrt{3}\cos\theta - \sin\theta$.

(*c*) $3\sin\theta + 4\cos\theta + 2$.

(*d*) $a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta$.

(*e*) $\cos\theta\,(\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha})$.

20. Solve the system for x and y:

$$\tan x + \tan y = (4/3)\sqrt{3}; \quad \tan 2x + \tan 2y = 0.$$

21. Draw the graphs of the following equations:

(*a*) $y = \cos x + \sin x$

(*b*) $y = \sqrt{3}\cos x - \sin x + 1$.

22. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, then show that

$$\theta = (1/4)\,[(2n+1)\pi \pm \cos^{-1}(1/8)],\ n \in Z.$$

6.9 PROPERTIES OF TRIANGLES

The triangle is one of the simplest geometrical figures and has many interesting properties. Associated with a triangle are some special points, circles and distances (lengths). We study the properties of these points and circles in this section. If ABC is a triangle, the six quantities namely the three angles A, B, C and the three sides $BC = a$, $CA = b$, $AB = c$ are called the *elements* of the triangle.

Also the semiperimeter $\dfrac{a+b+c}{2}$ of triangle ABC is denoted by s.

A. The circumcentre and the sine and cosine rules.

Theorem 10. (*The Sine Rule*) : In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof.

Case I. Suppose A is an acute angle in the acute-angled triangle ABC. Then the circumcentre S is in the interior of the triangle. Join BS and produce it to meet the circumcircle in D. Join DC. Then as A and D are on the same side of BC, we have $\angle BDC = \angle BAC = A$. But from the right triangle BDC in which $\angle BCD = 90°$, we have $\sin \angle BDC = BC/BD$.

That is, $\sin A = \dfrac{a}{2R}$. So $\dfrac{a}{\sin A} = 2R.$

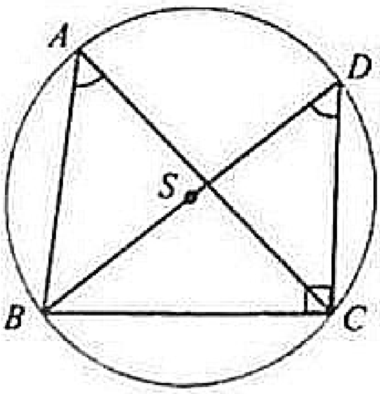


Fig. 6.30

Case II. Suppose A is an obtuse angle. Then the circumcentre S is outside the triangle ABC and in fact A and S lie on the opposite sides of BC (Fig. 6.31, 6.32). Produce BS to meet the circumcircle in D and join CD. Then as A and D are on the opposite sides of BC, we have $\angle BDC = 180° - \angle BAC = 180° - A$. Also from triangle BDC which is right-angled at C, we get $\sin \angle BDC = BC/BD$.

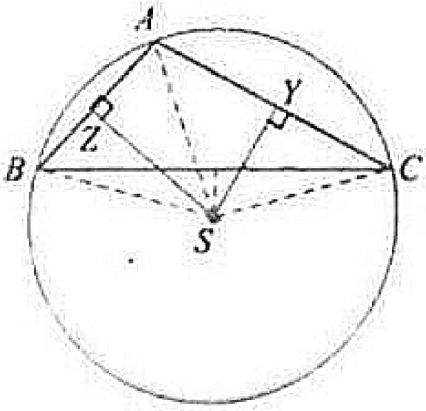


Fig. 6.31

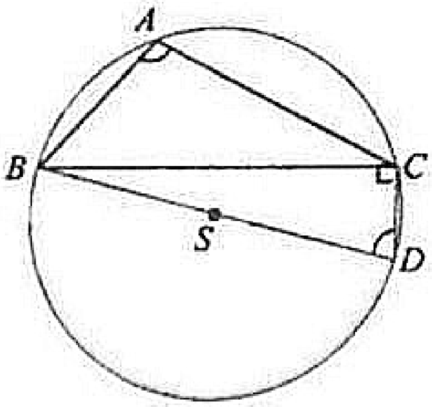


Fig. 6.32

That is, $\sin(180° - A) = \dfrac{a}{2R}$.

But $\sin(180° - A) = \sin A$. Hence $\sin A = \dfrac{a}{2R}$ and $\dfrac{a}{\sin A} = 2R.$

Case III. Let A be a right angle. Then S is the midpoint of the hypotenuse BC (Fig. 6.33, 6.34), which is a diameter of the circumcircle, so $a = BC = 2R$ and

$$\frac{a}{\sin A} = \frac{2R}{\sin 90^\circ} = \frac{2R}{1} = 2R.$$

Thus we have proved that $a/(\sin A) = 2R$ in all cases. Similarly one proves that

$$\frac{b}{\sin B} = 2R \text{ and } \frac{c}{\sin C} = 2R.$$

This is the Sine Rule (or the Sine Law). ☐

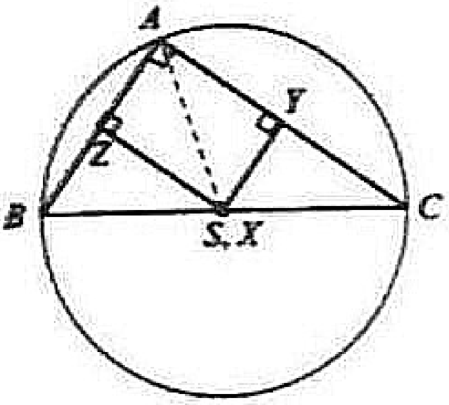


Fig. 6.33

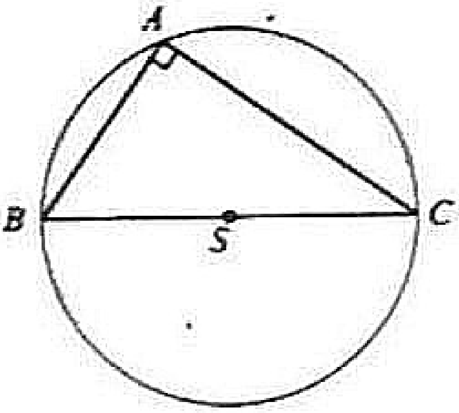


Fig. 6.34

Remark 1. The reader might have noticed a small lacuna in the proof in Case I. Though A is an acute angle, the triangle itself need not be acute-angled. The acute angle A may lie in a right-angled triangle or obtuse-angled triangle. These two cases may be dealt with similarly and the reader should convince himself of the validity of the sine rule.

From the above theorem, we have $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$.

EXAMPLE 1. *Show that* $a = b \cos C + c \cos B$.

SOLUTION. We have $b \cos C + c \cos B = 2R \sin B \cos C + 2R \sin C \cos B = 2R (\sin B \cos C + \cos B \sin C) = 2R \sin (B + C) = 2R \sin (\pi - A) = 2R \sin A = a$.

Remark 2. Similarly it follows that $b = c \cos A + a \cos C$, $C = a \cos B + b \cos A$. One can prove this relation geometrically using projections [see problem 13, Exercise 6.9].

EXAMPLE 2. Solve the triangle ABC, given $a = 6$, $B = 45^\circ$, $A = 75^\circ$. Find the circumradius of the triangle.

Remark 3. Solving a triangle means to find the three remaining elements, given three independent elements of the triangle.

SOLUTION. We have $C = 180^\circ - (A + B) = 60^\circ$.

From the Sine Rule,
$$b = \frac{a}{\sin A} \sin B$$
$$= \frac{6}{\sin 75^\circ} \sin 45^\circ = \frac{6}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} \cdot \frac{1}{\sqrt{2}} = \frac{12}{\sqrt{3}+1} = 6(\sqrt{3} - 1).$$

Again from the same rule.

$$c = \frac{a}{\sin A} \sin C = \frac{6}{\sin 75^\circ} \sin 60^\circ$$
$$= \frac{6}{\frac{\sqrt{3}-1}{2\sqrt{2}}} \frac{\sqrt{3}}{2} = \frac{6\sqrt{6}}{\sqrt{3}+1} = 3\sqrt{6}(\sqrt{3} - 1).$$

$$R = \frac{a}{2\sin A} = \frac{6}{2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} = \frac{6\sqrt{2}}{\sqrt{3}+1} = 3(\sqrt{6}-\sqrt{2}).$$

Theorem 11. (The *Cosine Rule*) In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A,$$
$$b^2 = c^2 + a^2 - 2ca\cos B,$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Proof.

Case I. Let A be an acute angle in acute-angled triangle ABC. Draw CD perpendicular to AB. The point D falls within the interior of side AB (Fig. 6.35). We have

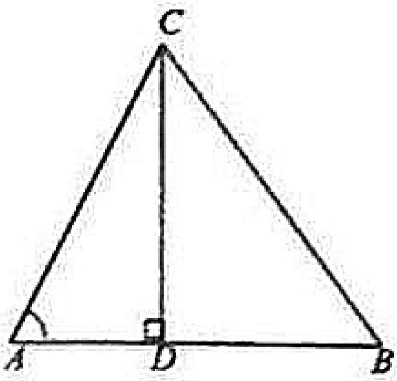


Fig. 6.35

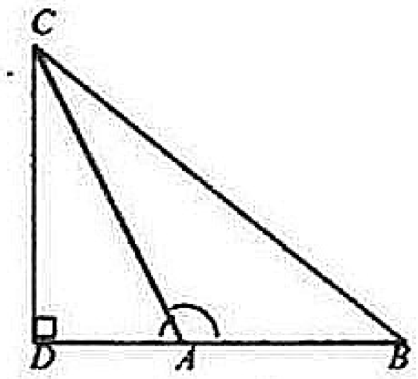


Fig. 6.36

$$\begin{aligned} BC^2 &= CD^2 + DB^2 && \text{(from the right triangle } BDC) \\ &= CD^2 + (AB - AD)^2 = CD^2 + AB^2 + AD^2 - 2AB \cdot AD \\ &= (AD^2 + CD^2) + AB^2 - 2AB \cdot AD \\ &= AC^2 + AB^2 - 2AB \cdot AD, && \text{(from the right triangle } ACD). \end{aligned}$$

Now from triangle ACD, $\cos \angle CAD = AD/DC$.

That is, $\quad AD = AC\cos A$.

So $\quad BC^2 = AC^2 + AB^2 - 2AB \cdot AC\cos A$.

That is, $\quad a^2 = b^2 + c^2 - 2\,bc\cos A$.

Case II. Let A be an obtuse angle. Draw CD perpendicular to BA extended. The point D falls outside the line segment AB (Fig. 6.36).

We have

$$\begin{aligned} BC^2 &= CD^2 + DB^2 && \text{(from the right triangle } BCD) \\ &= CD^2 + (DA + AB)^2 \\ &= CD^2 + DA^2 + AB^2 + 2DA \cdot AB \\ &= AC^2 + AB^2 + 2DA \cdot AB && \text{(from the right triangle } ACD) \end{aligned}$$

From triangle ACD, $\cos \angle DAC = AD/AC$.

So $\quad AD = AC\cos(180° - A) = -AC\cos A$.

Hence $\quad BC^2 = AC^2 + AB^2 - 2AB \cdot AC\cos A$.

That is, $\quad a^2 = b^2 + c^2 - 2bc\cos A$.

Case III. Let A be a right angle (Fig. 6.37). Straightaway

$$\begin{aligned} b^2 + c^2 - 2bc\cos A &= b^2 + c^2 - 2bc\cos 90° \\ &= b^2 + c^2 = a^2. \end{aligned}$$

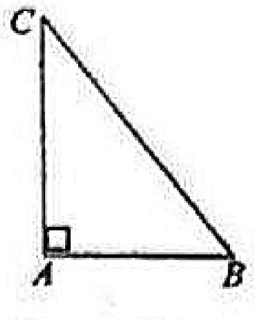


Fig. 6.37

Thus in all cases,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Similarly the other two relations can be proved.

This is the Cosine Rule (or the Cosine Law). ❑

Remark 4. As before the proof in case *I* is incomplete. The acute angle A may also lie in a right triangle or an obtuse triangle *ABC*. The cosine rule may be proved in these two cases also similarly.

From the cosine rule, one has

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

EXAMPLE 3. *If $C = 60°$, show that*

$$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}.$$

Prove the converse also.

SOLUTION. The given relation holds good

iff $(a + b + 2c)(a + b + c) = 3(a + c)(b + c)$,

i.e., iff $a^2 + b^2 - ab = c^2$,

i.e., iff $a^2 + b^2 - ab = a^2 + b^2 - 2ab \cos C$,

i.e., iff $\cos C = 1/2$,

i.e., iff $C = 60°$.

Note. The Exercises under this Section 6.9 A would form part of Exercise 6.9 B to come after the next Section.

B. Ratios of A/2 and the area of a triangle:

From the identity $\cos A = 1 - 2\sin^2 (A/2)$, we have

$$\sin^2 (A/2) = 1 - \cos A$$

$$= 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}.$$

Since $2s = a + b + c$, we have $a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c)$
and $a - b + c = 2(s - b)$, similarly.

So $$2\sin^2(A/2) = \frac{4(s-b)(s-c)}{2bc}.$$

Therefore $$\sin^2(A/2) = \frac{(s-b)(s-c)}{bc}.$$

Taking square roots on both sides and observing that $\sin (A/2)$ is always positive (because $0 < A/2 < 90°$), we get

$$\sin(A/2) = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Using the identity $\cos A = 2\cos^2(A/2) - 1$, one proves similarly that

$$\cos(A/2) = \sqrt{\frac{s(s-a)}{bc}}.$$

So $$\tan(A/2) = \frac{\sin(A/2)}{\cos(A/2)} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ and } \cot(A/2)$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.$$

The ratios of $B/2$ and $C/2$ are similarly obtained.

Again $$\sin A = 2\sin(A/2)\cos(A/2) = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c).}$$

The expressions for $\sin B$ and $\sin C$ are similarly written down.

Theorem 12. (Area of a triangle). If Δ denotes the area of triangle ABC, then

$$\Delta = (1/2)bc\sin A = (1/2)ca\sin B = (1/2)ab\sin C.$$

The proof is left to the reader. One has only to use the formula area = (1/2) (base) (height) for a triangle and consider the three cases A acute, A obtuse and A a right angle. The case A acute has three subcases as usual. □

Corollary 1. $$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \qquad \textit{(Heron's formula)}$$

$$= 2R^2 \sin A \sin B \sin C.$$

Proof. We have $$\Delta = (1/2)\, bc \sin A$$

$$= (1/2)\, bc \,.\, \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Also $$\Delta = (1/2)\, bc\sin A$$

$$= (1/2)\cdot 2R\sin B \cdot 2R\sin C \cdot \sin A$$

$$= 2R^2 \sin A \cdot \sin B \cdot \sin C.$$ □

Corollary 2. (*a*) $\Delta = \dfrac{abc}{4R}$; (*b*) $R = \dfrac{abc}{4\Delta}$.

The proofs of these simple relations are left to the reader. □

EXAMPLE 4. *If in triangle ABC,* $a = 13$, $b = 4$, $c = 15$ *find* Δ *and* R.

SOLUTION. We have $s = \dfrac{a+b+c}{2} = 16$; so $s-a = 16-13 = 3$; $s-b = 16-4 = 12$;

$s - c = 16 - 15 = 1$.

Hence $$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16\times 3\times 12\times 1} = 24 \text{ square units.}$$

$$R = \frac{abc}{4\Delta} = \frac{13\times 4\times 15}{4\times 24} = 8\frac{1}{8} \text{ units.}$$

EXAMPLE 5. *Show that, in a triangle ABC,*

$$\frac{\cot A+\cot B+\cot C}{\cot \frac{A}{2}+\cot \frac{B}{2}+\cot \frac{C}{2}}=\frac{a^2+b^2+c^2}{(a+b+c)^2}.$$

SOLUTION. We have
$$\cot A=\frac{\cos A}{\sin A}=\frac{b^2+c^2-a^2}{2bc\sin A}=\frac{b^2+c^2-a^2}{4\Delta}$$

Similar expressions for cot B and cot C can be written down.

So
$$\cot A+\cot B+\cot C$$
$$=\frac{b^2+c^2-a^2}{4\Delta}+\frac{c^2+a^2-b^2}{4\Delta}+\frac{a^2+b^2-c^2}{4\Delta}$$
$$=\frac{a^2+b^2+c^2}{4\Delta}$$

Again
$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$$
$$=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$
$$=\frac{\sqrt{s}[(s-a)+(s-b)+(s-c)]}{\sqrt{(s-a)(s-b)(s-c)}}=\frac{\sqrt{s}\sqrt{s}[3s-(a+b+c)]}{\sqrt{s(s-a)(s-b)(s-c)}}$$
$$=\frac{s(3s-2s)}{\Delta}=\frac{s^2}{\Delta}.$$

Hence L.H.S. of the given relation
$$=\frac{a^2+b^2+c^2}{4\Delta}\div\frac{s^2}{\Delta}=\frac{a^2+b^2+c^2}{4s^2}=\text{R.H.S.}$$

EXERCISE 6.9 B

1. Show that (*a*) $\dfrac{a+b}{c}=\dfrac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}$;

   (*b*) $\dfrac{a-b}{a+b}=\dfrac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}$; (*c*) $\dfrac{b-c}{b+c}\cot\dfrac{A}{2}=\tan\dfrac{B-C}{2}$.

2. Prove *Appollonius's Theorem* using the Cosine Rule : 'If ABC is a triangle and D is the midpoint of BC, then $AB^2+AC^2=2(AD^2+BD^2)$.' Hence evaluate the lengths of the medians of a triangle in terms of its sides. (You will find the answers in Theorem 28 of Chapter 4).

3. Prove the following generalisation of the result in Problem 2 above : 'If ABC is a triangle and D is a point on BC dividing it internally in the ratio $m:n$, then

$$(m+n)\cdot AD^2 = m\cdot AC^2 + n\cdot AB^2 - \frac{mn}{m+n}\cdot BC^2.$$

What is the corresponding result if D divides BC externally in the ratio $m : n$?

4. If the internal bisector of angle A meets the opposite side BC in D, show that

$$AD = \frac{2bc}{b+c}\cos\frac{A}{2}.$$

Hence or otherwise show that if the internal bisectors of two angles are equal in a triangle, then the triangle is isosceles.

5. If in a triangle ABC, $a = 13$, $b = 4$, $c = 15$, find its altitudes h_a, h_b, h_c.

6. If in a triangle ABC, h_a, h_b, h_c are its altitudes and $a \geq b \geq c$., show that $a + h_a \geq b + h_b \geq c + h_c$.

7. If SX, SY, SZ are the perpendiculars dropped on the sides BC, CA, AB of a triangle ABC from its circumcentre S, show that $SX = R\cos A$, $SY = R\cos B$, $SZ = R\cos C$.

8. Prove that, in any triangle ABC,
   (*i*) $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc$;
   (*ii*) $a^3\sin(B-C) + b^3\sin(C-A) + c^3\sin(A-B) = 0$.

9. Show that, in a triangle ABC,
   (*i*) $4\Delta = b^2\sin 2C + c^2\sin 2B$;
   (*ii*) $\Delta = \dfrac{a^2}{2(\cot B + \cot C)}$;
   (*iii*) $16\Delta^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$.

10. If P is an interior point in a triangle ABC, then
    $\sin\angle PAB\cdot\sin\angle PBC\cdot\sin\angle PCA = \sin\angle PBA\cdot\sin\angle PCB\cdot\sin\angle PAC$.
    State and prove the converse of this result.

11. If D is a point on the side BC of a triangle ABC such that $BD : DC = m : n$, and $\angle ADC = \theta$, $\angle DAB = \alpha$, $\angle DAC = \beta$, then show that
    (*i*) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$.
    (*ii*) $(m+n)\cot\theta = n\cot B - m\cot C$.

12. If P is an interior point of a triangle ABC, such that $\angle PAB = \angle PBC = \angle PCA = w$, then show that
    (*i*) $\cot w = \cot A + \cot B + \cot C$;
    (*ii*) $\operatorname{cosec}^2 w = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$.

13. (*a*) Prove geometrically that $a = (b\cos C) + (c\cos B)$.
    (*b*) Prove the cosine rule using the sine rule [Hint : First prove $a = b\cos C + c\cos B$ and two similar relations (see Example 1). Then solve for $\cos A$, $\cos B$, Cos C].

14. Given the numerical values of sides a, b, c of a triangle ABC, its angles A, B, C can be found by using any of the expressions for the sines, the cosines of the angles A, B, C, or the sines, the cosines, the tangents, of the angles $A/2$, $B/2$, $C/2$. But generally the expressions $\tan A/2$, $\tan B/2$, $\tan C/2$ are preferred to the others while using the logarithmic tables. Explain why.

15. Two triangles have the same perimeter and area. If the sides of one triangle are 51, 35, 26 and one side of the other triangle is 41, find the remaining sides of the latter.

16. Solve the triangle ABC, given side a, angle A and the product $k = bc$ of the other two sides. Under what conditions does the triangls exist?

17. Let an n-sided regular polygon of side a inscribed in a circle of radius R and circumscribed about a circle of radius r. Find the area of the polygon in terms of (*i*) a (*ii*) r (*iii*) R. Also express R and r in terms of a.

18. Suppose a triangle ABC is to be solved given b, c and B. Use the sine rule to show that (*i*) if B is acute; then no triangle, one right triangle, two triangles or one triangle exists according as $b < c \sin B$, $b = c \sin B$, $c > b > c \sin B$ or $b > c$. (*ii*) if B is greater than or equal to 90°, then exactly one solution exists if $b > c$ and none otherwise.

19. Discuss the ambiguous case of Question 21 above algebraically considering the relation $b^2 = c^2 + a^2 - 2\,c\,a \cos B$ as a quadratic equation in a and solving it for a.

20. Solve the following triangles:

(*i*) $a = 121, b = 94, c = 135$.
(*ii*) $b = 108, c = 77, A = 76°$.
(*iii*) $a = 55, A = 65°, B = 38°$.
(*iv*) $b = 37, c = 42, B = 52°$.
(*v*) $a = 125, b = 112, A = 40°$.
(*vi*) $c = 13.1, a = 17.5, C = 49°$.
(*vii*) $b = 1, c = \sqrt{6} - \sqrt{2}, B = 75°$.
(*viii*) $a = 100, b = 80, A = 130°$.
(*ix*) $a = 70,\ c = 56.3, C = 100°$.
(*x*) $a = 3, b = 5, c = 7$.

C. The Incentre and the Ex-centres:

From Theorem 24 of Chapter 3, we know that the internal bisectors of the angles of a triangle are concurrent and that the point of concurrence is equidistant from the sides. We recall that r denotes the inradius of a triangle.

Theorem 13. In a triangle ABC, the inradius is given by

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$$

$$= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Proof. (*i*) With reference to Figure 6.38

$$\Delta ABC = \Delta IBC + \Delta ICA + \Delta IAB$$

That is,
$$\Delta = (1/2)BC\cdot ID + (1/2)CA\cdot IE + (1/2)AB\cdot IF$$
$$= (1/2)(ar + br + cr)$$
$$= (1/2)(a + b + c)\,r = sr$$

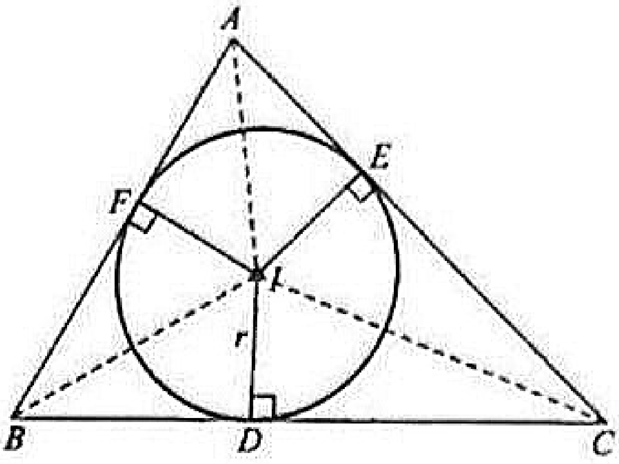


Fig. 6.38

Hence $r = \dfrac{\Delta}{s}$ (as already seen in Theorem 37 of Chapter 4).

(*ii*) Again from triangles IBD and ICD, we have

$$BD = ID \cot \angle IBD = r \cot(B/2),$$

and

$$CD = ID \cot \angle ICD = r \cot(C/2).$$

Adding we have

$$a = r(\cot B/2 + \cot C/2)$$

So

$$r = \frac{a}{\cot \frac{B}{2} + \cot \frac{C}{2}} = \frac{2R \sin A \sin(B/2) \sin(C/2)}{\sin \frac{B+C}{2}}$$

$$= (2R \cdot 2\sin(A/2)\cos(A/2)\sin(B/2)\sin(C/2))/\cos(A/2)$$

$$= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

(*iii*) Since AE and AF are tangents to the incircle we have $AE = AF$.

Similarly $EF = BD$, $CD = CF$.

Now

$$2s = a + b + c = a + AE + EC + AF + FB$$

$$= a + 2AE + CD + BD = a + 2AE + BC$$

$$= 2(a + AE).$$

So $AE = s - a = AF$ (as already seen in Theorem 33 of Chapter 4).

So

$$r = (s-a)\tan \frac{A}{2}.$$

Similarly from triangles IBD and ICD, we obtain

$$r = (s-b)\tan \frac{B}{2} \text{ and } r = (s-c)\tan \frac{C}{2}.$$

□

Again from Theorem 30 of Chapter 4, we know that the external bisectors of any two angles of a triangle and the internal bisector of the third angle are concurrent and that the point of concurrence is equidistant from all the three sides of the triangle. Let r_1, r_2, r_3 denote the ex-radii of the triangle. Note that in chapter 4 we have called these r_a, r_b, r_c.

Theorem 14. In a triangle ABC,

$$r_1 = \frac{\Delta}{s-a} = s \tan \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

Proof. (*i*) With reference to Figure 6.39,

quadrilateral

$$ABI_1C = \Delta ABI_1 + \Delta ACI_1$$

$$= \Delta ABC + \Delta I_1BC$$

Hence

$$\Delta ABC = \Delta I_1CA + \Delta I_1AB - \Delta I_1BC$$

That is,

$$\Delta = (1/2)AC.I_1F_1 + (1/2)AB.I_1F_1 - (1/2)BC.I_1D_1$$

$$= (1/2)(br_1 + cr_1 - ar_1)$$

$$= (1/2)(b + c - a)r_1 = (s-a)r_1$$

Therefore

$$r_1 = \frac{\Delta}{s-a}.$$

Similarly

$$r_2 = \frac{\Delta}{s-b} \text{ and } r_3 = \frac{\Delta}{s-c}.$$

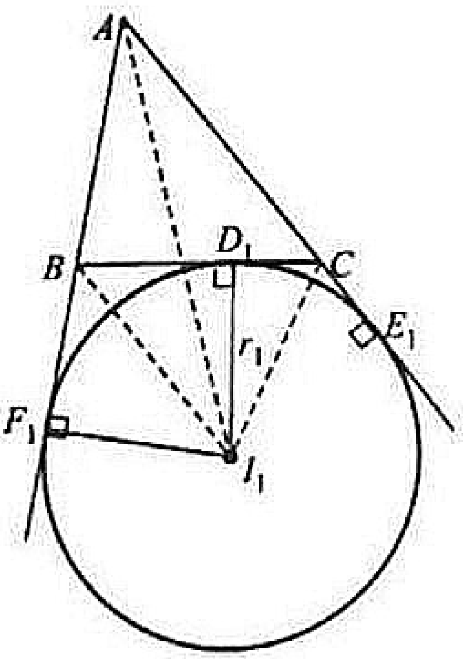


Fig. 6.39

Recall that this is Theorem 37(2) of Chapter 4.

(*ii*) Again from triangles $I_1\,BD$ and $I_1\,CD$,

$$BD_1 = I_1D_1 \cot \angle I_1BD$$
$$= r_1 \cot(90° - B/2)$$
$$= r_1 \tan (B/2);$$

and $$CD_1 = I_1D_1 \cot \angle I_1CD_1$$
$$= r_1 \tan(C/2).$$

Adding we get $$a = r_1(\tan B/2) + \tan(C/2))$$

So $$r_1 = \frac{a}{\tan\dfrac{B}{2} + \tan\dfrac{C}{2}}$$

$$= \frac{2R \sin A \cdot \cos(B/2) \cos (C/2)}{\sin \dfrac{(B+C)}{2}}$$

$$= 2R \cdot 2\sin\frac{A}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \div \cos\frac{A}{2}.$$

$$= 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

(Similarly $$r_2 = 4R \sin\frac{B}{2}\ \cos\frac{C}{2}\cos\frac{A}{2}$$

and $$r_3 = 4R \sin\frac{C}{2}\ \cos\frac{A}{2}\ \cos\ \frac{B}{2}.\Big)$$

(*iii*) Finally, since AE_1 and AF_1 are tangents to the excircle, we have $AE_1 = AF_1$. Similarly $BD_1 = BF_1$ and $CD_1 = CE_1$.

So $$2s = a + b + c = BC + CA + AB$$
$$= BD_1 + D_1C + CA + AB = (AB + BF_1) + (AC + CE_1)$$
$$= AF_1 + AE_1 = 2AE_1$$

That is, $$AE_1 = s = AF_1$$

[Note further that $BD_1 = BF_1 = AF_1 - AB = s - c$, and $CD_1 = CE_1 = AE_1 - AC = s - b$ Recall Theorem 34 and its corollary, from Chapter 4].

Hence from triangle I_1AE_1, we have

$$r_1 = I_1E_1 = AE_1 \tan IAE_1 = s \tan\frac{A}{2}.$$

$$\left(\text{Similarly } r_2 = s\tan\frac{B}{2} \text{ and } r_3 = s\tan\frac{C}{2}.\right)$$ ❑

EXAMPLE 6. *Show that* $r_1 - r_2 + r_3 + r = 4R \cos B$.

SOLUTION. We have $\quad r_1 - r_2 + r_3 + r = (r_1 + r_3) - (r_2 - r)$.

Now
$$r_1 + r_3 = 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$
$$= 4R\cos\frac{B}{2}\sin\frac{A+C}{2} = 4\text{R}\cos^2\frac{B}{2}.$$

Again,
$$r_2 - r = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} - 4R\sin\frac{B}{2}\sin\frac{A}{2}\sin\frac{C}{2}$$
$$= 4R\sin\frac{A}{2}\cos\frac{A+C}{2} = 4R\sin^2\frac{B}{2}$$

Hence
$$r_1 - r_2 + r_3 + r = 4R\cos^2\frac{B}{2} - 4R\sin^2\frac{B}{2}$$
$$= 4R\left(\cos^2\frac{B}{2} - \sin^2\frac{B}{2}\right) = 4R\cos B.$$

EXAMPLE 7. *If one of the ex-radii of a triangle is equal to its semiperimeter, then the triangle is right-angled.*

SOLUTION. Let $\quad r_1 = s$, We know that $r_1 = s\tan\frac{A}{2}$.

Hence $\quad \tan\frac{A}{2} = 1$ which means $\frac{A}{2} = 45°$. That is, A = 90°.

EXAMPLE 8. *If* $r : b + c : a = 2 : 17 : 13$, *then the triangle is right-angled.*

SOLUTION. Let $\quad r = 2\lambda,\ b + c = 17\lambda,\ a = 13\lambda$.

We have
$$r = (s-a)\tan\frac{A}{2} = \frac{1}{2}(b+c-a)\tan\frac{A}{2}$$

$$\therefore \quad \tan\frac{A}{2} = \frac{2r}{b+c-a} = \frac{4\lambda}{17\lambda - 13\lambda} = 1. \text{ Hence } A = 90°.$$

EXAMPLE 9. *Show that*

$$\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{D^2}.$$

SOLUTION. We have
$$\text{L.H.S.} = \frac{s^2}{\Delta^2} + \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2}$$
$$= \frac{1}{\Delta^2}[s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2]$$
$$= \frac{1}{\Delta^2}[4s^2 - 2(a+b+c)s + a^2 + b^2 + c^2]$$
$$= \frac{a^2+b^2+c^2}{\Delta^2} = \text{R.H.S.}$$

EXAMPLE 10. *Show that* $R \geq 2r$ *and that equality holds good iff the triangle is equilateral.*

SOLUTION. The given statement is equivalent to the following:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8} \quad \text{with equality iff } A=B=C.$$

We have

$$\begin{aligned}
&\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\
&=\frac{1}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]\sin\frac{C}{2}\\
&=\frac{1}{2}\left[\cos\frac{A-B}{2}-\sin\frac{C}{2}\right]\sin\frac{C}{2}\\
&\le\frac{1}{2}\left[1-\sin\frac{C}{2}\right]\sin\frac{C}{2}\\
&=\frac{1}{2}\left[\sin\frac{C}{2}-\sin^2\frac{C}{2}\right]\\
&=\frac{1}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-\sin\frac{C}{2}\right)^2\right]\le\frac{1}{8}
\end{aligned}$$

The equality holds good iff $\cos\frac{A-B}{2}=1$ and $\sin\frac{C}{2}=\frac{1}{2}$ which happens iff $A=B=C\,(=60^\circ)$.

EXERCISE 6.9 C

Prove the following relations for a triangle ABC.

1. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.
2. $rr_1r_2r_3=\Delta^2$.
3. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$.
4. $r_2r_3+r_3r_1+r_1r_2=s^2$.
5. $rr_1+rr_2+rr_3=ab+bc+ca-s^2$.
6. $r_1+r_2+r_3-r=4R$.
7. $r_1^2+r_2^2+r_3^2+r^2=16R^2-a^2-b^2-c^2$.
8. $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$.
9. $a\cot A+b\cot B+c\cot C=2(R+r)$.
10. $(r_2+r_3)(r_3+r_1)(r_1+r_2)=4Rs^2$.
11. $r_1r_2r_3=rs^2$.
12. $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$.
13. $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}+\frac{1}{2R}$.

14. $r_1^3 + r_2^3 + r_3^3 - r^3 = 64R^3 - 6R(a^2 + b^2 + c^2)$.

15. (The excentral triangle). The triangle formed by joining the excentres I_1, I_2, I_3 of a triangle ABC in pairs is called the *excentral triangle* of triangle ABC. Show that

   (*i*) its angles are $\dfrac{\pi - A}{2}, \dfrac{\pi - B}{2}, \dfrac{\pi - C}{2}$;

   (*ii*) its sides are $4R\cos\dfrac{A}{2}, 4R\cos\dfrac{B}{2}, 4R\cos\dfrac{C}{2}$;

   (*iii*) its circumradius is $2R$;

   (*iv*) its inradius is $2R\left(\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} - 1\right)$

   (*v*) its area is $2Rs$.

16. If the internal bisectors of angles A, B, C, on a triangle meet its circumcircle in A', B', C' respectively, then show that (*i*) A' is the circumcentre of triangle IBC; (*ii*) the area of triangle $A'B'C'$ is $\dfrac{1}{2}Rs$.

17. If D, E, F are the points of contact of the incircle of a triangle ABC with its sides then show that

   (*i*) the sides of triangle DEF are $2r\cos\dfrac{A}{2}, 2r\cos\dfrac{B}{2}, 2r\cos\dfrac{C}{2}$;

   (*ii*) its angles are $\dfrac{\pi - A}{2}, \dfrac{\pi - B}{2}, \dfrac{\pi - C}{2}$;

   (*iii*) its area $= Rr\sin A\sin B\sin C = \dfrac{\Delta r}{2R}$.

18. If d_1, d_2, d_3 are the diameters of the excircles of a triangle ABC, then

$$\frac{a}{d_1} + \frac{b}{d_2} + \frac{c}{d_3} = \frac{d_1 + d_2 + d_3}{a + b + c}.$$

19. If the ex-radii r_1, r_2, r_3 of a triangle are given, explain how to find its sides. Hence find a, b, c given $r_1 = 21, r_2 = 24, r_3 = 28$.

20. If the incircle of a triangle passes through its circumcentre, then show that $\cos A + \cos B + \cos C = \sqrt{2}$.

21. (*a*) Show that r_1, r_2, r_3 and $-r$ are the roots of the equation

$$x^4 - 4Rx^3 + \frac{1}{2}(a^2 + b^2 + c^2)x^2 - \Delta^2 = 0.$$

   (*b*) Hence deduce that

   (*i*) $r_1^3 + r_2^3 + r_3^3 - r^3 = 64R^3 - 6R(a^2 + b^2 + c^2)$;

   (*ii*) $r_1^4 + r_2^4 + r_3^4 + r^4 = 256R^4 - 32R^2(a^2 + b^2 + c^2)$
$$+ \frac{1}{4}(a^2 + b^2 + c^2) + (b^2c^2 + c^2a^2 + a^2b^2).$$

22. In a triangle ABC, show that

   (*i*) $IA.IB.IC = 4Rr^2$.

   (*ii*) $II_1.II_2.II_3 = 16R^2r$.

(*iii*) $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$.

(*iv*) $I_1A.I_1B.I_1C = 4Rs^2$.

D. The Orthocentre, the Pedal Triangle, the Centroid, the Circumcentre and the Incentre

(*a*) Let ABC be a triangle. We know from Theorem 26 of Chapter 3 that its altitudes are concurrent. The point of concurrence is called the *orthocentre* of the triangle and is denoted by O. The orthocentre falls within or outside the triangle according as the triangle is acute- or obtuse-angled (figures 6.40 and 6.41). If ABC is right-angled, say at A, then its orthocentre O coincides with A, as do the feet of perpendiculars from B and C (figure 6.42). In the first two cases (that is, when triangle ABC is acute — or obtuse-angled) it can also be observed that A, B, C are repectively the orthocentres of triangles OBC, OCA, OCB. Let us hereafter consider only acute-angled triangles. If the altitudes through A, B, C meet the opposite sides in X, Y, Z respectively, then the triangle XYZ formed by the feet of these altitudes is called *the pedal triangle* of triangle ABC. Also we have six cyclic quadrilaterals namely $OYAZ$. $OZBX$, $OXCY$ $BCYZ$, $CAZX$ and $ABXY$. (see Fig. 6.43) Further the triangles AYZ, BZX, CXY are all similar to triangle ABC. From these facts it can be deduced that

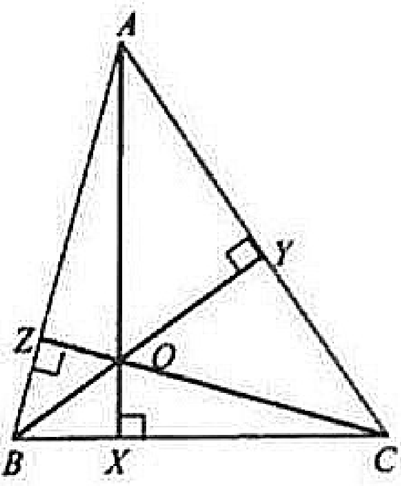


Fig. 6.40

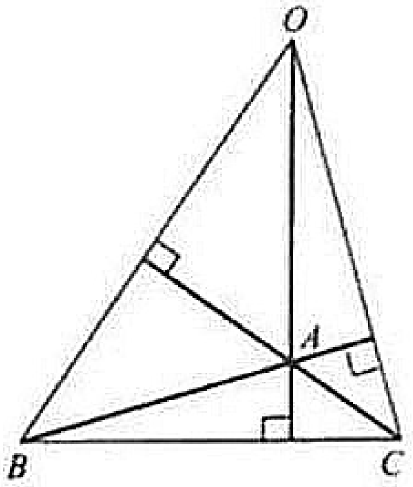


Fig. 6.41

(*i*) the sides of the pedal triangle are $a \cos A = R \sin 2A$; $b \cos B = R \sin 2B$; $c \cos C = R \sin 2C$.

(*ii*) its angles are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$;

(*iii*) the orthocentre O of triangle ABC is the incentre of its pedal triangle XYZ.

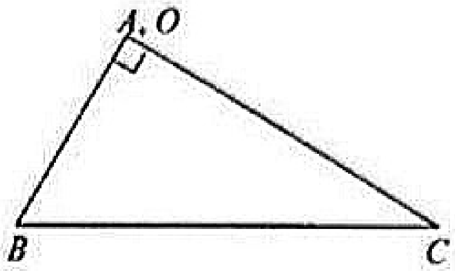


Fig. 6.42

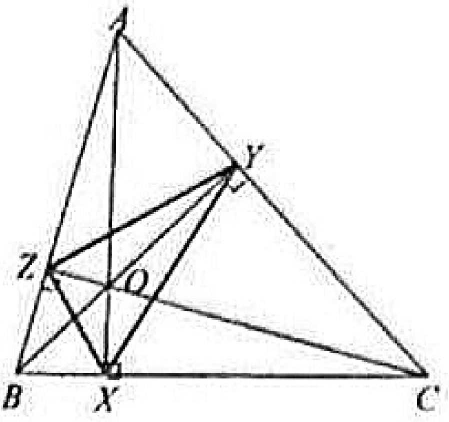


Fig. 6.43

The proofs of these statements are left to the reader. Observe that any triangle ABC is itself the pedal triangle of its excentral triangle and consequently the incentre of a triangle ABC is the orthocentre of its ex-central triangle.

(*b*) If ABC is a triangle and AA_1, BB_1, CC_1, are its medians, we know from Theorem 25 of chapter 3 that these are concurrent at a point usually denoted by G and called the centroid of triangle ABC. It is also known that G divides each median in the ratio 2:1, that is, G is a point of trisection of each of the medians.

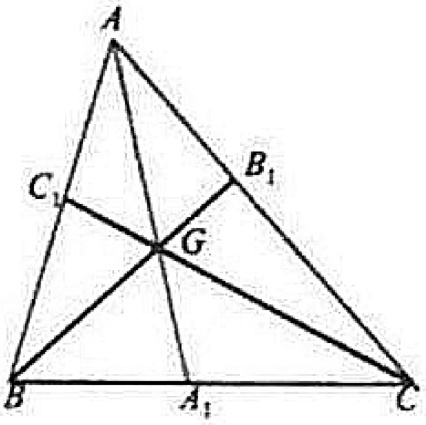


Fig. 6.44

Specifically, $AG : GA_1 = BG : GB_1 = CG : GC_1 = 2{:}1$. (see Fig. 6.44)

The lengths of the medians are given by $AA_1 = (1/2)\ \sqrt{2b^2 + 2c^2 - a^2}$, $BB_1 = (1/2)$ $\sqrt{2c^2 + 2a^2 - b^2}$, $CC_1 = (1/2)\ \sqrt{2a^2 + 2b^2 - c^2}$ (see Problem 2 of Exercise 6.9 B). It is known that from Theorem 56 of Chapter 4 while the centroid G divides the line segment joining the orthocentre O and the circumcentre S in the ratio 2:1, the nine-point centre N divides the same line segment OS in the ratio 1:1, that is, N is the mid-point of OS.

(*c*) Let S be the circumcentre and I the incentre of a triangle ABC. In Fig. 6.45, S is shown as the point of intersection of the perpendicular bisectors of BC and CA, and I as the point of intersection of the internal bisectors of angles A and B. Join SI. We use the cosine rule in triangle ASI to evaluate SI. For this, we need to know AI, AS and $\angle IAS$. If IF is perpendicular to AB, then from triangle IAF,

$$AI = \frac{IF}{(\sin A/2)} = \frac{r}{(\sin A/2)} = 4R \sin(B/2) \sin(C/2).$$

Clearly $AS = R$.

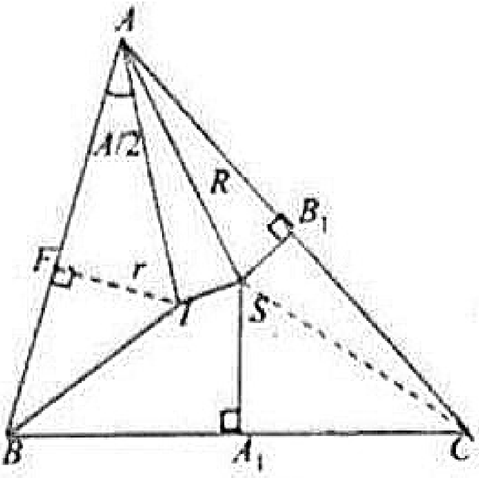


Fig. 6.45

Also $\angle SAB_1 = 90° - \angle ASB_1 = 90° - (1/2)\angle ASC$
$= 90° - \angle ABC = 90° - B.$

So $\angle IAS = \angle IAC - \angle SAC = \dfrac{A}{2} - (90° - B) = \dfrac{B-C}{2}.$

Hence
$$SI^2 = AS^2 + AI^2 - 2AS \cdot AI\cos\angle IAS$$
$$= R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 2\cdot R\cdot 4R\sin\frac{B}{2}\sin\frac{A}{2}\cos\frac{B-C}{2}$$
$$= R^2\left[1 + 8\sin\frac{B}{2}\sin\frac{C}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2} - \cos\frac{B-C}{2}\right)\right]$$
$$= R^2\left[1 - 8\sin\frac{B}{2}\sin\frac{C}{2}.\cos\frac{B+C}{2}\right]$$
$$= R^2\left[1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right] = R^2 - 2Rr.$$

Thus we have proved Euler's theorem (Theorem 4.3 of chapter 4) trigonometrically: $SI^2 = R^2 - 2Rr$.

Similarly, using the cosine rule, we can find expressions for SI_1^2, SO^2, IO^2, I_1O^2 etc.

EXERCISE 6.9 D

1. The altitude through A on BC meets BC in D and the circumcircle of triangle ABC in E when produced. Show that $OD = OE$.
2. Show that the distances from the orthocentre to the vertices of a triangle ABC are $2R\cos A$, $2R\cos B$, $2R\cos C$ and its distances from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$, $2R\cos A\cos B$.
3. Show that in a triangle the distance from the orthocentre to any vertex is twice the distance from the circumcentre to the opposite side.
4. Let ABC be a triangle, O its orthocentre and XYZ be its pedal triangle.

   Show that
   - (*a*) the sides of the pedal triangle are $a\cos A$, $b\cos B$, $c\cos C$.
   - (*b*) its angles are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$.
   - (*c*) its area is $2\Delta\cos A\cos B\cos C$.
   - (*d*) its circumradius is $R/2$ and its inradius is $2R\cos A\cos B\cos C$.
   - (*e*) its incentre is O.
5. Prove that the circumcentre of a triangle ABC is the orthocentre of the triangle formed by the midpoints of the sides of triangle ABC.
6. Prove that the median through A divides angle A into two parts whose cotangents are $2\cot A + \cot C$ and $2\cot A + \cot B$ and makes, with BC an angle whose cotangent is $(1/2)(\cot B - \cot C)$.
7. Four geometrical proofs of the famous Feuerbach's Theorem have been given in Chapter 4 (Theorem 59). Using the methods of this chapter, give a fifth proof.
8. Prove the following for any triangle ABC. (Some of these have already been proved geometrically in Chapter 4. The corresponding references are given in parenthesis)

   (*i*) $SI_1^2 = R^2[1 + 8\sin(A/2)\cos(B/2)\cos(C/2)] = R^2 + 2Rr_1$.

(*ii*) $IO^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

(*iii*) $SO^2 = R^2(1 - 8 \cos A \cos B \cos C) = 9R^2 - a^2 - b^2 - c^2$.

(Corollary 1 of Theorem 48)

(*iv*) $I_1O^2 = 2r_1^2 - 4R^2 \cos A \cos B \cos C$.

(*v*) $IN = (1/2)R - r$.

(*vi*) $I_1N = (1/2)R + r_1$.

(*vii*) $SG^2 = R^2 - (1/9)(a^2 + b^2 + c^2)$. (Cor. 3 of Theorem 28)

(*viii*) $AO^2 + BO^2 + CO^2 - SO^2 = 3R^2$ (Cor. 2 of Theorem 48)

(*ix*) $SI^2 + SI_1^2 + SI_2^2 + SI_3^2 = 12R^2$. (Theorems 43, 44)

(*x*) $AN^2 = \frac{1}{4}R^2(1 + 8 \cos A \sin B \sin C) = \frac{1}{4}(R^2 - a^2 + b^2 + c^2)$.

(Cor. of Theorem 57 gives $AN^2 + BN^2 + CN^2$).

(*xi*) $a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2 = abc$.

(*xii*) $a \cdot AI_1^2 - b \cdot BI_1^2 - c \cdot CI_1^2 = abc$.

(*xiii*) Area of triangle SOI is $2R^2 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}$.

9. If any two of the four points S, I, G, O coincide then the triangle is equilateral.

10. If $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius R with $AB = a$, $BC = b$, $CD = c$, $DA = d$, semiperimeter $s = \frac{a+b+c+d}{2}$, then show that

(*i*) $\cos A = \frac{2}{ad + bc}(a^2 + d^2 - b^2 - c^2)$.

(*ii*) $\sin A = \frac{2}{ad + bc}\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

(*iii*) $\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

(*iv*) $AC = \sqrt{\frac{(ac+bd)(ad+bc)}{ad+cd}}$,

$BD = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$. (Brahmagupta's Theorem)

(*v*) $ac + bd = AC.BD$ (Ptolemy's Theorem).

(*vi*) $R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$.

(*vii*) $\tan \frac{A}{2} = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}$.

(*viii*) The product of the segments into which either diagonal is divided by the other is

$$\frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}.$$

11. Interpret $\sqrt{\frac{(ab+cd)(cd+bc)}{ac+bd}}$ with reference to the above problem.

12. (*a*) If $ABCD$ is a quadrilateral with the same notation for its sides, semiperimeter and area as in problem 10 above, and $A + C = 2\alpha$, then

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}.$$

(*b*) The area of a quadrilateral is half the product of its diagonals and the sine of the angle between them.

6.10 HEIGHTS AND DISTANCES

In this section, we consider some useful problems in which we calculate the distance between certain points or heights of objects such as towers, buildings, and mountains, which are not directly accessible. Of course we need to know certain other measurements such as lengths and angles, which can be found in practice with the help of instruments.

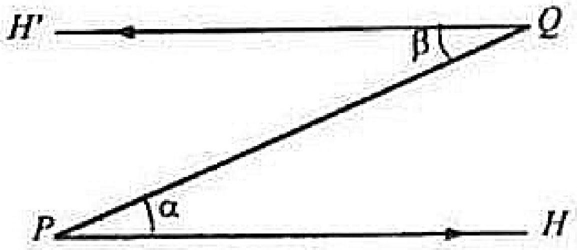


Fig. 6.46

Suppose P and Q are two points in a vertical plane at different horizontal levels. Suppose PH and QH are the horizontal lines in the plane through P and Q as shown in, Fig. 6.46, then angle $HPQ = \alpha$ is called the *angle of elevation* of Q relative to P (or as seen from P) and angle $H'QP = \beta$ is called the *angle of depression* of P relative to Q. Obviously these two angles are equal as the horizontal lines are in the same vertical plane.

EXAMPLE 1. *The angle of elevation of the top of a tower is observed to be 30° from a point on the ground. After walking a distance of 50 metres towards the tower the angle of elevation is found to be 60°. Find the height of the tower.*

SOLUTION. Let PQ be the tower and A and B be first and second points of observation respectively. The whole observation is taking place in one vertical plane as indicated by the data of the problem (A, B, Q are collinear)

Also $\angle PAQ = 30°$, $\angle PBQ = 60°$, $AB = 50$ metres.

Let $PQ = x$ metres.

From triangles PAQ and PBQ we have

$$AQ = x\cot 30° \quad \text{and} \quad BQ = x\cot 60°$$

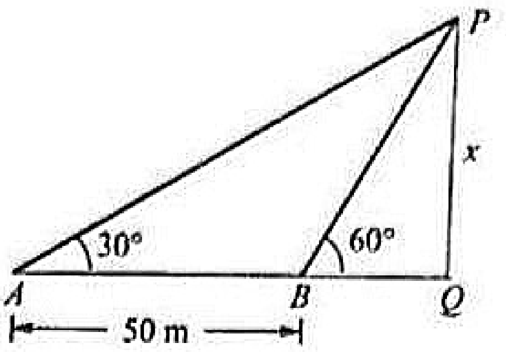


Fig. 6.47

Hence $$x = \frac{AB}{\cot 30^\circ - \cot 60^\circ} = \frac{50}{\sqrt{3} - 1/\sqrt{3}} = 25\sqrt{3}.$$

Thus the height of the tower is $25\sqrt{3}$ metres.

EXAMPLE 2. *From the top of a mountain the angles of depression of three consecutive milestones on a straight road are observed to be α, β, γ respectively. Find the height of the mountain.*

SOLUTION. The point of observation (that is, the peak of the mountain and the mile stones are not in the same vertical plane, in this case, Let P be the point of observation and PQ be the perpendicular drawn from P to the ground. Join P and Q to each of three points A, B, C which represent three collinear mile stones. From the hypothesis $\angle PAQ = \alpha$, $\angle PBQ = \beta$, $\angle PCQ = \gamma$. Let $PQ = h$. Then $AQ = h \cot \alpha$, $BQ = h \cot \beta$, $CQ = h \cot \gamma$. We look at the triangle QAC (in the horizontal plane) for which QB is the median through Q. We apply Appollonius's Theorem to get $QA^2 + QC^2 = 2\,(QB^2 + AB^2)$. That is, $h^2 \cot^2 \alpha + h^2 \cot^2 \gamma = 2(h^2 \cot^2 \beta + 1)$. Solving for h, we get

$$h = \frac{2}{(\cot^2 \alpha + \cot^2 \gamma - 2 \cot^2 \beta)^{1/2}}.$$

Hence the height of the mountain is h miles where h is as given above.

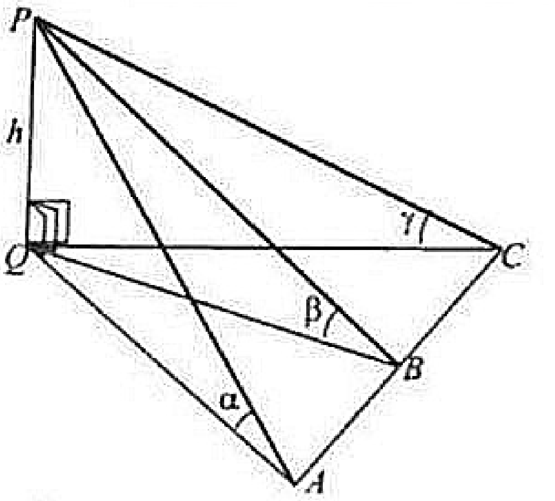


Fig. 6.48

EXAMPLE 3. *A man wishing to ascertain the distance between two objects in a horizontal plane walks along a straight road and observes that at a certain point on the road the two objects subtend the greatest angle α; he walks a distance c along the road, and finds that the objects are in a straight line with his position and that this line makes an angle β with the road. Prove that the distance between the objects is*

$$\frac{2c \sin a \sin b}{\cos a + \cos b}$$

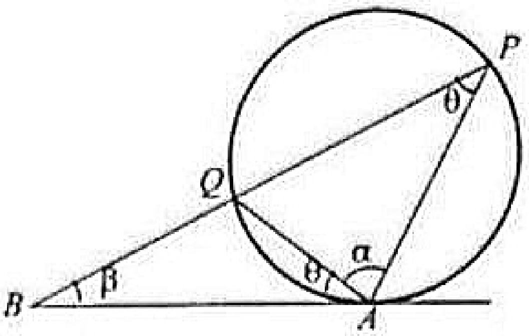


Fig. 6.49

SOLUTION. Let P and Q be the two objects and A and B be the two points of observation. When PQ subtends maximum angle α at A, AB is tangent to the circle passing through P, Q, A. Also $\angle PBA = \beta$ and $AB = c$. If $\angle BAQ = \theta$, then $\angle APB = \theta$ (angle in the alternate segment). Then from ΔPBA,

we get $\beta + \theta + (\alpha + \theta) = 180°$.

Therefore $\theta = (180° - \alpha - \beta)/2$. From triangles PQA and BQA, we have

$$\frac{PQ}{\sin\alpha} = \frac{AQ}{\sin\theta} \text{ and } \frac{AB}{\sin(\alpha+\theta)} = \frac{AQ}{\sin\beta}.$$

Dividing the first relation by the second, we get

$$PQ = \frac{c\sin\alpha\sin\beta}{\sin\theta\sin(\alpha+\theta)}$$

$$= \frac{c\sin\alpha\sin\beta}{\sin\left(\frac{\pi}{2} - \frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\pi}{2} - \frac{\alpha-\beta}{2}\right)}$$

$$= \frac{c\sin\alpha\sin\beta}{\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} = \frac{2c\sin\alpha\sin\beta}{\cos\alpha + \cos\beta}.$$

EXAMPLE 4. *A flagstaff on the top of a tower is observed to subtend the same angle α at two points, distant 2 a from each other and lying in a line through the base of the tower in the horizontal plane; and an angle β at a point half way between them. Prove that the height of the flagstaff is*

$$a \sin\alpha \sqrt{\frac{2\sin b}{\cos a \sin(b-a)}}.$$

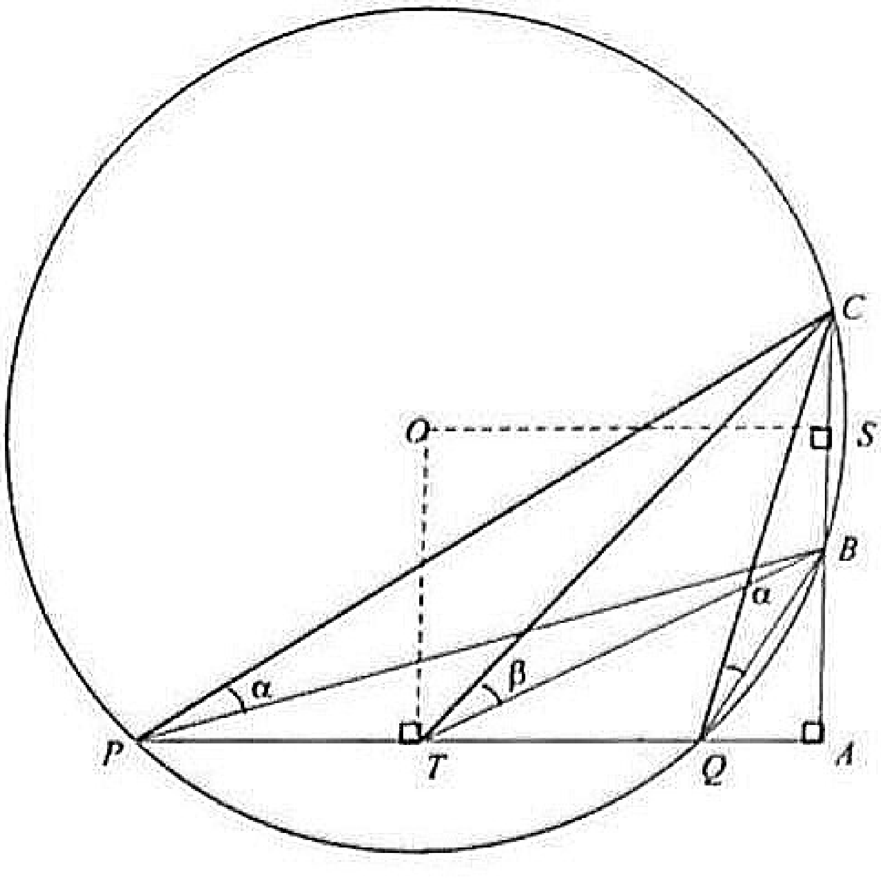


Fig. 6.50

SOLUTION. Let AB be the tower and BC the flagstaff; P, T, Q be the points of observation at which the flagstaff BC subtends angles α, β. a respectively. Since BC subtends equal angles at P and Q, it follows that B, C, P, Q are concyclic. Let O be the centre and r the radius of the circle passing through these four points. Let S be the midpoint of BC. Then $ASOT$ is a rectangle. Let $BC = x$, $AB = y$, $QA = z$. Then from triangle BPC,

$$r = \frac{BC}{2\sin\angle BPC} = \frac{x}{2\sin\alpha}.$$

Now $$\angle BOS = (1/2)\angle BOC = (1/2)(2\angle BPC) = \alpha.$$

So $$TA = OS = BS\cot\alpha = (x/2)\cot\alpha.$$

That is, $$a + z = (x/2)\cot\alpha.$$

If $$\angle BTA = \theta, \text{ then } \angle ATC = \beta + \theta, \text{ and so}$$

$$\tan\beta = \tan[(\beta+\theta)-\theta] = \frac{\tan(\beta+\theta) - \tan\theta}{1 + \tan(\beta+\theta)\tan\theta}$$

$$\frac{\dfrac{x+y}{a+z} - \dfrac{y}{a+z}}{1 + \dfrac{(x+y)y}{(a+z)^2}} = \frac{x(a+z)}{(a+z)^2 + (x+y)y}$$

Also from triangles OTQ and OSB,

$$OT^2 + TQ^2 = OQ^2 = OB^2 = OS^2 + SB^2.$$

That is, $$(y + x/2)^2 + a^2 = ((x/2)\cot\alpha)^2 + \tfrac{x^2}{4}.$$

Therefore $$xy + y^2 = (x^2/4)\cot^2\alpha - a^2.$$

Hence $$\tan\beta = \frac{x(x/2)\cot\alpha}{\left(\frac{x^2}{4}\right)\cot^2\alpha + \left(\frac{x^2}{4}\right)\cot^2\alpha - a^2}$$

Simplifying, we get $$x^2 = \frac{2a^2\sin\beta\sin^2\alpha}{\sin(\beta-\alpha)\cos\alpha},$$ as desired.

EXERCISE 6.10

1. The angle of elevation on the top of a mountain from a point on the ground is found to be α. After walking a distance a along a slope of inclination β towards the cliff, the elevation is found to be γ. Show that the height of the mountain is
$$\frac{a\sin\alpha\sin(\alpha-\beta)}{\sin(\gamma-\alpha)}.$$
2. A man walking on a straight road finds that the line joining two objects on the same side of the road subtends an angle α at some point on the road. After walking a distance b along the road he finds that the line joining the two objects again subtends the same angle α. After walking a further distance a he finds that he is in a line with the objects making an angle α with the road. Find the distance between the two objects.
3. A tower stands vertically in the interior of a field which has the shape of an equilateral triangle of side a. If the angles of elevation of the top of the tower are α, β, γ from the corners of the field find the height of the tower.

Also from (5) and (6),

$$l^2 + m^2 = 2 + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$
$$= 2 + 2\cos(\alpha - \beta) = 4\cos^2(\alpha - \beta)/2.$$

From (6) alone,

$$2m = 4\cos[(\alpha + \beta)/2]\cos[(\alpha - \beta)/2].$$

Hence $$\frac{2m}{(l^2 + m^2)} = \frac{\cos(\alpha + \beta)/2}{\cos(\alpha - \beta)/2}. \tag{9}$$

From (8) and (9), we obtain

$$(l^2 + m^2)(1 - n) = 2m(1 + n).$$

EXERCISE 6.11

Eliminate θ from the equations [(1) – (15)]

1. $a\cos\theta + b\sin\theta = c$; $a\sin\theta - b\cos\theta = d$.
2. $a\cos(\theta + \alpha) = x$; $b\cos(\theta - \beta) = y$.
3. $x\cos\theta - y\sin\theta = \cos 2\theta$; $x\sin\theta + y\cos\theta = 2\sin 2\theta$.
4. $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$; $\dfrac{ax\sin\theta}{\cos^2\theta} + \dfrac{by\cos\theta}{\sin^2\theta} = 0$.
5. $\lambda\cos 2\theta = \cos(\theta + \alpha)$, $\lambda\sin 2\theta = 2\sin(\theta + \alpha)$.
6. $\cos^2\theta = (m^2 - 1)/3$; $\tan^3(\theta/2) = \tan\alpha$.
7. $2\cos\theta - \cos 2\theta = a$; $2\sin\theta - \sin 2\theta = b$.
8. $x\cos 3\theta + y\sin 3\theta = a\cos\theta$;
   $x\sin 3\theta + y\cos 3\theta = a\cos[\theta + (\pi/6)]$.
9. $\operatorname{cosec}\theta - \sin\theta = a$; $\sin\theta - \cos\theta = b$.
10. $x\cos(\theta + \alpha) + y\sin(\theta + \alpha) = a\sin 2\theta$;
    $y\cos(\theta + \alpha) - x\sin(\theta + \alpha) = 2a\cos 2\theta$.
11. $\sin\theta + \sin 2\theta = x$; $\cos\theta + \cos 2\theta = y$.
12. $\tan\theta - \tan 2\theta = a$; $\cot\theta - \cot 2\theta = b$.
13. $x\sin\theta - y\cos\theta = -\sin 4\theta$;
    $x\cos\theta + y\sin\theta = (5/2) - (3/2)\cos 4\theta$.
14. $\tan(\theta - \alpha) + \tan(\theta - \beta) = x$; $\cot(\theta - \alpha) + \cot(\theta - \beta) = y$.
15. $\dfrac{\cos(\alpha - 3\theta)}{\cos^3\theta} = \dfrac{\sin(\alpha - 3\theta)}{\sin^3\theta} = m$.

Eliminate θ and φ from the equations [(16) – (21)].

16. $\sin\theta + \sin\phi = x$; $\cos\theta + \cos\phi = y$; $\theta + \phi = \alpha$.
17. $\tan\theta - \tan\phi = a$; $\cot\theta - \cot\phi = b$; $\theta - \phi - \alpha$.
18. $\sin\theta + \sin\phi = a$; $\cos\theta + \cos\phi = b$; $\tan\theta + \tan\phi = c$.
19. $x\cos\theta + y\sin\theta = 1$, $x\cos\theta + y\sin\phi = 1$,
    $p\cos\theta\cos\phi + q\sin\theta\sin\phi = 0$.
20. $\cos\theta + \cos\phi = a$; $\cot\theta + \cot\phi = b$; $\operatorname{cosec}\theta + \operatorname{cosec}\phi = c$.
21. $\cos\theta + \cos\phi = x$; $\cos 2\theta + \cos 2\phi = y$; $\cos 3\theta + \cos 3\phi = 3$.

PROBLEMS

1. If $m^2 + m'^2 + 2mm' \cos\theta = 1$,
$$n^2 + n'^2 + 2nn' \cos\theta = 1,$$
$$mn + m'n' + (m'n + m'n)\cos\theta = 0,$$
then show that $$m^2 + n^2 = \operatorname{cosec}^2\theta.$$

2. If $\tan(\pi/4 + y/2) = \tan^3(\pi/4 + x/2)$,

then show that $\sin y = \dfrac{3\sin x + \sin^3 x}{1 + 3\sin^2 x}$.

3. If α, β are acute angles and
$[\sin(\alpha - \beta) + \cos(\alpha + 2\beta)\sin\beta]^2 = 4\cos\alpha\cos\beta\sin(\alpha + \beta)$,

then show that $\tan\alpha = \tan\beta\left[\dfrac{1}{(\sqrt{2}\cos\beta - 1)^2} - 1\right]$.

4. Show that
$$\sin^2 12° + \sin^2 21° + \sin^2 39° + \sin^2 48° = 1 + \sin^2 9° + \sin^2 18°.$$

5. Prove that $\dfrac{\tan(x + \alpha)}{\tan(x - \alpha)}$ cannot lie between
$$\tan^2(\pi/4 - \alpha) \text{ and } \tan^2(\pi/4 + \alpha).$$

6. Show that $\cos\theta\,(\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha})$ always lies between $\pm\sqrt{1 + \sin^2\alpha}$.

7. Prove that
$\sin^3(\alpha - \beta)\sin^3(\gamma - \delta) + \sin^3(\beta - \gamma)\sin^3(\alpha - \delta) + \sin^3(\gamma - \alpha)\sin^3(\beta - \delta)$
$= 3\sin(\alpha - \beta)\sin(\beta - \gamma)\sin(\gamma - \alpha)\sin(\alpha - \delta)\sin(\beta - \delta)\sin(\gamma - \delta)$.

8. Prove that $\dfrac{\cot 3x}{\cot x}$ never lies between 1/3 and 3.

9. If $\sin x = K\sin(A - x)$, show that
$$\tan(x - A/2) = [(k - 1)/(k + 1)]\tan A/2.$$

10. If in triangle ABC, $\cot A + \cot B + \cot C = \sqrt{3}$, show that the triangle is equilateral.

11. In triangle ABC show that
$$-2 \le \sin 3A + \sin 3B + \sin 3C \le 3\sqrt{3}/2.$$
When does equality hold good on either side?

12. Solve the equation $\cos^n x - \sin^n x = 1$, where n is a positive integer.

13. Solve the equation : $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

14. Find all x in $[0, 2\pi]$, such that
$$2\cos x \le \left|\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}\right| \le \sqrt{2}.$$

15. If in triangle ABC,
$$a + b = \tan(C/2)\,(a\tan A + b\tan B), \text{ then } a = b.$$

16. If $f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta$ and $f(\theta) \ge 0$ for all real θ, then show that $a^2 + b^2 \le 2$, $A^2 + B^2 \le 1$. (Here a, b, A, B are real numbers).

17. If $\tan(A/2)$, $\tan(B/2)$, $\tan(C/2)$ are in A.P. Then so are $\cos A$, $\cos B$, $\cos C$. Here A, B, C are the angles of a triangle.

18. If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \theta$ then show that

$$\prod_{i=1}^{4} \cos(\theta - \alpha_i) + \prod_{i=1}^{4} \sin(\theta - \alpha_i) = \prod_{i=1}^{4} \cos \alpha_i + \prod_{i=1}^{4} \sin \alpha_i.$$

19. Determine the number of real solutions of $\sin x = x/100$.

20. If $\cot^2 (\beta/2) = \cot(\theta + \alpha)/2 \cdot \cot (\theta - \alpha)/2$, then show that $\cos\theta = \cos \alpha \cos \beta$.

21. If $\sqrt{2} \cos A = \cos B + \cos^3 B$, $\sqrt{2} \sin A = \sin B - \sin^3 B$, prove that $\sin (A - B) = \pm 1/3$.

22. In triangle ABC, M is an interior point on side BC. If r, r', r'' are the inradii and r_1, r'_1, r''_1 are the ex-radii opposite A of triangles ABC, ABM, ACM, then show that

$$\frac{r'r''}{r'_1 r''_1} = \frac{r}{r_1}.$$

23. If x_1, x_2, x_3, x_4 are the roots of the equation
$x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - x \sin \beta = 0$,

show that $\sum\limits_{i=1}^{4} \tan^{-1} x_i = n\pi + \pi/2 - \beta$, where n is an integer.

24. If A, B, C are the angles of a triangle, then $\tan^{-1} (\cot B \cot C) + \tan^{-1} (\cot C \cot A) + \tan^{-1} (\cot A \cot B)$

$$= \tan^{-1} \left\{1 + \frac{8 \cos A \cos B \cos C}{\sin^2 2A + \sin^2 2B + \sin^2 2C}\right\}.$$

25. Suppose ABC is an acute triangle. Consider the triangle formed by the three direct (external) common tangents (which are not the sides of triangle ABC) drawn to the excircles of triangle ABC taken pairwise. Find the angles, the sides, the area, the circumradius and the inradius of the triangle so formed.

26. If $ABCD$ is a quadrilateral with $AB + CD = AD + BC$, then show that whether the quadrilateral is convex or not, there is a circle touching all its sides (produced if necessary). When is the radius of the 'incircle' of such a quadrilateral maximum for given lengths of sides?

27. Eliminate α, β, γ from the equations

$$a \cos \alpha + b \cos \beta + c \cos \gamma = 0,$$
$$a \sin \alpha + b \sin \beta + c \sin \gamma = 0,$$
$$a \sec \alpha + b \sec \beta + c \sec \gamma = 0.$$

28. If θ is an angle expressed in radians and $0 < \theta < \pi/2$, then $\sin \theta < \theta < \tan \theta$.

29. Suppose ABC is an acute-angled triangle in a horizontal plane and P, Q, R are three points directly below A, B, C respectively such that $AP = x$, $BQ = x + y$, $CR = x + y + z$. If θ is the angle between the planes containing triangle ABC and triangle PQR, show that $\tan^2 \theta \sin^2 A = y^2/c^2 + z^2/b^2 - 2\,(yz/bc) \cos A$.

30. The line joining two objects lying on the same side of a straight road on a horizontal plane subtends two maximal angles α and β at two points on the road distant c from each other. Show that the distance between the objects is
$c \sec[(\alpha + \beta/2] (\sin \alpha \sin \beta)^{1/2}$.

[If the line joining the objects meets the road in P, then on either side of P, there is a point on the road at which the line subtends a maximum angle.]

CHAP

7

COORDINATE GEOMETRY OF STRAIGHT LINES AND CIRCLES

7.1 INTRODUCTION

Descartes (1596 – 1650) re-created Geometry by using algebraic formulations and methods. The Geometry that arose thus has been called *Cartesian Geometry* for that very reason. It is also called *Analytic Geometry* or *Coordinate Geometry* for reasons which will be obvious in this Chapter. The application of algebra to geometry which became the fashion after Descartes's time may well be named as the key step for all exploration of natural phenomena by mathematics in the past three centuries.

We saw in the first chapter that points on a geometric straight line can be conveniently described by the set of real numbers, once we fix our points corresponding to zero and the number one. We saw through examples that certain geometric problems on straight lines have simple solutions if we describe the points on the line by real numbers and use the algebraic properties of real numbers. Likewise, certain algebraic problems in the set of real numbers have nice geometric solutions via the above correspondence. Suppose now that one wants to describe points on a given plane by means of our familiar real numbers. Imagine that we have a vehicle which can travel in only one direction, but which can move both forward and backward in that direction. If we start from a point, say O on a given plane, we can at best cover all the points on that straight line through O, in the 'direction' of the vehicle. On the other hand, suppose our vehicle can travel in two different directions, say Ox and Oy directions. Of course we assume that the vehicle can go forwards and backwards in the above two direction. If P is any point in the plane, we may draw the lines through P parallel to the given two directions. Let them meet Ox, Oy at A, B respectively. We start from O, move through OA units in the direction of Ox and then AP units in the direction of Oy. This takes us to the point P. In the figure 7.1 that we have drawn, our movements are in the forward direction, both in Ox and Oy directions.

From figures 7.2, 7.3 and 7.4, it is clear that we may have to go forward or backward along the two 'directions' of the vehicle, depending upon the position of P relative to Ox and Oy.

If we have to move x units in the forward Ox direction and y units in the forward Oy direction to reach P from O (as in the case of Fig. 7.1) we may associate the ordered pair (x, y) of real numbers with P. Suppose we have to go x units backwards in the Ox-direction and y units forward in the Oy-direction (as in Fig. 7.2) to reach P, then we associate $(-x, y)$ with P. Similarly $(-x, -y)$ corresponds to P in Fig. 7.3 and $(x, -y)$ to

18. If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \theta$ then show that

$$\mathop{\pi}_{i=1}^{4} \cos(\theta - \alpha_i) + \mathop{\pi}_{i=1}^{4} \sin(\theta - \alpha_i) = \mathop{\pi}_{i=1}^{4} \cos \alpha_i + \mathop{\pi}_{i=1}^{4} \sin \alpha_i.$$

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Thus $CA^2 = 205 = 125 + 80 = AB^2 + BC^2$ and hence ABC is right angled at B.

EXAMPLE 3. *Prove that the points A(3, – 5), B(– 5, – 4), C(7, 10) and D(15, 9) taken in order are the vertices of a parallelogram.*

SOLUTION. We have

$$\begin{aligned} AB^2 &= (3-(-5))^2 + (-5-(-4))^2 = 65 \\ BC^2 &= (-5-7)^2 + (-4-10)^2 = 340 \\ CD^2 &= (7-15)^2 + (10-9)^2 = 65 \\ DA^2 &= (15-3)^2 + (9-(-5))^2 = 340. \end{aligned}$$

Thus in the quadrilateral $ABCD$, the opposite sides AB, CD and BC, DA are equal in length, which implies that the quadrilateral $ABCD$ is a parallelogram.

EXAMPLE 4. *Find the circumcentre and circumradius of the triangle ABC whose vertices are A(1, 1), B(2, – 1) and C(3, 2).*

SOLUTION. Let $S(x, y)$ be the circumcentre and r be the circumradius of ΔABC. Then $AS^2 = BS^2 = CS^2 = r^2$. This gives

$$(x-1)^2 + (y-1)^2 = (x-2)^2 + (y+1)^2 = (x-3)^2 + (y-1)^2 = r^2.)$$

This in turn gives $-2x - 2y + 2 = -4x + 2y + 5 \Rightarrow 2x - 4y = 3$

$$-4x + 2y + 5 = -6x - 4y + 13 \Rightarrow 2x + 6y = 8$$

Solving, we get $x = 5/2$, $y = 1/2$. Therefore S is $(5/2\ .\ 1/2)$ and the circumradius is

$$r = \sqrt{AS^2} = \sqrt{\left(\tfrac{5}{2}-1\right)^2 + \left(\tfrac{1}{2}-1\right)^2} = \tfrac{\sqrt{10}}{2} \text{ units.}$$

To find the area of the triangle ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$

We have area of ABC = area of trapezium $APRC$

+ area of trapezium $CRQB$

– area trapezium $APQB$ (see Fig. 7.9)

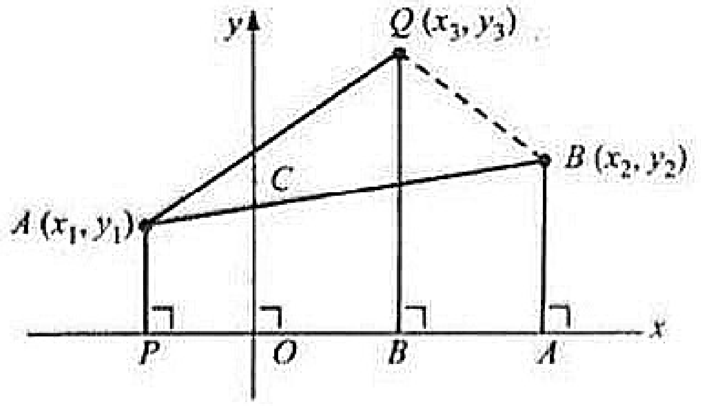


Fig. 7.9

$$\begin{aligned} &= \frac{1}{2}(AP + CR)PR + \frac{1}{2}(CR + QB)RQ - \frac{1}{2}(AP + QB)\,PQ \\ &= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_2 + y_3)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1) \\ &= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} \\ &= \frac{1}{2}\Sigma x_1(y_2 - y_3). \end{aligned}$$

We follow the convention the area bounded by an oriented closed curve r is positive if the area enclosed by r lies to the left as one travels on r along its orientation; otherwise the area is negative. Thus the area of $\Delta ABC = -$ area of ΔACB. The area of the triangle formed by three distinct points A, B and C is zero if and only if they lie on a straight line. Thus a necessary and sufficient condition that three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1)\, x_3(y_1 - y_2)\} = 0$.

EXAMPLE 5. *Find the area of the triangle formed by A(2, 3), B(3, 0) and C(− 4, 2).*

SOLUTION. Applying the formula for the area of a triangle in terms of the coordinates of its vertices, we get

$$\text{Area of } \Delta ABC = \tfrac{1}{2}\{2(0-2) + 3(2-3) + (-4)(3-0)\} = -19/2.$$

Note that we get area of $\Delta ABC < 0$ since the ΔABC has the negative orientation, namely the clockwise orientation and as one travels around ΔABC, the area is to his right.

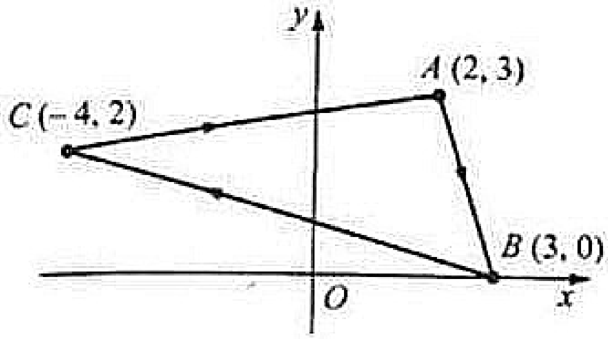


Fig. 7.10

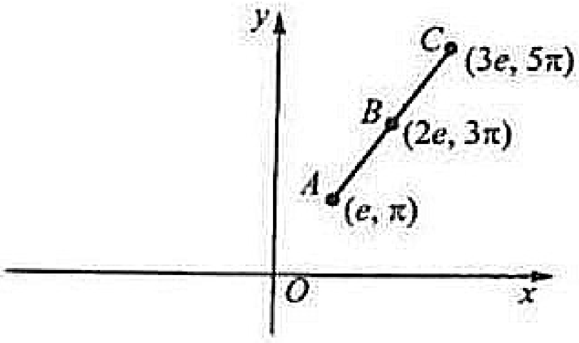


Fig. 7.11

EXAMPLE 6. *Find the area of ΔABC where A is (e, π), B is $(2\,e, 3\,\pi)$ and C is $(3\,e, 5\pi)$.*

SOLUTION. Area of ΔABC

$$= \frac{1}{2}\{e\,(3\pi - 5\pi) + 2\,e(5\pi - \pi) + 3\,e(\pi - 3\pi)\}$$

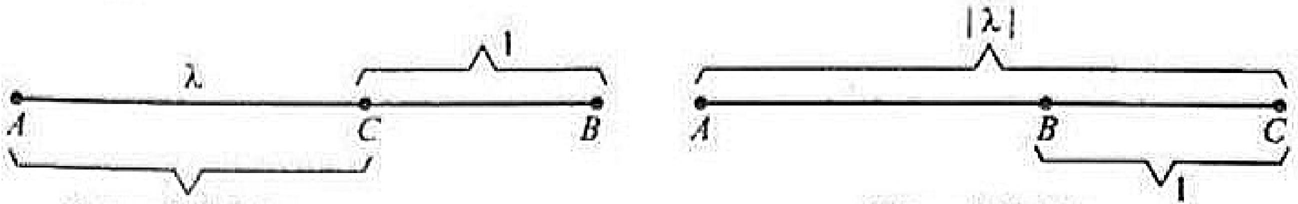
$$= \frac{1}{2}\{-2\pi\,e + 8\pi\,e - 6\pi e\} = 0$$

This implies that A, B, C all lie on a straight line.

Section Formula See Fig. 7.12. We know that given two points A, B there is only one point on the straight line AB which divides the line segment AB in a given ratio. If the given ratio is $\lambda > 0$ then the point of division C lies between A and B such that $AC/CB = \lambda$; on the other hand if $\lambda < 0$, it is external division and the point C lies outside the segment AB such that $AC/CB = \lambda$.

Now let A and B be the points (x_1, y_1) and (x_2, y_2) respectively. Let $\lambda = m/n$ be a given ratio. The problem is to find the coordinates of the point C on the straight line AB such that $AC/CB = m/n = \lambda$.

Case (*i*) $m/n = \lambda > 0$



Internal division

External division

Fig. 7.12

Let C be the point dividing AB in the ratio $m : n$ Then $AC/CB = m/n$. The triangles ADC and CEB are similar (Fig. 7.13). Hence the corresponding sides are proportional.

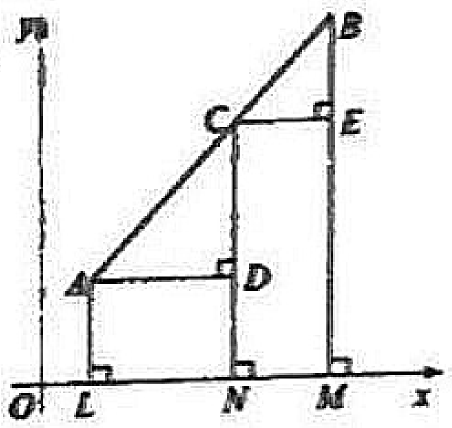


Fig. 7.13

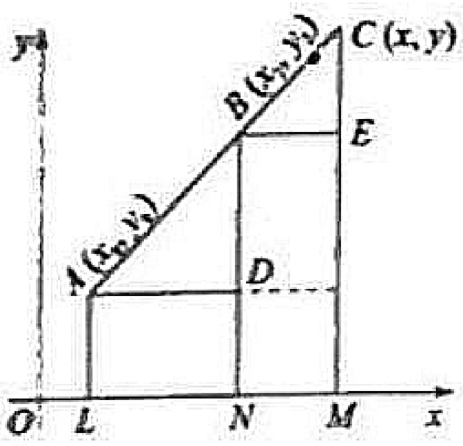


Fig. 7.14

$$\therefore \qquad \frac{m}{n} = \frac{AC}{CB} = \frac{AD}{CE} = \frac{x - x_1}{x_2 - x}$$

and this gives $x = \dfrac{mx_2 + nx_1}{m+n}$. Similarly

$$\frac{m}{n} = \frac{AC}{CB} = \frac{DC}{EB} = \frac{y - y_1}{y_2 - y} \text{ gives } y = \frac{my_2 + ny_1}{m+n}$$

where C is (x, y). Thus the required point C is $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$

Case (*ii*) $\lambda = -m/n = m/-n < 0$ with m, n being positive. Changing n to $-n$ in the above formula we get the corresponding point of division as

$$C = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

Also $\qquad \dfrac{AC}{CB} = \dfrac{m}{n} < 0$

implies that AC and CB are of opposite orientations and hence C lies outside the segment AB.

The triangles AFC and BEC are similar. Therefore

$$|\lambda| = \left|\frac{-m}{n}\right| = \frac{m}{n} = \frac{AC}{BC} = \frac{AF}{BE} = \frac{x - x_1}{x - x_2}$$

Solving we get $x = \dfrac{mx_2 - nx_1}{m-n}$ and similarly $\dfrac{m}{n} = \dfrac{AC}{BC} = \dfrac{CF}{CE} = \dfrac{y - y_1}{y - y_2}$ where $C(x, y)$ is the required point of division.

gives $y = \dfrac{my_2 - ny_1}{m-n}$. Thus the point C dividing AB internally in the ratio m/n is $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$; and the point C dividing AB externally in the same ratio is $\left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}\right)$

Remark. Any point P on the straight line AB divides the line segment AB in the ratio $AP/PB = \lambda$. The ratio λ is positive when P lies between A and B, and λ is negative when P lies outside the segment AB. (In this case AP and PB are of opposite orientations and hence $AP/PB < 0$). In either case P has coordinates $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$.

As we vary the parameter λ, we get the various points on the straight line. Thus, the points on the line AB are given by the set $\left\{\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right) : \lambda \text{ is any real number}, \lambda \neq -1\right\}$. When $\lambda = -1$, the point $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ is not defined. In other words, there is no point on the straight line AB which divides the line segment AB externally in the ratio 1 : 1.

As an immediate corollary of the section formula, we observe that the midpoint of the line segment AB joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ by taking $\lambda = 1$ in the section formula.

To find the coordinates of the centroid of the triangle ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

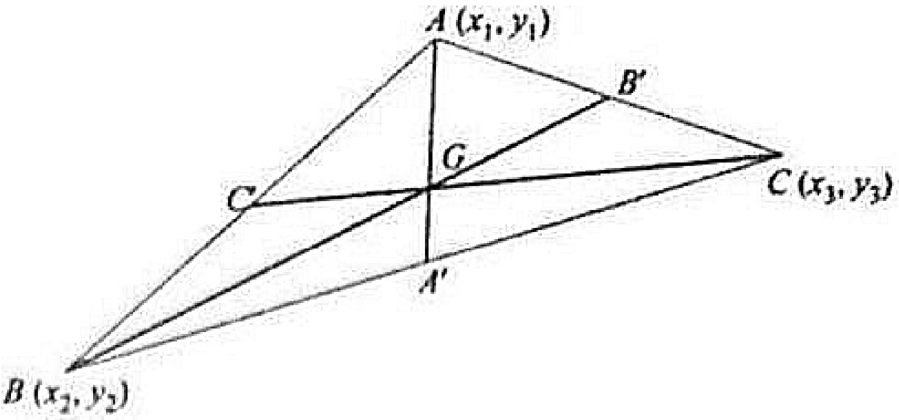


Fig. 7.15

Let A', B', C' be the midpoints of BC, CA, AB respectively. Then A' is $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$, B' is $\left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$ and C' is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

We know that the medians AA', BB' and CC' meet at the centroid G of ΔABC and that G divides each median in the ratio 2 : 1

Therefore $\quad G$ is $\left(\frac{2(x_2 + x_3) + 1.x_1}{2 + 1}, \frac{2(y_2 + y_3) + 1.y_1}{2 + 1}\right)$

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Thus the centroid of the triangle with vertices $(x_i, y_i)_{i=1,2,3}$ is the point whose x, y coordinates are the arithmetic averages of x_1, x_2, x_3 and y_1, y_2, y_3 respectively.

Remark. The point dividing AA' in the ratio 2 : 1 was found to be $((x_1 + x_2 + x_3)/3, (y_1 + y_2 + y_3)/3$ which is symmetric in x_i and symmetric in y_i. This implies that this point of division also divides the other two medians in the same ratio; this incidentally proves that the medians of a triangle are concurrent and the point of concurrence divides each median in the ratio 2 : 1.

To find the incentre of the triangle *ABC* with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

The incentre I of the triangle ABC is the point of concurrence of the angular bisectors of the internal angles of triangle ABC. Let AD, BE and CF be the internal angular bisectors of ΔABC, meeting the opposite sides at D, E, F (Fig. 7.16). As we have seen in Chapter 3, by the bisector theorem, we have $BD/DC = AB/AC = c/b$; $CE/EA = a/c$ and $AF/FB = b/a$ where a, b, c stand, as usual for the lengths of the sides BC, CA, AB respectively. Hence D has coordinates

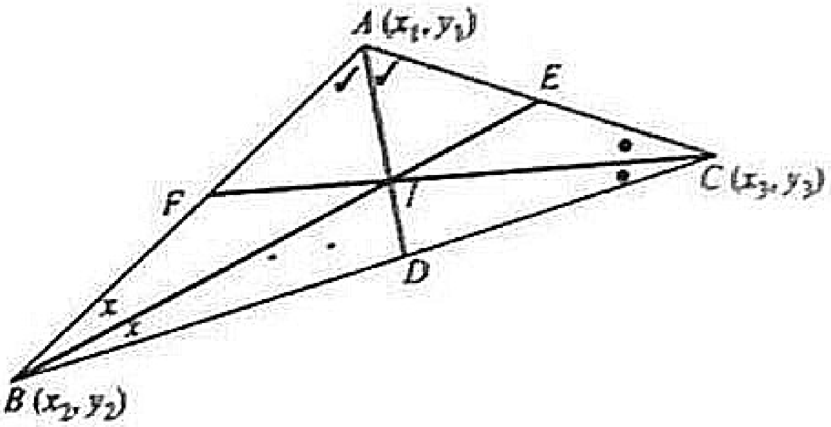


Fig. 7.16

$$D = \left(\frac{cx_3 + bx_2}{c+b}, \frac{cy_3 + by_2}{c+b}\right) \text{(using the section formula)}$$

Again in ΔABD, the angular bisector BI meets AD at I.

$$\frac{AI}{ID} = \frac{AB}{BD}. \text{ Now } \frac{BD}{DC} = \frac{c}{b} \text{ gives}$$

$$\frac{BD}{BD+DC} = \frac{c}{c+b} \text{ or } \frac{BD}{a} = \frac{c}{c+b}$$

$$\therefore \qquad AI/ID = \frac{c+b}{a}.$$

Using the section formula once more we get

$$I = \left(\frac{(c+b)\frac{(cx_3+bx_2)}{c+b} + ax_1}{c+b+a}, \frac{(c+b)\frac{(cy_3+by_2)}{c+b} + ay_1}{c+b+a}\right)$$

$$= \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

Remark. We have taken I as the intersection of the bisectors AD and BE and found its coordinates to be $\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$; the symmetry again implies that the other angular bisector CF also passes through I. In other words, this gives another proof of the fact that the internal bisectors of the angles of a triangle are concurrent.

Equation of a Curve In coordinate geometry, we study the geometry of points, straight lines, curves and surfaces using the representation of points by their coordinates, which are real numbers, and using their algebraic and other properties to the maximum extent possible. As we have seen in this introductory section, points in a plane are in one-one correspondence with ordered pairs of real numbers, once we fix our origin and two coordinate directions. The points on a given curve will have their coordinates satisfying certain constraints. In other words, the two coordinates x, y of points on a given curve cannot both be independent. For example, consider the x-axis, namely the straight line $x'Ox$. A point (x, y) lies on this x-axis if and only if $y = 0$. Similarly a point (x, y) will lie on the y-axis if and only if $x = 0$. Again for the straight line L bisecting the angle between the positive coordinate axis, we observe that $P(x, y)$ lies on L if and only if $y = x$. (see Fig. 7.17).

As yet another example, consider the straight line AB where A is $(1, 0)$ and B is $(0, 1)$. The ΔAOB is isosceles and $\angle OAB = \angle OBA = \pi/4$. Let $P(x, y)$ be any point on the straight line AB. If M is the foot of the perpendicular from P on the x-axis, then $OM = x$ and $MP = y$. We have $x + y = OM + MP = OM + MA = OA = 1$ (see Fig. 7.18).

The reader can easily check that $x + y = 1$ wherever be the point P on the staight line AB. Also one can check that if $x + y = 1$ then (x, y) lies on AB. Finally consider the circle C with centre $(0, 0)$ and radius a. If $P(x, y)$ is any point on this circle then its distance from the centre must be the radius a. In other words $\sqrt{(x^2 + y^2)} = a$ or $x^2 + y^2 = a^2$. Conversely, if $x^2 + y^2 = a^2$ then (x, y) is at a distance 'a' from the centre $(0, 0)$ and hence is a point on the circle C.

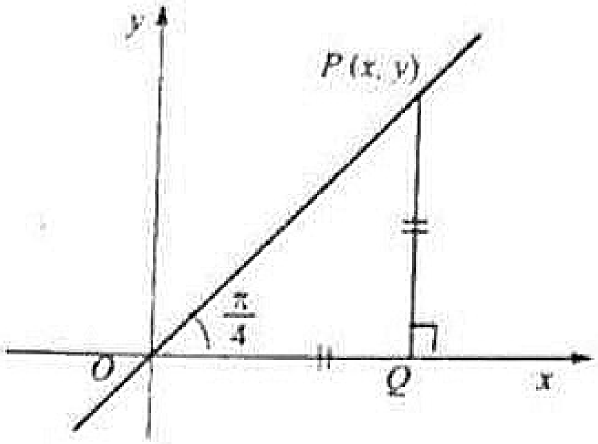


Fig. 7.17

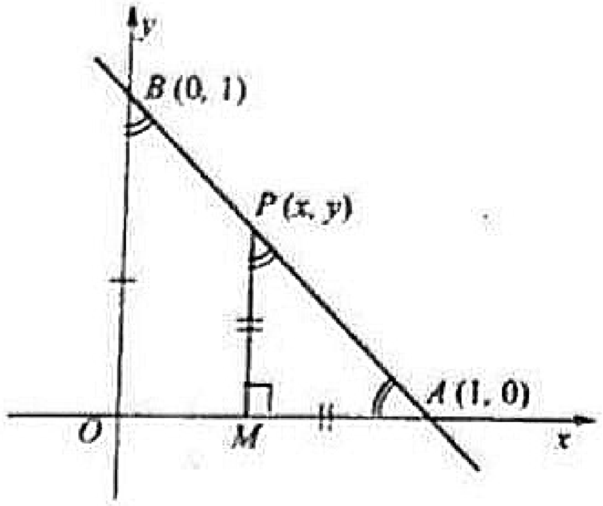


Fig. 7.18

The above examples tell us that points on each of those curves satisfy relations of the form $f(x, y) = 0$ which we call 'the equation of the corresponding curve'. Thus the equation of the x-axis is $y = 0$; the equation of the y-axis is $x = 0$; the equation to the straight line bisecting the angle between Ox and Oy is $y = x$; the equation of the straight line through (1, 0) and (0, 1) is $x + y = 1$; the equation of the circle with centre (0, 0) and radius a is $x^2 + y^2 = a^2$.

The equation of a curve C is an equation of the form $f(x, y) = 0$ which is satisfied by the coordinates (x, y) of every point on the curve and by no other points.

When a point moves in accordance with certain given conditions, its path is called the *locus of the moving point*. For instance, the locus of a point which moves in a plane such that its distance from a given point A is always a constant r is the circle with centre A and radius r.

EXERCISE 7.1

1. Find the distance between
   (*i*) (2, – 3) and (–3, –6) : (*ii*) (3, 4) and (7, 11) : (*iii*) $(at_1^2, 2at_2)$ and $(at_2^2, 2at_2)$: (*iv*) $(a \cos \theta, b \sin \theta)$ and $(a \cos \phi), b \sin \phi)$: (*v*) $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$.
2. Prove that (4, – 4), (–2, 4) and (6, 10) are the vertices of an isosceles triangle.
3. Calculate the lengths of the sides of the triangle whose vertices are (8, 9), (– 4, 4), (4, – 2).
4. Prove that each of the following sets of points forms a rhombus
   (*i*) (2, 5), (6, 2), (2, –1), (–2, 2)
   (*ii*) (3, 4), (–2, 3), (–3, –2), (2, –1).
5. Prove that each of the following sets of points forms a square.
   (*i*) (–3, 1), (–2, –3), (2, –2), (1, 2)
   (*ii*) (0, 2), (3, 8), (9, 5), (6, –1).
6. Calculate the area of the triangle whose vertices are
   (*i*) (2, 4), (7, 9) and (9, 2) (*ii*) (–2, 3), (–7, 5) and (3, –5).
7. Show by area that the following points are collinear:
   (2, 2), (4, – 4), (3, –1).
8. Write down the coordinates of the points dividing the join of (– 8, 3) and (4, 9) in the ratio (*i*) 2:1 (*ii*) 1:5 (*iii*) 4:1 externally.
9. Prove that $P(6, 2)$ is collinear with $A(-2, 2)$ and $B(12, 5)$; find the ratio in which P divides AB.
10. Find the ratio in which the diagonals of the following quadrilaterals $ABCD$ divide one another (*i*) $A(1, 5)$, $B(4, 1)$, $C(7, 5)$, $D(4, 9)$
    (*ii*) $A(10, 10)$, $B(14, 2)$, $C(7, -2)$, $D(2, 2)$.
11. Find the ratios in which the join of $A(-2, 2)$ and $B(4, 5)$ is cut by the axes.
12. Calculate the sides and the perimeter of ΔABC where A is (20, 50), B is (–20, –46) and C is (48, 5). Calculate the area of ΔABC and verify $\Delta^2 = s(s - a)(s - b)(s - c)$.
13. Prove that the four points (– 3, 11), (5, 9), (8, 0) and (6, 8) lie on a circle with center (–1, 2).
14. What are the coordinates of B if $P(3, 5)$ divides the join of $A(-1, 3)$ and B in the ratio 2:3?
15. Prove that $P(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the join of (x_1, y_1) and (x_2, y_2) in the ratio $t : (1 - t)$.
16. Show that (2, –1) is the centre of the circumcircle of ΔABC, where A is (–3, –1), B is (–1, 3) and C is (6, 2). Find the circumradius.

17. Given ΔABC with vertices $A(1, 2)$, $B(8, 4)$ and $C(4, 10)$ find the coordinates of a point P such that ΔPCB, ΔPCA and ΔPAB have the same areas in magnitude and sign.

18. Prove that the lines joining the midpoints of the opposite sides of a quadrilateral bisect one another.

19. Apply Ptolemy's theorem to check that the points $(1, -2)$, $(-2, -1)$, $(4, 7)$ and $(6, 3)$ are concyclic.

20. Apply Ptolemy's theorem as in problem 19 to the four points $(-1, -5)$, $(1, -1)$, $(2, 1)$, $(3, 3)$. Show that the four points are collinear.

21. Show that $(7, 5)$ divides AB and CD in the same ratio where A is $(1, 2)$; B is $(5, 4)$; C is $(-5, -1)$ and D is $(3, 3)$.

22. If G is the centroid of ΔABC, prove that
    (*i*) $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$
    (*ii*) $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$ (where O is any point in the plane ABC).

23. Find the incentre of the triangle whose vertices are $(0, 0)$, $(20, 15)$ and $(36, 15)$.

24. In ΔABC, D is the midpoint of BC. Prove that $AB^2 + AC^2 = 2AD^2 + 2DC^2$.

25. If O is the origin and A, B are the points (x_1, y_1) and (x_2, y_2) prove that
    $OA \cdot OB \cos \angle AOB = x_1x_2 + y_1y_2$.

26. A point P moves so that its distance from the point $(-1, 0)$ is always three times its distance from $(0, 2)$. Find the locus of P.

27. If A is $(a, 0)$ and B is $(-a, 0)$ find the locus of P when
    (*i*) $PA^2 - PB^2 = 2k^2 =$ constant.
    (*ii*) $PA + PB = c =$ constant.
    (*iii*) $PB^2 + PC^2 = 2PA^2$ where C is $(c, 0)$.

7.2 STRAIGHT LINES

Consider the family of straight lines in the xy-plane. We would like to study this family of straight lines by means of their equations. A straight line is completely determined by any two points on it or by its direction and a point lying on it.

If a straight line L makes an angle θ with the positive x-axis then $\tan\theta$ is defined as the slope of the straight line with respect to the rectangular axes Ox, Oy. By definition, all parallel straight lines have same slope. In fact, two straight lines are parallel if and only if they have the same slope. All straight lines parallel to the x-axis have slope zero; and all straight lines parallel to the y-axis have slope infinity. The straight line $y = x$ bisecting the angle xOy has slope $\tan \pi/4 = 1$.

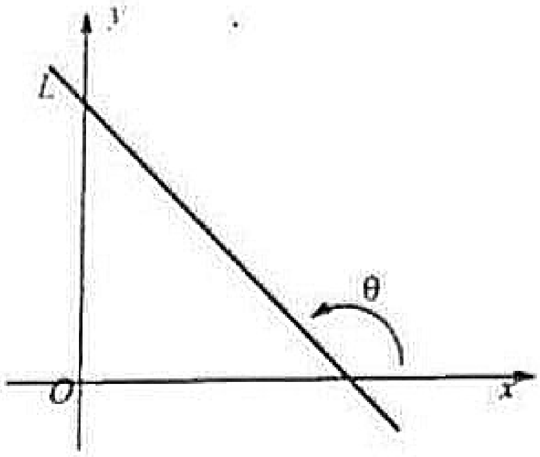


Fig. 7.19

Equation of a straight line passing through two given points.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given two points and let $p(x', y')$ be any point on the straight line. In the adjoining figure, the triangles BRA and ASP are similar.

$$\frac{AS}{BR} = \frac{PS}{AR} \text{ and hence } \frac{x' - x_1}{x_1 - x_2} = \frac{y' - y_1}{y_1 - y_2}$$

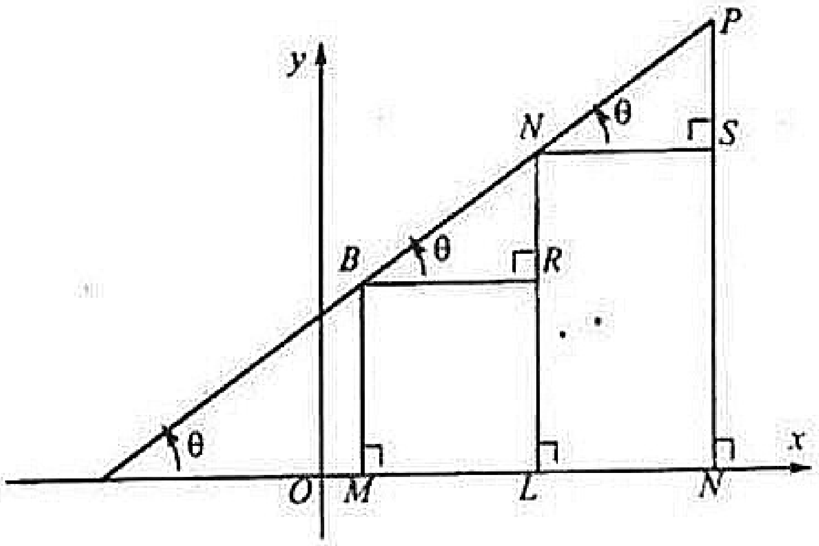


Fig. 7.20

Also conversely if $(x'\ y')$ satisfies $\frac{x' - x_1}{x_1 - x_2} = \frac{y' - y_1}{y_1 - y_2}$ then (x', y') should lie on AB. (Why?) In other words, the equation to the straight line AB joining the given points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ is given by $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$.

Note. (1) If P is any point on the straight line AB, then the area of the triangle PAB must be zero. So, if P is (x, y), then

$$0 = x\,(y_1 - y_2) + x_1(y_2 - y) + x_2(y - y_1)$$

This implies that $(x - x_1)\,(y_1 - y_2) = (x_1 - x_2)\,(y - y_1)$

i.e. $$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

Also, if area of $\Delta\ PAB = 0$ then P lies on the straight line AB. Therefore the equation to the straight line AB is $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$ which is in agreement with what we have got already.

(2) P lies on AB if and only if slope of PA = slope of PB = slope of the straight line AB = $\tan\theta$.

Therefore $$\tan\theta = \frac{PS}{AS} = \frac{AR}{BR}\ i.e. \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

(see Fig. 7.20 and observe that slope of $PA = \frac{PS}{AS}$ slope of $PB = \frac{AR}{BR}$).

Thus $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$ is the equation to the straight line AB. Incidentally, slope of AB $= \tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$.

Equation of a straight line in terms of the intercepts it makes with the coordinate axes.

Let L be a straight line making intercepts $OA = a$ and $OB = b$ on the coordinate axes. In our figure (Fig. 7.21). We have $a > 0$, $b > 0$. If $a < 0$, the corresponding point A $(a, 0)$ will be on the negative x-axis. Similarly if $b < 0$ the corresponding point $(0, b)$ will be on the negative y-axis. If L makes intercepts a and b on the x and y axes respectively, then L passes through $(a, 0)$ and $(0, b)$. Therefore the equation to the straight line L is

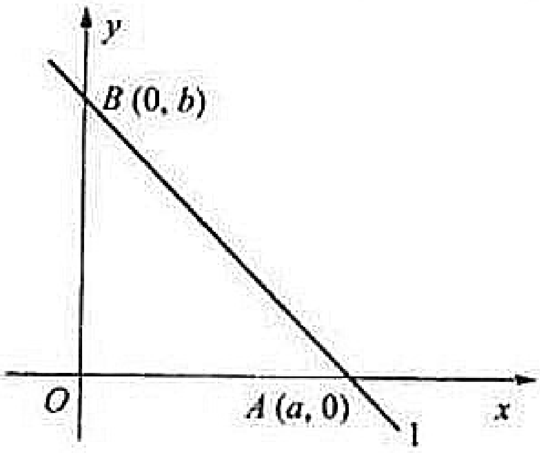


Fig. 7.21

$$\frac{x-a}{a-0} = \frac{y-0}{0-b} \quad \text{or} \quad \frac{x}{a} - 1 = \frac{y}{b} \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 1.$$

Note. The equation of the straight line L making intercepts 'a' and 'b' can be directly derived, without using the 2 point-formula.

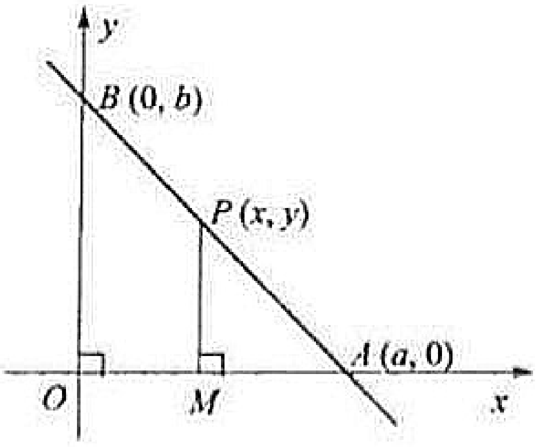


Fig. 7.22

Let P (x, y) be any point on AB, and let $PM \perp OX$ as in Fig. 7.22. The triangles AOB and AMP are similar.

$$\frac{OM}{OA} = \frac{BP}{BA} \quad \text{or} \quad \frac{x}{a} = \frac{PB}{AB}$$

and $$\frac{PM}{BO} = \frac{AP}{AB} \quad \text{gives} \quad \frac{y}{b} = \frac{AP}{AB}$$

Adding we get $$\frac{x}{a} + \frac{y}{b} = \frac{AP + PB}{AB} = 1.$$

We have taken P between A and B, the reader may check the validity of the equation in all the other cases. Thus, equation of the straight line AB is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Again, $P(x, y)$ lies on AB if and only if the area of the triangle PAB is zero.

$$0 = x(0 - b) + a(b - y) + 0(y - 0)$$

i.e. $$bx + ay = ab$$

Dividing by ab, $$\frac{x}{a} + \frac{y}{b} = 1.$$

Equation of a straight line in terms of its slope and y-intercept

Let L be the straight line with slope $m = \tan\theta$ and passing through C $(0, c)$, in other words making a y-intercept of 'c' on the y-axis. From the right triangle PNC (Fig. 7.23), we get $\tan(\pi - \theta) = -\tan\theta = -m = CN/NP$. $-m = \dfrac{c - y}{x}$, $y = mx + c$ is the equation to the straight line L with slope m and y-intercept 'c'.

One may check the validity of the equation for different positions of P on the line L.

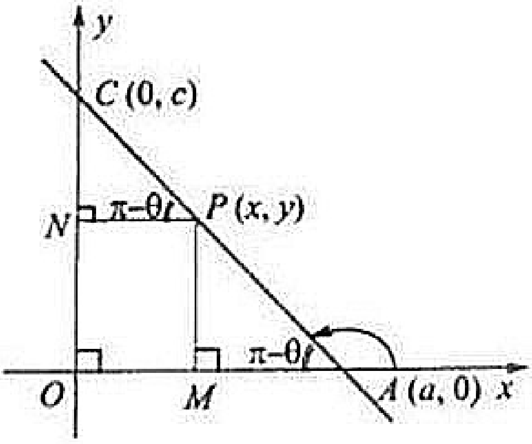


Fig. 7.23

Note. If the above line L meets the x-axis at A $(a, 0)$, then

$$\tan\angle OAC = \tan(\pi - \theta) = -\tan\theta = -m = \frac{OC}{OA} = \frac{c}{a},\ a = -\frac{c}{m}.$$

The line L makes intercepts $a = -c/m$ and c with the x and y axes respectively and therefore its equation is given by

$$\frac{x}{-(c/m)} + \frac{y}{c} = 1 \quad \text{or} \quad y = mx + c.$$

Equation of a straight line in terms of the length of the perpendicular from the origin and the angle which the perpendicular makes with the positive x-axis.

Let the straight line L be at a distance p from $O(0, 0)$, and let the perpendicular from $(0, 0)$ make an angle α with Ox (Fig. 7.24). From the right triangles OAM and OMB we observe that $OA = p\sec\alpha$ and $OB = p\operatorname{cosec}\alpha$. Using the intercept form of a straight line, the equation to L is

$$\frac{x}{p\sec\alpha} + \frac{y}{p\operatorname{cosec}\alpha} = 1 \text{ or } x\cos\alpha + y\sin\alpha = p$$

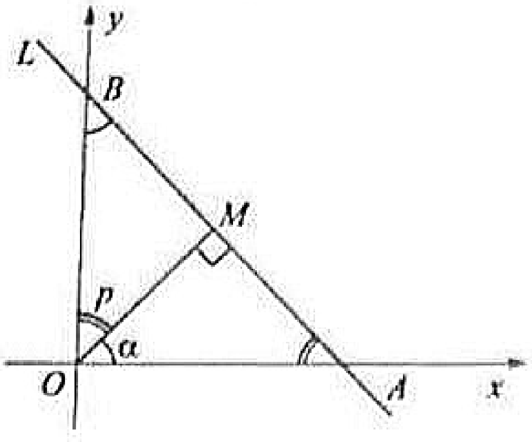


Fig. 7.24

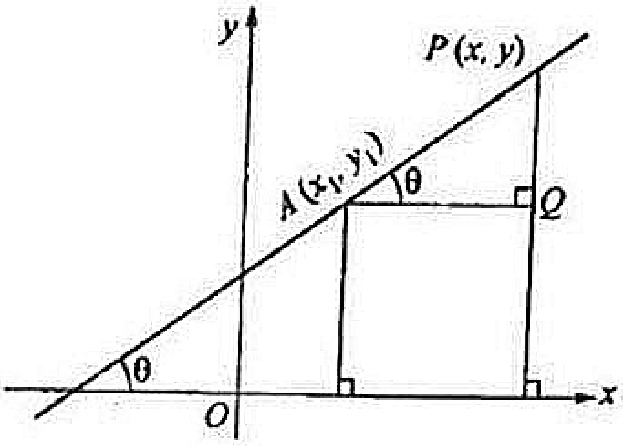


Fig. 7.25

One can easily check that $x\cos\alpha + y\sin\alpha = p$ is satisfied in all possible positions of the straight line.

Equation of a straight line passing through the point (x_1, y_1) and making an angle θ with the positive x-axis.

Let L be the straight line making an angle θ such that $m = \tan\theta$ and passing through $A(x_1, y_1)$. From ΔAQP we get $m = \tan\theta = \frac{PQ}{AQ} = \frac{y - y_1}{x - x_1}$. This in turn implies that the equation to the straight line L is $y - y_1 = m(x - x_1)$.

Parametric form of a straight line

Let L be the straight line passing through $A(x_1, y_1)$ and making an angle θ with the positive x-axis. From ΔAQP of Fig. 7.25 we get $\cos\theta = \cos\theta = \frac{AQ}{AP}$, $\sin\theta = \frac{PQ}{AP}$. If we denote the algebraic distance AP as r (for points on L on one side of A the algebraic distance is taken as positive and for points on the other side, the algebraic distance is taken as negative) then

We get $$\frac{AQ}{\cos\theta} = \frac{PQ}{\sin\theta} = r \quad \text{or} \quad \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r.$$

This is the parametric equation of the line L. Any point P on L is of the form $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ where r is the algebraic distance of P from $A(x_1, y_1)$. $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$ is the parametric equation to L. When one varies r over the real numbers one gets all the points on the straight line L.

We have now derived equation of a straight line in various different forms like.

1. $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$ (2 point–formula)
2. $\frac{x}{a} + \frac{y}{b} = 1$ (intercept form)
3. $y = mx + c$ (Slope–intercept form)
4. $x\cos\alpha + y\sin\alpha = p$ (normal form)
5. $y - y_1 = m(x - x_1)$ (Slope–one point form)
6. $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ (Parametric form)

$$x = x_1 + r\cos\theta,\ y = y_1 + r\sin\theta.$$

All these equations are linear equations in x, y of the form $Ax + By + C = 0$. This prompts the question:

"Does a linear equation of the form $Ax + By + C = 0$ always represent a straight line?"

The answer is yes.

$Ax + By + C = 0$ always represents a straight line unless $A = B = 0$ in which case C also becomes zero. Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be any three points on $Ax + By + C = 0$. Then we have

$$Ax_1 + By_1 + C = 0 \tag{1}$$

$$Ax_2 + By_2 + C = 0 \tag{2}$$

$$Ax_3 + By_3 + C = 0 \tag{3}$$

Multiply the equations (1), (2) and (3) by $y_2 - y_3$, $y_3 - y_1$ and $y_1 - y_2$ respectively and add the resultant equations to get

$$A\,\Sigma x_1(y_2 - y_3) + B\,\Sigma y_1(y_2 - y_3) + C\,\Sigma(y_2 - y_3) = 0$$

But $\quad \Sigma y_1(y_2 - y_3) = 0 = \Sigma(y_2 - y_3)$ and therefore we get

$$A\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} = 0. \tag{4}$$

If $A = 0$ then $Ax + By + C = 0$ becomes $By + C = 0$ which is the straight line $y = -C/B$ parallel to the x-axis, provided $B \neq 0$. If $A = 0 = B$ then $C = 0$. If $A \neq 0$, then the above equation (4) implies that $\Sigma x_1(y_2 - y_3) = 0$ which in turn implies that the area of the triangle formed by (x_1, y_1), (x_2, y_2) and (x_3, y_3) is zero. This means that any three points on $Ax + By + C = 0$ are collinear; in other words $Ax + By + C = 0$ represents a straight line.

EXAMPLE 1. *Find the equations of the medians of the triangle with vertices at A (1, 2), B (3, 4) and C (– 2, – 5).*

SOLUTION. Let A', B', C' be the midpoints of the sides BC, CA, AB respectively. Then

$$A' \text{ is } \left(\frac{3+(-2)}{2}, \frac{4+(-5)}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right)$$

$$B' \text{ is } \left(\frac{-2+1)}{2}, \frac{-5+2}{2}\right) = \left(\frac{1}{2}, -\frac{3}{2}\right)$$

$$C' \text{ is } \left(\frac{1+3}{2}, \frac{2+4}{2}\right) = (2, 3)$$

Equation to the median AA' is $\dfrac{x-1}{1-\frac{1}{2}} = \dfrac{y-2}{2-\left(-\frac{1}{2}\right)}$ which gives on simplification $5x - y - 3 = 0$. Similarly the median BB' has the equation $11x - 7y - 5 = 0$ and the median CC' has the equation $2x - y - 1 = 0$. The centroid G of the given triangle is given by

$$\left(\frac{1+3+(-2)}{3}, \frac{2+4+(-5)}{3}\right) = \left(\frac{2}{3}, \frac{1}{3}\right).$$

One readily checks that the centroid $\left(\dfrac{2}{3}, \dfrac{1}{3}\right)$ lies on all the three medians

$$5x - y - 3 = 0$$
$$11x - 7y - 5 = 0$$
$$2x - y - 1 = 0.$$

EXAMPLE 2. *Find the equation of the straight line when the portion of it intercepted between the axes is divided by the point (3, 1) in the ratio 1 : 3.*

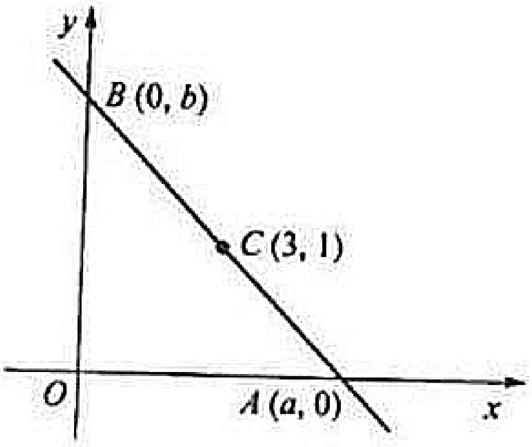


Fig. 7.26

SOLUTION. Let the required straight line meet the x-axis is at A $(a, 0)$ and meet the y-axis at B $(0, b)$. It is given that the point C (3, 1) divides AB or BA in the ratio 1 : 3. Therefore by the section formula, C must be the point

$$\left(\frac{1.0 + 3.a}{4}, \frac{1.b + 3.0}{4}\right) = \left(\frac{3a}{4}, \frac{b}{4}\right) \quad \text{if } \frac{AC}{CB} = \frac{1}{3}$$

or the point $\left(\frac{1.a + 3.0}{4}, \frac{1.0 + 3b}{4}\right) = \left(\frac{a}{4}, \frac{3b}{4}\right) \quad \text{if } \frac{BC}{CA} = \frac{1}{3}$

$C(3, 1) = \left(\frac{3a}{4}, \frac{b}{4}\right)$ gives $a = 4$, $b = 4$ and $C(3, 1) = \left(\frac{a}{4}, \frac{3b}{4}\right)$

gives $a = 12$, $b = 4/3$. Hence the required line is either

$\frac{x}{4} + \frac{y}{4} = 1$ or $\frac{x}{12} + \frac{y}{4/3} = 1$ *i.e.*, either

$x + y = 4$ or $x + 9y = 12$.

EXAMPLE 3. *Find the distance of the line $3x - y = 0$ from the point (4, 1) measured along a line making an angle of 135° with the x-axis.*

SOLUTION. The straight line L throgh (4, 1) making an angle of 135° with the x-axis is

$$\frac{x - 4}{\cos 135^\circ} = \frac{y - 1}{\sin 135^\circ} = r. \ i.e. \ \frac{x - 4}{-1/\sqrt{2}} = \frac{y - 1}{1/\sqrt{2}} = r.$$

(See Fig. 7.27) Any point P on this straight line is of the form $x = 4 - r/\sqrt{2}$, $y = 1 + r/\sqrt{2}$. Where r is the algebraic distance AP. If this point $(4 - r/\sqrt{2}, 1 + r/\sqrt{2})$ were to be on $3x - y = 0$ then $3(4 - r/\sqrt{2}) - (1 + r/\sqrt{2}) = 0$, $11 = 4\, r/\sqrt{2} = 2\sqrt{2}\, r$. Therefore $r = 11/2\sqrt{2} = 11\sqrt{2}/4$ units. Thus the distance of $3x - y = 0$ from (4, 1) measured along L is $11\sqrt{2}/4$ units.

EXAMPLE 4. *Find the equations of the straight lines through the origin whose intercepts between the straight lines $2x + 3y = 12$ and $2x + 3y = 15$ are each equal to three.*

SOLUTION. Assume the equation of the required straight line to be $y = mx$. Here m is to be determined. The intersection of this with $2x + 3y = 12$ is

$$p = \left(\frac{12}{2+3m}, \frac{12m}{2+3m}\right) \text{ and with } 2x + 3y = 15 \text{ is } Q = \left(\frac{15}{2+3m}, \frac{15m}{2+3m}\right).$$

The requirement is that the distance $PQ = 3$. This means

$$9 = \left(\frac{3}{2+3m}\right)^2 + \left(\frac{3m}{2+3m}\right)^2$$

which leads to $$m = \frac{-3 \pm \sqrt{3}}{4}.$$

Thus the required lines are

$$y = \frac{-3+\sqrt{3}}{4}x \quad \text{and} \quad y = \frac{-3-\sqrt{3}}{4}x.$$

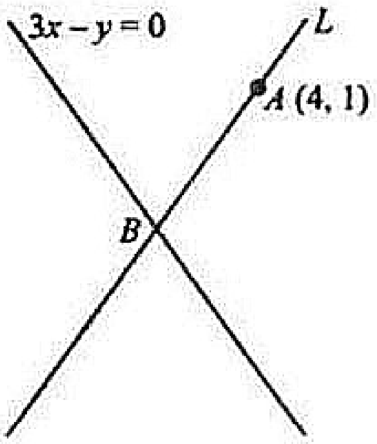


Fig. 7.27

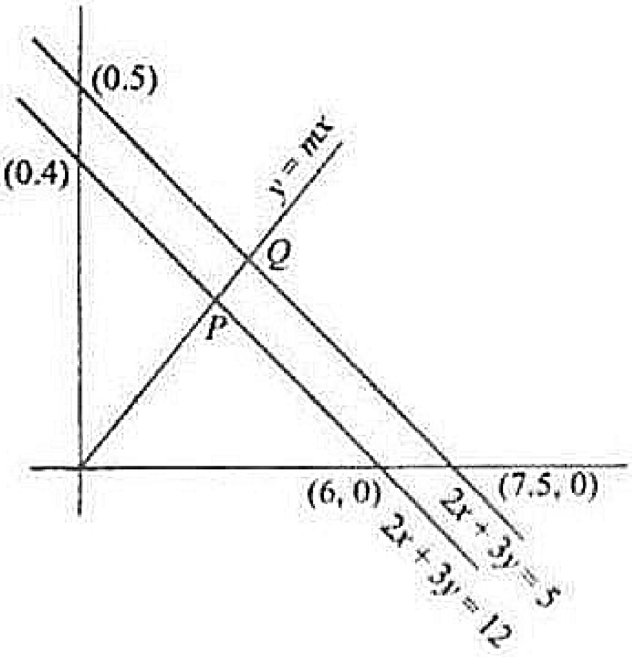


Fig. 7.28

Angle between straight lines

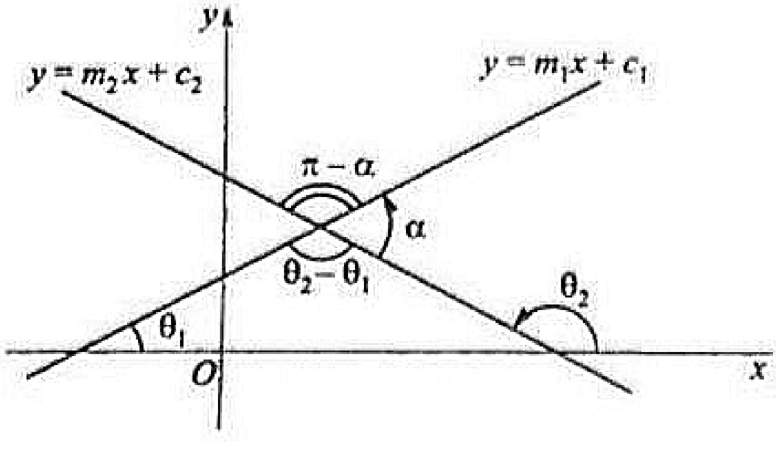


Fig. 7.29

Consider two straight lines $y = m_1x + C_1$ and $y = m_2x + C_2$. Then the two angles between these straight lines are supplementary angles. From Fig. 7.29 it is clear that the two angles between the two straight lines are $\theta_2 - \theta_1$ and $\pi - (\theta_2 - \theta_1)$ where θ_1 and θ_2 are the angles made by the given lines with the positive x-axis. Therefore we have $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$ and the angles between the straight lines are given by the equation

$$\tan\theta = \pm\tan(\theta_1 - \theta_2) = \pm\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\,\tan\theta_2} = \pm\frac{m_1 - m_2}{1 + m_1m_2}$$

If $\dfrac{m_1 - m_2}{1 + m_1m_2} > 0$ then $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1m_2}$ gives the acute angle between the two straight lines and if $\dfrac{m_1 - m_2}{1 + m_1m_2} < 0$ then $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1m_2}$ gives the obtuse angle between the straight lines.

Note. (1) Two straight lines are parallel iff their slopes are equal. The straight line $Ax + By + C = 0$ has slope $-A/B$ (compare it with $y = mx + C$) and hence two straight lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are parallel iff $b_1/A_1 = b_2/A_2$ or equivalently $A_1/A_2 = B_1/B_2$; in which case $A_2 = kA_1$, and $B_2 = kA_2$ for some constant k. This makes $A_2x + B_2y + C_2 = 0$ as $k(A_1x + B_1y + C_2/k) = 0$ which is the same as $A_1x + B_1y + C_2/k = 0$. Thus any two parallel straight lines can be put in the form $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ so that they differ only in the constant terms.

(2) Two straight lines $y = m_1x + C_1$ and $y = m_2x + C_2$ are perpendicular if and only if $\dfrac{m_1 - m_2}{1 + m_1m_2} = \tan\pi/2 = \infty$ which happens if and only if $m_1m_2 = -1$. Thus two straight lines are perpendicular if and only if the product of their slopes is -1. The straight line $ax + by + c = 0$ has slope $-a/b$ and hence any straight line perpendicular to it must have slope b/a. Therefore any straight line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + C' = 0$ for some constant C'.

EXAMPLE 5. *An equilateral triangle has its centroid at the origin and one side is $x + y = 1$. Find the other sides of the triangle.*

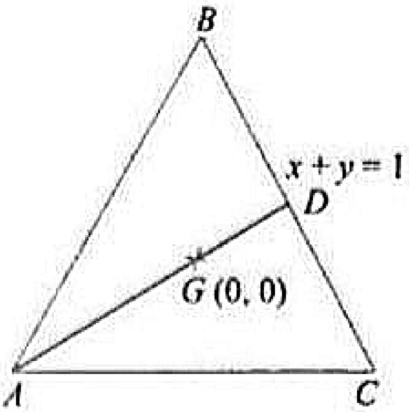


Fig. 7.30

SOLUTION. Let the vertex not on the line $x + y = 1$ be $A(x_1, y_1)$. Let the other two vertices be $B(b, 1 - b)$ and $C(c, 1 - c)$. Note that we have here used the fact that B and C lie on $x + y = 1$. We are given that G is $(0, 0)$. Let AG meet $x + y = 1$ at D. Since G is the centroid, we have

$$b + c + x_1 = 0 = 1 - b + 1 - c + y_1 \qquad (1)$$

But $AD \perp BC$ ($\because$ the triangle is equilateral). $\therefore$ 'm' of AD × 'm' of $BC = -1$

i.e., $\frac{y_1}{x_1} \times (-1) = -1$ so that $y_1 = x_1$.

Substituting this in (1) we get $b + c = 1$. This gives $c = 1 - b$. Thus the three vertices of the triangle are $A(x_1, x_1)$, $B(b, 1 - b)$ and $C(1 - b, b)$. Again equation (1) gives $x_1 + b + 1 - b = 0$, which means $x_1 = -1$. Hence A is $(-1, -1)$.

Now equation to AC is

$$y + 1 = \frac{b+1}{2-b}(x+1)$$

which reduces to

$$y = \frac{b+1}{2-b}x + \frac{2b-1}{2-b} \tag{2}$$

Similarly, equation to AB is

$$y = \frac{2-b}{b+1}x + \frac{1-2b}{b+1} \tag{3}$$

Now we have only to find b. We know AC and BC make an angle of 60°. So

$$\sqrt{3} = \frac{\frac{b+1}{2-b}+1}{1+\left(\frac{b+1}{2-b}\right)(-1)} \tag{*}$$

This gives, $b = -\frac{\sqrt{3}-1}{2}$. Since this gives a positive value for the 'm' of AC (check !), we accept this value and do not proceed with $-\sqrt{3}$ on the L.H.S. of (*). Substituting the value of b in (2) and (3) we get the required lines.

EXAMPLE 6. *If the image of the point* (h_1, k_1) *with respect to the line* $ax + by + c = 0$ *is the point* (h_2, k_2) *then show that*

$$\frac{h_2 - h_1}{a} = \frac{k_2 - k_1}{b} = -2\,\frac{(ah_1 + bk_1 + c)}{a^2 + b^2}$$

SOLUTION. As $Q\,(h_2, k_2)$ is the image of $P(x_1, y_1)$ with respect to the line $ax + by + c = 0$ we must have PQ perpendicular to $ax + by + c = 0$ and $PR = RQ$ (see Fig. 7.31) where R is the point of intersection of PQ with the straight line $ax + by + c = 0$.

$\therefore$ Slope PQ × Slope $(ax + by + c) = -1$

i.e., $\frac{k_2 - k_1}{h_2 - h_1} \times \frac{-a}{b} = -1$

This implies that $\frac{k_2 - k_1}{b} = \frac{h_2 - h_1}{a} = \lambda$ (say).

Then $h_2 = h_1 + a\lambda$ and $k_2 = k_1 + b\lambda$. This gives the midpoint R of PQ as

$$\left(\frac{h_1 + h_1 + a\lambda}{2}, \frac{k_1 + k_1 + b\lambda}{2}\right).$$

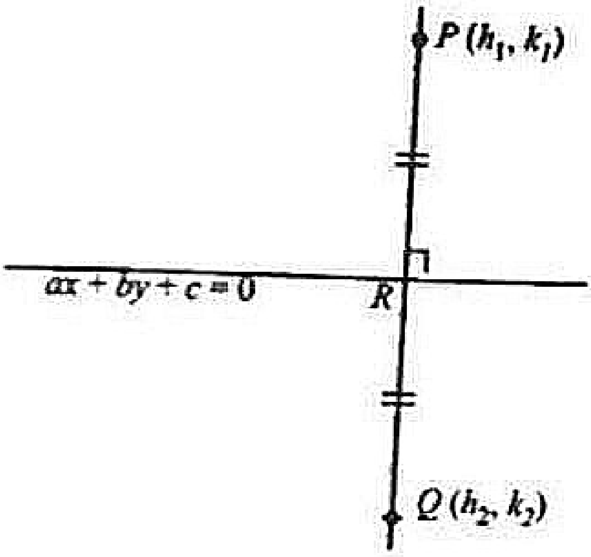


Fig. 7.31

$\therefore$ R is the point $\left(\frac{2h_1 + a\lambda}{2}, \frac{2k_1 + b\lambda}{2}\right)$.

Now R lies on $ax + by + c = 0$ gives

$$a\left(\frac{2h_1 + a\lambda}{2}\right) + b\left(\frac{2k_1 + b\lambda}{2}\right) + c = 0$$

$\therefore$ $$(a^2 + b^2)\lambda = -2(ah_1 + bk_1 + c)$$

or $$\lambda = -2\,\frac{(ah_1 + bk_1 + c)}{a^2 + b^2}$$

Thus $$\frac{h_2 - h_1}{a} = \frac{k_2 - k_1}{b} = \lambda = -2\,\frac{ah_1 + bk_1 + c}{a^2 + b^2}.$$

Equation of a family of straight lines passing through the intersection of two given lines.

Consider two straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv a_2x + b_2y + c_2 = 0$ intersecting at a point $P(h, k)$. Now

$$L_1 + \lambda L_2 \equiv (a_1x + b_1y + c_1) + \lambda\,(a_2x + b_2y + c_2) = 0 \qquad (*)$$

is again a linear equation, namely

$$(a_1 + \lambda a_2)\,x + (b_1 + \lambda\, b_2)\,y + (c_1 + \lambda\, c_2) = 0$$

and hence is also a straight line. Further the point (h, k) satisfies $(a_1h + b_1k + c_1) + \lambda\,(a_2h + b_2k + c_2) = 0 + \lambda.\,0 = 0$. Therefore, (*) is a straight line passing through the point of intersection of L_1 and L_2. Conversely, suppose L is any straight line passing through the point of intersection of L_1 and L_2. Let $px + qy + r = 0$ be the straight line L.

Then L can be written as $y - k = m(x - h)$ where m is its slope. Solving $a_1h + b_1k + c_1 = 0$, $a_2h + b_2k + c_2 = 0$ we get

$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } k = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

$\therefore$ $y - \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = m\left(x - \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right)$ is the equation of L.

Simplifying we get $m(a_1b_2 - a_2b_1)x - y(a_1b_2 - a_2b_1) + (a_2c_1 - a_1c_2) - m(b_1c_2 - b_2c_1) = 0$ *i.e.*, $(a_1 + mb_2)(a_1x + b_1y + c_1) - (a_1 + mb_1)(a_2x + b_2y + c_2) = 0$ which is of the form $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$

where $$\lambda = -\frac{a_1 + mb_1}{a_2 + mb_2}$$

Note. When we very λ in $L_1 + \lambda L_2 = 0$ we get the family of straight lines passing through the intersection of $L_1. = 0$ and $L_2 = 0$.

EXAMPLE 7. *The equation of the sides of a triangle are $x + 2y = 0$, $4x + 3y = 5$ and $3x + y = 0$. Find the orthocentre of the triangle.*

SOLUTION. Let AB be $x + 2y = 0$, BC be $4x + 3y = 5$ and CA be $3x + y = 0$. The slope of BC is $-4/3$ and therefore the slope of AD is $3/4$. Hence the equation to AD is $3x - 4y = 0$. Similarly one finds that the altitude BE must be of the form $x + 2y + \lambda(4x + 3y - 5) = 0$ or $(1 + 4\lambda)x + (2 + 3\lambda)y - 5\lambda = 0$.

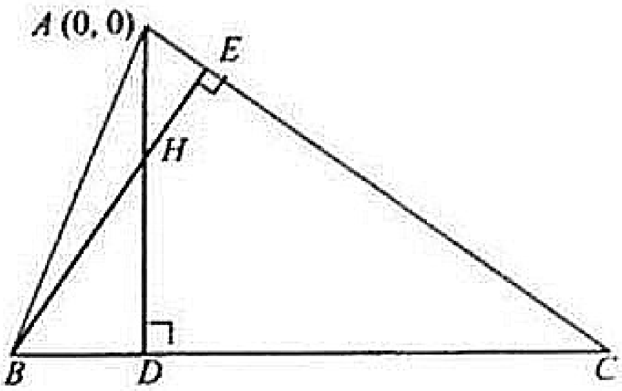


Fig. 7.32

$$-1 = (\text{Slope of } BE)(\text{Slope of } CA) = -\left(\frac{1 + 4\lambda}{2 + 3\lambda}\right)(-3) \text{ or } 3(1 + 4\lambda) = -(2 + 3\lambda).$$

This gives $\lambda = -1/3$ or BE is given by $x - 3y - 5 = 0$.

The orthocentre H of $\Delta\, ABC$ is got by solving $3x - 4y = 0$, $x - 3y = 5$. We get H as $(-4, -3)$.

EXAMPLE 8. *Prove that the diagonals of the parallelogram formed by the lines $ax + by + c = 0$, $ax + by + c' = 0$, $a'x + b'y + c = 0$, $a'x + b'y + c' = 0$ will be at right angles if $a^2 + b^2 = a'^2 + b'^2$.*

SOLUTION. Now the diagonal AC is of the form $ax + by + c + \lambda(a'x + b'y + c) = 0$ as it passes through the intersection of $ax + by + c = 0$ and $a'x + b'y + c' = 0$ (Fig. 7.33).

$$\text{Slope of } AC = -\left(\frac{a + \lambda a'}{b + \lambda b'}\right)$$

Similarly, the diagonal Ac is also of the form $(ax + by + c') + \mu(a'x + b'y + c' = 0)$. Thus AC is given by $(a + \lambda a')x + (b + \lambda b')y + (1 + \lambda)c = 0$ and $(a + \mu a')x + (b + \mu b')y + (1 + \mu)c' = 0$.

We must have $$\frac{a + \lambda a'}{a + \mu a'} = \frac{b + \lambda b'}{b + \mu b'} = \left(\frac{1 + \lambda}{1 + \mu}\right)\frac{c}{c'}$$

$$\lambda(a'b - ab') = \mu(a'b - ab')$$

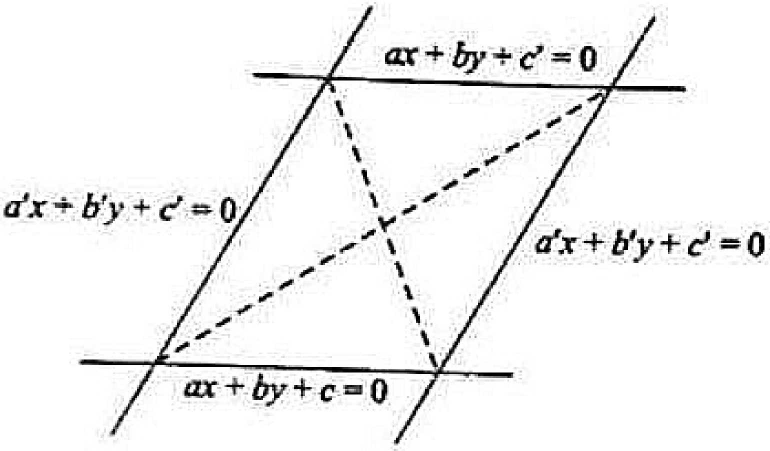


Fig. 7.33

Now $\frac{a}{b} \neq \frac{a'}{b'}$ (Why?) and hence $\lambda = \mu$. This means that $1 = \frac{1+\lambda}{1+\lambda\frac{c}{c'}}$.

But $c \neq c'$ (why?) and hence $\lambda = -1$. Thus $\lambda_1 = k = -1$.

Therefore equation to AC is $(a - a')x + (b - b')y = 0$. Similarly equation to BD is $(a + a')x + (b + b')y + c + c' = 0$. (It is easily observed that $(ax + by + c) - (a'x + b'y + c) = 0$ passes through A and C; and $(ax + by + c) + (a'x + b'y + c') = 0$ passes through B and D).

Now AC is perpendicular to BD if and only if the product of their slopes is -1; which happens iff

$$\left(-\frac{a-a'}{b-b'}\right)\cdot\left(-\frac{a+a'}{b+b'}\right) = -1 \text{ or simplifying } a^2 + b^2 = a'^1 + b'^2$$

Remark. If the sides of parallelogram are $L_1 \equiv ax + by + c = 0$, $L_2 \equiv a'x + b'y + c = 0$, $L_3 \equiv ax + by + c' = 0$ and $L_4 \equiv a'x + b'y + c' = 0$, the diagonals are given by $L_1 - L_2 = 0$ and $L_1 + L_2 = 0$.

EXAMPLE 9. *The sides of a triangle are* $U_r \equiv x \cos \alpha r + y \sin \alpha r - p_r = 0$ *for* $r = 1, 2, 3$.

Show that its orthocentre is given by

$$U_1 \cos(\alpha_2 - \alpha_3) = U_2 \cos(\alpha_3 - \alpha_1) = U_3 \cos(\alpha_1 - \alpha_2).$$

SOLUTION. Let BC be $U_1 \equiv x \cos \alpha_1 + y \sin \alpha_1 - p_1 = 0_1$,
CA be $U_2 = 0$ and AB be $U_3 = 0$.

Then the altitude AD through A is of the form $U_2 + \lambda U_3 = (x \cos \alpha_2 + y \sin \alpha_2 - p_2) + \lambda(x \cos \alpha_3 + y \sin \alpha_3) - p_3 = 0$ *i.e.*, AD is given by $(\cos \alpha_2 + \lambda \cos \alpha_3)x + (\sin \alpha_2 + \lambda \sin \alpha_3)y - (p_2 + \lambda p_3) = 0$.

$$\text{Slope of } AD = -\frac{\cos \alpha_2 + \lambda \cos \alpha_3}{\sin \alpha_2 + \lambda \sin \alpha_3}. \quad \text{Now } AD \perp BC \text{ gives}$$

$$\left(-\frac{\cos \alpha_2 + \lambda \cos \alpha_3}{\sin \alpha_2 + \lambda \sin \alpha_3}\right)(-\cot \alpha_1) = -1,$$

or $$\cos \alpha_1(\cos \alpha_2 + \lambda \cos \alpha_3) + \sin \alpha_1(\sin \alpha_2 + \lambda \sin \alpha_3) = 0,$$

This gives $$(\cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2 + \lambda(\cos \alpha_1 \cos \alpha_3 + \sin \alpha_1 \sin \alpha_3) = 0$$

or $$\text{equivalents } -\lambda = -\frac{\cos(\alpha_1 - \alpha_2)}{\cos(\alpha_3 - \alpha_1)}.$$

(0, 0) and A are on the same side of BC whose equation is $x + 7y + 11 = 0$ since $2 + 7 + 11 = 20 > 0$ and $0 + 7(0) + 11 = 11 > 0$ are of the same sign.

Similarly, B and (0, 0) are on the same side of CA because $3 - 3(-2) + 1 = 10 > 0$ and $0 - 3(0) + 1 = 1 > 0$. Finally, substituting the coordinates of C in $3x + y - 7$ we get $3(-4) + (-1) - 7 = -20 < 0$; also $3(0) + (0) - 7 = -7 < 0$. Hence (0, 0) lies within ΔABC.

Alternately we can do the above example as follows.

Equation to OA is $y = x/2$ or $x - 2y = 0$.

Substituting the coordinats of $B(3, -2)$ and $C(-4, -1)$ in the equation to OA we see that $3 - 2(-2) = 7 > 0$ and $-4 - 2(-1) = -2 < 0$. So B and C are on the opposite sides of OA. This means that $O(0, 0)$ lies within either the region 1, 4 or 7 (Fig. 7.35). Similarly C and A are on the opposite sides of OB. Again this means that O lies within the region either 2, 5 or 7. Hence O (0, 0) lies within the region 7; *i.e.*, O lies with ΔABC.

To find the length of the perpendicular from the origin to the straight line $ax + by + c = 0$.

If ON is the perpendicular from O on $ax + by + c = 0$ then the straight line ON may be put in the normal form $x \cos \alpha + y \sin \alpha = p$ where $p = |ON|$ (Fig. 7.36).

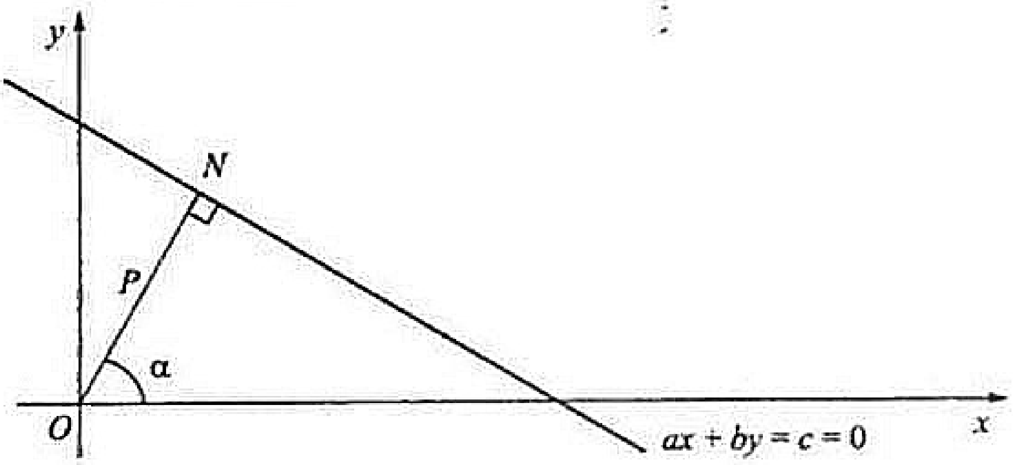


Fig. 7.36

Now $ax + by + c = 0$ and $x \cos \alpha + y \sin \alpha = p$ represent the same straight line implies that

$$\frac{a}{\cos \alpha} = \frac{b}{\sin \alpha} = \frac{c}{(-p)}.$$

Eliminating α using $\cos^2 \alpha + \sin^2 \alpha = 1$ we get $p = \pm \dfrac{c}{\sqrt{a^2 + b^2}}$.

Thus the length of the perpendicular from the origin is $p = \left| \dfrac{c}{\sqrt{a^2 + b^2}} \right|$

Note. If $c < 0$ in $ax + by + c = 0$, then $p = \dfrac{-c}{\sqrt{a^2 + b^2}}$ and if $c > 0$ then $p = \dfrac{c}{\sqrt{a^2 + b^2}}$.

SHIFTING OF ORIGIN

Suppose we fix a Cartesian frame of reference OX, OY and A is a point with coordinates (x_1, y_1). Consider a new pair of axes AX', AY' through A parallel to the original axes OX, OY respectively (Fig. 7.37).

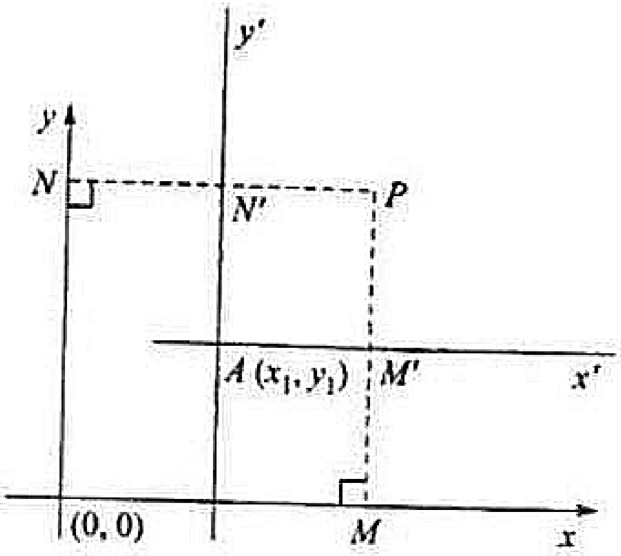


Fig. 7.37

With respect to the new axes, whose origin now is A, the coordinates of A are $(0, 0)$. Let P be any point in the plane whose coordinates are $P(x, y)$ and $P(x', y')$ with respect to OX, OY and AX', AY' axes respectively. Then we have $OM = x, AM' = x', ON = y, AN' = y'$ (see Fig. 7.37 for the explanation of the symbols M, M' etc. We observe that $OM = x = x_1 + x'$ and $ON = y = y_1 + y'$. Thus the transformation equations from the old set of axes to the new set of axes are $x' = x - x_1, y' = y - y_1$.

For example, if we shift the origin to $(-1, 3)$ with the axes remaining parallel, the transformation equations are $x' = x - (-1) = x + 1, y' = y - 3$. This means that if P is $(3, -2)$ with respect to the old axes, then P is $(3 + 1, -2 - 3) = (4, -5)$ with respect to the new axes.

To find the perpendicular distance of $A(x_1, y_1)$ from $ax + by + c = 0$:

Shifting the origin to $A(x_1, y_1)$ with the axes remaining parallel, the equation to the given line becomes $a(x' + x_1) + b(y' + y_1) + c = 0$ or $ax' + by' + (ax_1 + by_1 + c) = 0$, with respect to the new axes. A is the origin of the new axes and hence the perpendicular distance of A from $ax' + by' + (ax_1 + by_1 + c) = 0$ is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

EXAMPLE 12. *Find the locus of a point which moves such that the sum of the perpendicular distances from it on two given straight lines is a constant.*

SOLUTION. We may take one of the two given straight lines to be our x-axis and the point of intersection of the given lines as our origin. Observe that we have the freedom of choosing our axes, depending on the problem. (Reader, this is where we intelligently exploit the convenience of coordinate geometry). By our choice of the axes one of the given lines is $y = 0$ and let the other be $mx - y = 0$. If $P(x', y')$ is the variable point, its distances

from the two given lines are $|y'|$ and $\dfrac{|mx' - y'|}{\sqrt{1 + m^2}}$

We are given that P moves such that $|y'| + \frac{|mx' - y'|}{\sqrt{1+m^2}} = k =$ constant.

$\therefore$ The locus of p is $\sqrt{1 + m^2}\,|y| + |mx - y| = k\sqrt{1 + m^2}$.

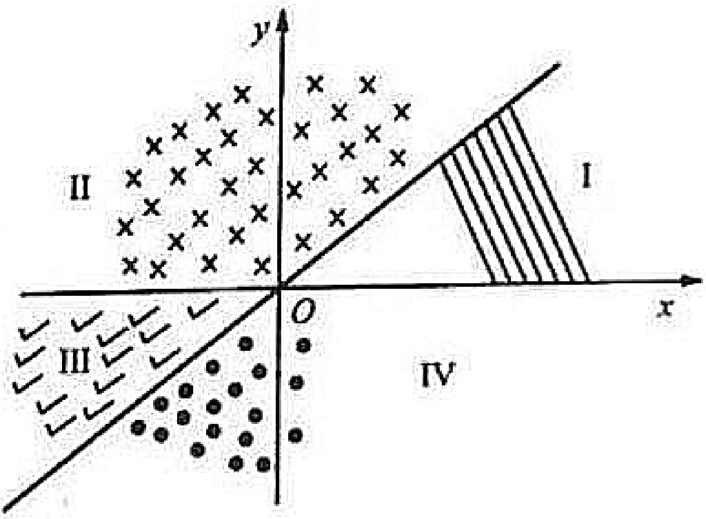


Fig. 7.38

In region I where $y > 0$, $mx > y$ the locus is the straight line $mx + (\sqrt{1+m^2 - 1})y - k\sqrt{1+m^2} = 0$. In region II where $y > 0$, $mx < y$ the locus is the straight line $(1 + \sqrt{1+m^2})y - mx = k\sqrt{1+m^2}$. In region III, where $y < 0$, $mx < y$ the locus is the straight line $(1 - (1 - \sqrt{1+m^2})y - mx = k\sqrt{1+m^2}$. In region IV, the locus is $mx - (1 + \sqrt{1+m^2})y = k\sqrt{1+m^2}$.

EXAMPLE 13. *Find the incentre of the triangle whose sides have the equations*

$$x + y - 7 = 0, \quad x - y + 1 = 0 \quad \text{and} \quad x - 3y + 5 = 0.$$

SOLUTION. Let I (x_1, y_1) be the incentre of Δ ABC whose sides have the given equations. Then the perpendicular distance of I from BC is $= \dfrac{|x_1 + y_1 - 7|}{\sqrt{2}}$.

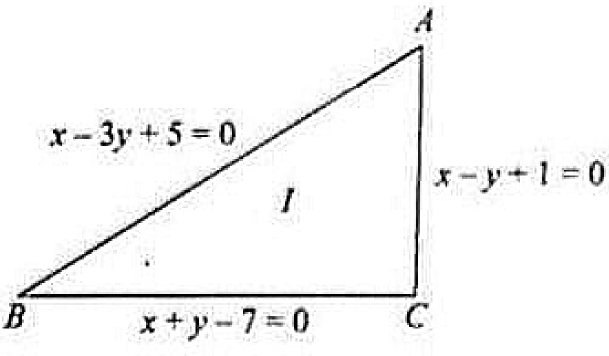


Fig. 7.39

From Fig. 7.39, it is clear that (0, 0) and I are on the opposite sides of BC. Hence $x_1 + y_1 - 7 > 0$.

$\therefore$ $\quad r = +\dfrac{x_1 + y_1 - 7}{\sqrt{2}}$. Similarly the distance of I from CA is

$$= \frac{|x_1 + y_1 + 1|}{\sqrt{2}} = -\frac{(x_1 - y_1 + 1)}{\sqrt{2}}$$

since O and I are on the opposite sides of $x - y + 1 = 0$. The distance of I from AB is

$$r = \frac{|x_1 - 3y_1 + 5|}{\sqrt{1+9}} = \frac{x_1 - 3y_1 + 5}{\sqrt{10}} \cdot$$

since O and I are on the same side of AB (Fig. 7.39).

Thus $$r = \frac{x_1 + y_1 - 7}{\sqrt{2}} = -\frac{(x_1 - y_1 + 1)}{\sqrt{2}} = \frac{x_1 - 3y_1 + 5}{\sqrt{10}}.$$

which leads to $$x_1 = 3, y_1 = 1 + \sqrt{5}.$$

$\therefore$ The incentre is $(3, 1 + \sqrt{5})$.

EXAMPLE 14. *Prove that the area of a parallelogram is $p_1p_2/\sin\alpha$ where p_1, and p_2 are the distances between the parallel sides and α is any angle of the parallelogram. Hence prove that the area of the parallelogram formed by $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_2x + b_2y + d_2 = 0$ is*

$$\left|\frac{(d_1 - c_1)(d_2 - c_2)}{(a_1b_2 - a_2b_1)}\right|.$$

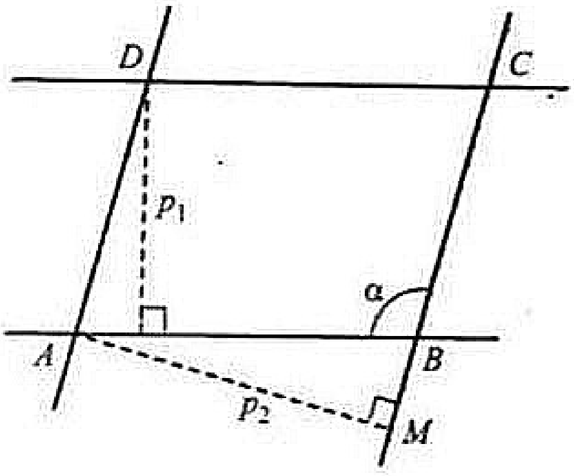


Fig. 7.40

SOLUTION. We have, area of the parallelogram $ABCD = AB \cdot p_1 \cdot p_1$ (From ΔABM) $= (p_2/\sin(180 - \alpha))$

Thus area of parallelogram $ABCD = (p_1 p_2)/\sin\alpha$.

Let AB be $a_1x + b_1y + c_1 = 0$, BC be $a_2x + b_2y + c_2 = 0$, CA be $a_1x + b_1y + d_1 = 0$, and DA be $a_2x + b_2y + d_2 = 0$. Then the distance between the parallel lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_1y + d_1 = 0$ is given by

$$p_1 = \frac{|c_1 - d_1|}{\sqrt{a_1^2 + b_1^2}}. \text{ Similarly } p_2 = \frac{|c_2 - d_2|}{\sqrt{a_2^2 + b_2^2}}$$

The acute angle between AB and BC is given by $\tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$ where m_1, m_2 are the slopes of AB, BC.

$$\therefore \quad \tan^2\alpha = \frac{\left(-\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)^2}{\left(1 + \frac{a_1a_2}{b_1b_2}\right)^2} = \frac{(a_2b_1 - a_1b_2)^2}{(b_1b_2 + a_1a_2)^2}.$$

Solving $x + 2y = 9$ and $3x - 5y = 5$ we get the point $(5, 2)$. If $ax + by = 1$ passes through $(5,2)$ then $5a + 2b = 1$. This means that $5x + 2y = 1$ passes through (a, b).

EXAMPLE 19. *Prove that* $(a + b)x - aby = c(a^2 + ab + b^2)$, $(b + c)x - bcy = a(b^2 + bc + c^2)$, $(c + a)x - cay = b(c^2 + ca + a^2)$ *are concurrent.*

SOLUTION. We note that

$$\begin{aligned} & c^2(a-b)\{(a+b)x - aby - c(a^2+ab+b^2)\} \\ & + a^2(b-c)\{(b+c)x - bcy - a(b^2+bc+c^2)\} \\ & + b^2(c-a)\{(c+a)x - cay - b(c^2+ca+a^2)\} \\ & = (\Sigma c^2(a^2-b^2))x - abc(\Sigma c(a-b))y - \Sigma c^3(a^3-b^3) \\ & \equiv 0 \text{ (identically zero).} \end{aligned}$$

Hence the three lines are concurrent.

EXAMPLE 20. *Find the equations of the diagonals formed by the lines* $2x - y + 7 = 0$, $2x - y - 5 = 0$, $3x + 2y - 5 = 0$ *and* $3x + 2y + 4 = 0$.

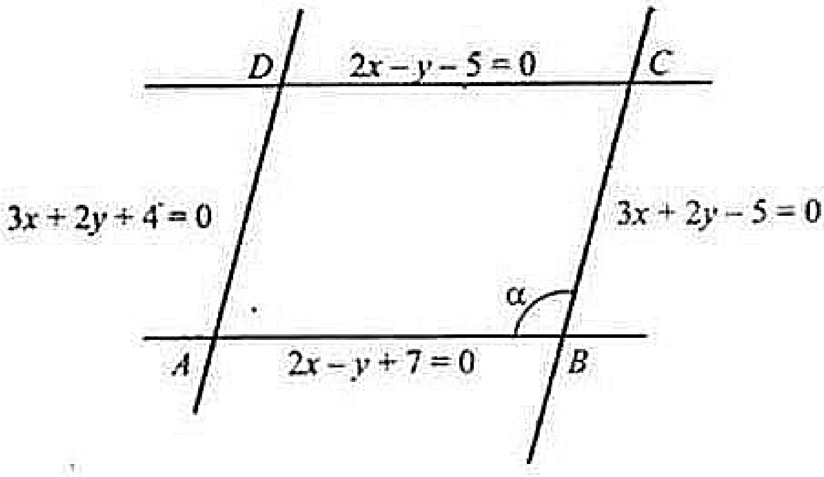


Fig. 7.43

SOLUTION. Equation to AC must be of both the forms

$$2x - y + 7 + \lambda(3x + 2y + 4) = 0 \text{ and}$$
$$2x - y - 5 + \mu(3x + 2y - 5) = 0.$$

This gives $$\frac{2+3\lambda}{2+3\mu} = \frac{-1+2\lambda}{-1+2\mu} = \frac{7+4\lambda}{-5-5\mu}$$

$$\frac{2+3\lambda}{2+3\mu} = \frac{-1+2\lambda}{-1+2\mu} \text{ gives } \lambda = \mu$$

$\therefore$ $$7 + 4\lambda = -5 - 5\lambda \text{ or } \lambda = -(4/3)$$

Hence equation to AC is

$$2x - y + 7 - (4/3)(3x + 2y + 4) = 0 \text{ or}$$
$$6x + 11y - 5 = 0$$

Similarly BD is of both the forms

$$2x - y + 7 + k(3x + 2y - 5) = 0$$
$$2x - y - 5 + l(3x + 2y + 4) = 0.$$

This gives $$\frac{2+3k}{2+3l} = \frac{-1+2k}{-1+2l} = \frac{7-5k}{-5+4l}.$$

Again, these equations give $k = 1$ and $7 - 5k = -5 + 4k$ or $k = (4/3)$.

Equation to BD is $2x - y + 7 + (4/3)(3x + 2y - 5) = 0$

or
$$18x + 5y + 1 = 0.$$

Thus the diagonals have the equations $6x + 11y - 5 = 0$, $18x + 5y + 1 = 0$.

EXAMPLE 21. *A straight line moves so that the sum of the reciprocals of its intercepts on the coordinate axes is constant. Show that it passes through a fixed point.*

SOLUTION. Let the variable line be $\frac{x}{a} + \frac{y}{b} = 1$.

Then we are given that $\frac{1}{a} + \frac{1}{b} = K = a$ constant.

Therefore, the variable line takes the form

$$\frac{x}{a} + \left(k - \frac{1}{a}\right)y - 1 = 0 \quad \text{or}$$

$\frac{1}{a}(x - y) + (Ky - 1) = 0$. This represents a straight line a through the intersection of $x - y = 0$ and $Ky - 1 = 0$. They intersect at $(1/K, 1/K)$ and hence $\frac{x}{a} + \frac{y}{b} = 1$ always passes through the fixed point $\left(\frac{1}{K}, \frac{1}{K}\right)$.

EXAMPLE 22. *$ABCD$ is a variable rectangle having its sides parallel to fixed directions. The vertices B and D lie on $x = a$ and $x = -a$ and A lies on the line $y = 0$. Find the locus of C.*

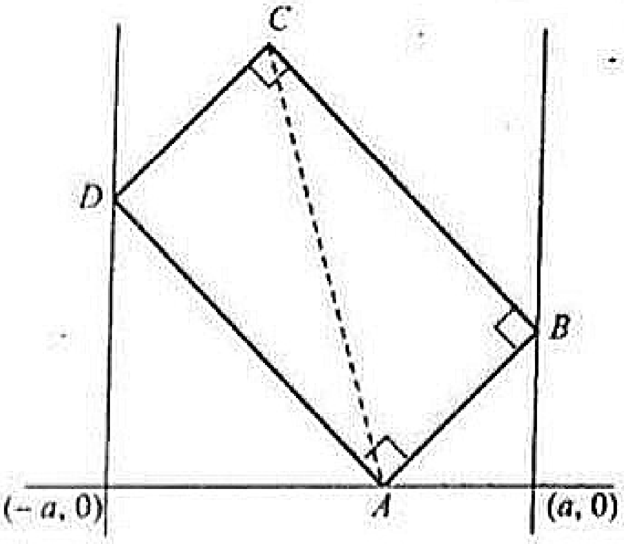


Fig. 7.44

SOLUTION. Let A be $(x_1, 0)$, B be (a, y_2) and D be $(-a, y_4)$. We are given AB and AD have fixed directions and hence their slopes are constants, say m_1 and m_2.

$$\therefore \qquad \frac{y_2}{a - x_1} = m_1, \text{ and } \frac{y_4}{-a - x_1} = m_2.$$

Further $m_1 m_2 = -1$ since $ABCD$ is a rectangle.

$$\frac{y_2}{a-x_1} = m_1 \text{ and } \frac{y_4}{-a-x_1} = -\frac{1}{m_1}.$$

The midpoint of BD is $\left(0, \frac{y_2+y_4}{2}\right) = \left(\frac{x_1+x}{2}, \frac{y}{2}\right)$

= midpoint of AC where C is taken to be (x, y).

This gives $x = -x_1$ and $y = y_2 + y_4$. So C is $(-x_1, y_2 + y_4)$.

Also $\frac{y_2}{a-x_1} = m_1$ and $\frac{y_4}{a+x_1} = +\frac{1}{m_1}$ gives

the locus of C as

$$(m_1^2 - 1)x + m_1 y = (m_1^2 + 1).$$

EXAMPLE 23. *Each side of a square is of length 6 units and the centre of the square is $(-1, 2)$. One of its diagonals is parallel to $x + y = 0$. Find the coordinates of the vertices of the square.*

SOLUTION. Let $ABCD$ be the given square with centre $(-1, 2)$ and side of length 6. BD is parallel to $x + y = 0$. Equation to BD is $x + y = 1$. Hence the equation to AC is $x - y + 3 = 0$ (note that $AC \perp BD$).

We have $|OC| = |OB| = |OA| = |OD| = 3\sqrt{2}$ units.

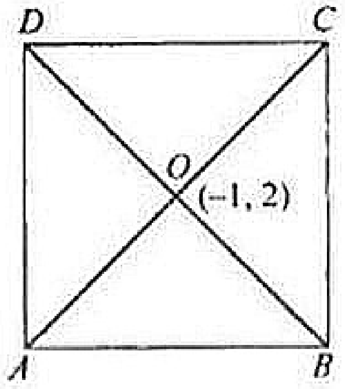


Fig. 7.45

We may write AC as $\frac{x-(-1)}{\cos 45^\circ} = \frac{y-2}{\sin 45^\circ} = r$ where r is the algebraic distance of (x, y) from $(-1, 2)$. Therefore A and C are given by

$\frac{x+1}{1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = \pm 3\sqrt{2}$ or A is $(2, 5)$ and C is $(-4, -1)$. Again, we may

write BD as $\frac{x+1}{-1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = r.$

B and D are given by $\frac{x+1}{-1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = \pm 3\sqrt{2}$

B is $(-4, 5)$ and D is $(2, -1)$.

The vertices of the square are $(2, 5)$, $(-4, 5)$, $(-4, -1)$ and $(2, -1)$.

EXERCISE 7.2

1. Find the slopes and the intercepts upon the axes of the following lines and reduce each to normal form.
   - (*i*) $3x - 4y + 12 = 0$
   - (*ii*) $12x + 5y = 39$
   - (*iii*) $15x - 8y + 34 = 0$
   - (*iv*) $11x + 60y = 61$
   - (*v*) $x - y = 8$.
2. Write the following straight lines in the parametric form
$$(x - x_1)/\cos\theta = (y - y_1)/\sin\theta = r.$$
   - (*i*) through $(2, 3)$ with slope 2
   - (*ii*) through $(1, 4)$ with slope $-1/3$
   - (*iii*) through $(1, 3)$ and $(4, 2)$.
3. Find the equations to the straight lines which join the origin and the points of trisection of the portion of the line $x + 3y - 12 = 0$ intercepted between the coordinate axes.
4. If $(x_1, y_1,)$ is the midpoint of the portion of a straight line intercepted between the coordinate axes, prove that the equation of the line is $x/2x_1 + y/2y_1 = 1$.
5. Find the equations of the straight lines through the origin whose intercepts between the lines $5x + 12y = 15$ and $5x + 12y = 30$ are each equal to three.
6. Show that the lines $3x + y + 4 = 0$, $3x + 4y - 15 = 0$ and $24x - 7y - 3 = 0$ form an isosceles triangle.
7. Find the area of the triangle formed by the lines $2x - y + 4 = 0$, $3x + 2y - 5 = 0$, $x + y + 1 = 0$.
8. Straight lines are drawn from A $(3, 2)$ to meet the line $6x + 7y = 30$ and these straight lines are bisected. Find the locus of the midpoints.
9. Find the area of the triangle formed by the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$.
10. Find the acute angle between $3x - 2y + 3 = 0$ and $2x + y - 5 = 0$.
11. Find the equations of the lines through $(2, 3)$ which make 45° with $3x - y + 5 = 0$.
12. If (h, k) is the foot of the perpendicular from (x_1, y_1) to the straight line $lx + my + n = 0$, show that $(h - x_1)/l = (k - y_1)/m$ $(lx_1 + my_1 + n)/(l^2 + m^2)$.
13. A vertex of an equilateral triangle is at $(2, 3)$ and the opposite side is $x + y = 2$. Find the equations to the other sides of the triangle.
14. A triangle is formed by the lines $ax + by + c = 0$, $lx + my + n = 0$, $px + qy + r = 0$. Show that $(ax + by + c) / (ap + bq) = (lx + my + n) / (lp + mq)$ passes through the orthocentre.
15. Show that the origin is within the triangle formed by the lines $4x + 7y + 19 = 0$, $4x + y - 11 = 0$ and $4x - 5y + 7 = 0$.
16. Find the in radius of the triangle formed by the lines $x = 0$, $y = 0$ and $x/3 + y/4 = 1$.
17. Show that the following pair of equations represents the same family of straight lines $2x + 3y - 8 + \lambda(4x - 7y + 10) = 0$ and $3x + 4y - 11 + \nu(2x - 5y + 8) = 0$.
18. Write down the equations of the bisectors of the angles between the lines.
   - (*i*) $x + 2y + 3 = 0$ and $2x - y - 5 = 0$
   - (*ii*) $4x + 3y + 10 = 0$ and $12x - 5y + 2 = 0$.
19. Prove that $x + y + 2 = 0$, $x - 7y = 2$, and $6x + 8y + 13 = 0$, $2y + 1 = 0$ have the same angular bisectors.
20. Find the directions in which a straight line must be drawn through $(1, 2)$ so that its point of intersection with $x + y = 4$ may be at a distance $(1/3)\sqrt{6}$ from the point.
21. Show that $2x - 3y + 5 = 0$, $3x + 4y - 7 = 0$ and $9x - 5y + 8 = 0$ are concurrent.

7.3 CIRCLES

The equation to the circle with centre origin and radius r is $x^2 + y^2 = r^2$. In fact any point $P(x, y)$ lying on the circle satisfies $OP^2 = r^2$ or $x^2 + y^2 = r^2$.

Conversely, if $x^2 + y^2 = r^2$ then (x, y) lies on the circle. If the centre is (a, b) instead of $(0, 0)$ then the equation to the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$. In general, the equation to any circle is of the form $x^2 + y^2 + 2\,gx + 2fy + c = 0$. We have already seen that the circle with centre (a, b) and radius r has the equation

$$(x - a)^2 + (y - b)^2 = r^2 \text{ or } x^2 + y^2 - 2\,ax - 2\,by + a^2 + b^2 - r^2 = 0$$

which is of the above mentioned form. Conversely, consider the set of points (x, y) satisfying $x^2 + y^2 + 2\,gx + 2\,fy + c = 0$. We may write this equation in the form $(x + g)^2 + (y + f)^2 + (c - g^2 - f^2) = 0$ or $(x - g)^2 + (y + f)^2 = g^2 + f^2 - c$ which is the equation to the circle with centre $(-g, -f)$ and radius $\sqrt{(g^2 + f^2 - c)}$ whenever $g^2 + f^2 - c \geq 0$. Thus we have

Proposition 1. $x^2 + y^2 + 2\,gx + 2fy + c = 0$ represents a circle whenever $g^2 + f^2 - c = 0$ and any circle can be put in the form $x^2 + y^2 + 2\,gx + 2\,fy + c = 0$. In fact $(-g, -f)$ is the centre and $\sqrt{(g^2 + f^2 - c)}$ is the radius. Preposition 1 says that the general second degree equation $ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0$ is a circle if and only $a = b$ and $h = 0$.

Some immediate observations

1. A circle is a second degree curve.
2. In the general circle $x^2 + y^2 + 2\,gx + 2\,fy + c = 0$ there are three independent constants g, f and c. Any three independent conditions enable us to fix g, f, c and hence the circle. In particular any three non-collinear points determine a circle.
3. A straight line is given by a linear equation of the form $ax + by + d = 0$ and a circle has the second degree equation $x^2 + y^2 + 2gx + 2fy + c = 0$. Therefore if we solve $ax + by + d = 0$ and $x^2 + y^2 + 2gx + 2fy + c = 0$ we have (*i*) two distinct points of intersection or (*ii*) two coincident points of intersection or (*iii*) two imaginary points of intersection. In other words a straight line either cuts a circle at two distinct points or touches a circle at two coincident points or never meets the circle at all. When the two points of intersection are coincident, the straight line is a tangent to the circle.
4. Two circles $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ intersect in general at two points. The points of intersection satisfy both $S_1 = 0$, $S_2 = 0$ and hence

$$S_1 - S_2 = 2\,(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0.$$

This is a straight line which becomes the common chord when the two circles intersect.

Note In general two quadratic curves

$$a_1x^2 + 2\,h_1xy + b_1y^2 + 2\,g_1x + 2f_1y + c_1 = 0 \text{ and}$$
$$a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$$

have four points in common ! (as seen in algebra).

5. A point $P(x_1, y_1,)$ lies inside circle, on the circle, outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1, + c_1 < 0$, $S_1 = 0$ or $S_1 > 0$. In particular, the origin lies within $x^2 + y^2 + 2gx + 2fy + c = 0$ if and only if $c < 0$.

Proposition 2. The length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)}$.

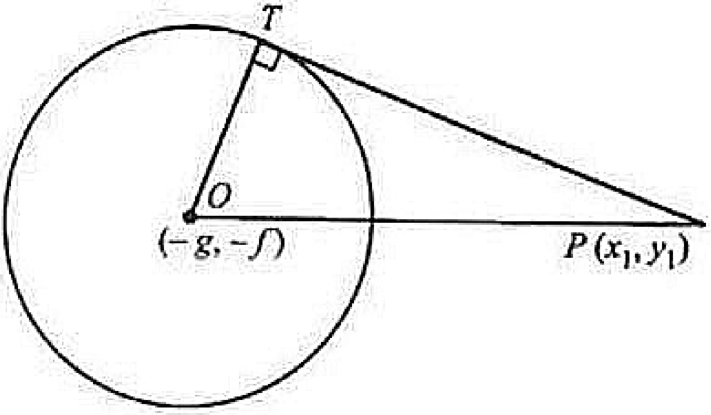


Fig. 7.46

Proof. Let P be (x_1, y_1) and PT be a tangent from P to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then ΔOTP is a right angled triangle and $PT^2 = OP^2 - OT^2 = OP^2 - (\text{radius})^2$

$$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$ □

Definition. The *power* of $P(x_1, y_1)$ with respect to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $OP^2 - r^2$ i.e., $x_1^2 + y_1^2 + 2gx_1, + 2fy_1 + c$.

Proposition 3. The tangent at (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ has the equation $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Proof. The centre O has coordinates $(-g, -f)$ and hence the slope of the radius OP is $\frac{y_1 + f}{x_1 + g}$. For a circle, the tangent at P is perpendicular to the radius OP and hence the tangent at $P(x_1, y_1)$ has the slope $-\frac{x_1 + g}{y_1 + f}$.

Therefore, the equation to the tangent at P is

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

i.e., $$(y - y_1)(y_1 + f) + (x_1 + g)(x - x_1) = 0$$

i.e., $$xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1.$$

Adding $gx_1 + fy_1 + c$ we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = S_1 = 0$$

(where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$) since (x_1, y_1) lies on the circle.

Thus the tangent at $P(x_1, y_1)$ to $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$ □

Note. By definition the tangent at P is the limiting position of the chord QP when Q approaches P on the circle. Let $Q(x_2, y_2)$ be a neighbouring point to $P(x_1, y_1)$ on the circle. Then the chord QP has the equation

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}. \tag{1}$$

Also P, Q are points on the circle gives

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0=x_2^2+y_2^2+2gx_2+2fy_2+c$$

$$\therefore \quad (x_1^2-x_2^2)+(y_1^2-y_2^2)+2g(x_1-x_2)+2f(y_1-y_2)=0$$

or
$$(x_1-x_2)(x_1+x_2+2g)=-(y_1-y_2)(y_1+y_2+2f)$$

or
$$\frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2+2g}{y_1+y_2+2f} \tag{2}$$

This makes (1) as $y-y_1=-\dfrac{x_1+x_2+2g}{y_1+y_2+2f}(x-x_1)$ (3)

As $Q \to P$, $x_2 \to x_1$ and $y_2 \to y_1$ and the chord QP becomes the tangent at P. Therefore from (3) we get the tangent at P

as
$$y-y_1=-\frac{2x_1+2g}{2y_1+2f}(x-x_1)$$

i.e.,
$$y-y_1=-\frac{x_1+g}{y_1+f}(x-x_1).$$

As seen earlier, this may be rewritten as $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

Proposition 4. The straight line $y=mx+c$ is a tangent to the circle $x^2+y^2=a^2$ if and only if $c^2=a^2(1+m^2)$.

Proof. Solving $y=mx+c$ with $x^2+y^2=a^2$ algebraically, we get the quadratic equation $x^2+(mx+c)^2=a^2$ or $(1+m^2)x^2+2mcx+c^2-a^2=0$.

This quadratic equation has equal roots if and only if its discriminant is zero; in other words $m^2c^2=(1+m^2)(c^2-a^2)$ or $c^2=a^2(1+m^2)$. Hence $y=mx+c$ is a tangent to $x^2+y^2=a^2$ iff $c^2=a^2(1+m^2)$. ❑

Corollary The point of contact of the tangent $y=mx+c$ with the circle

$$x^2+y^2=a^2 \text{ is } \left(\frac{-a^2m}{c}, \frac{a^2}{c}\right).$$

Proof. As in the proof of the proposition the equal roots for x satisfy $(1+m^2)x^2+2mcx+c^2-a^2=0$ with $c^2=a^2(1+m^2)$.

$$\therefore \quad x=\frac{-mc}{1+m^2}=\frac{-a^2m}{c}.$$

This gives the point of contact as $\left(\dfrac{-a^2m}{c}, \dfrac{a^2}{c}\right)$

$$=\left(\frac{-am}{\sqrt{(1+m^2)}}, \frac{a}{\sqrt{(1+m^2)}}\right) \text{ or } \left(\frac{am}{\sqrt{(1+m^2)}}, \frac{-a}{\sqrt{(1+m^2)}}\right)$$

according as $c=\pm a\sqrt{(1+m^2)}$ ❑

Note. For a circle, a straight line is a tangent iff the perpendicular distance of the centre from the line is equal to the radius. Thus $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$ iff

$$\frac{c^2}{1+m^2} = a^2 \text{ or } c^2 = a^2(1+m^2).$$

Proposition 5. From a given point P outside a circle S two tangents can be drawn to the circle S.

Proof. We may take S to be $x^2 + y^2 = a^2$ and P to be (x_1, y_1). Any tangent to S is of the form $y = mx \pm a\sqrt{(1+m^2)}$. If it passes through (x_1, y_1) then we have $y_1 = mx_1 \pm a\sqrt{(1+m^2)}$. This gives

$$(y_1 - mx_1)^2 = a^2(1+m^2).$$

or $$m^2(x_1^2 - a^2) - 2x_1y_1m + y_1^2 - a^2 = 0.$$

This is a quadratic in m with discriminant

$$4(x_1^2y_1^2 - (x_1^2 - a^2)(y_1^2 - a^2)) = 4(a^2(x_1^2 + y_1^2) - a^4)$$
$$= 4a^2(x_1^2 + y_1^2 - a^2) > 0$$

since $P(x_1, y_1)$ is outside the circle.

∴ It has two distinct roots giving rise to two tangents from P to the circle. ❑

Proposition 6. The equation to the chord of contact of tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point outside it

is $$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Proof. Let A (x_2, y_2) and B (x_3, y_3) be the points of contact of the tangents from P (x_1, y_1) to the given circle. Then the tangent at A has the equation $xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0$. The tangent at B (x_3, y_3) has the equation $xx_3 + yy_3 + g(x + x_3) + f(y + y_3) + c = 0$. Now (x_1, y_1) lies on both these tangents. Hence, we have

(*) $$x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$$

and $$x_1x_3 + y_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0.$$

But (*) implies that (x_2, y_2) and (x_3, y_3) lie on the straight line

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Thus the equation to the chord of contact of tangents from P (x_1, y_1) to the circle is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$. ❑

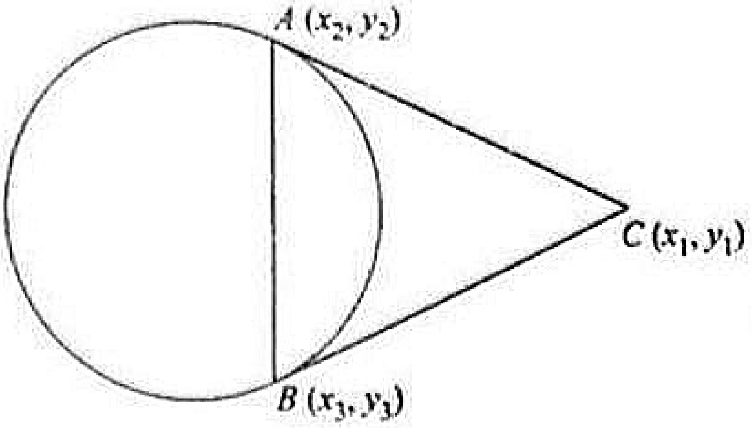


Fig. 7.47

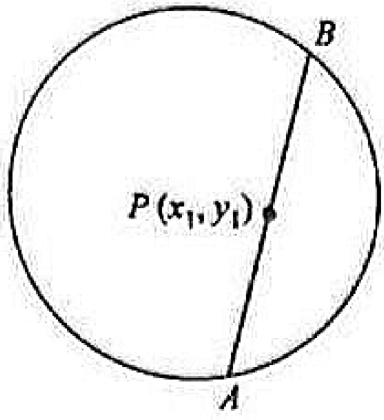


Fig. 7.48

Solving these three equations for g, f and c we get

$$g = -1/3, f = -10/3 \text{ and } c = 17/3$$

Hence the circumcircle has the equation $x^2 + y^2 - 2x - 20/3y + 17/3 = 0$

or $\quad 3x^2 + 3y^2 - 2x - 20y + 17 = 0$

The circumcentre is $(1/3, 10/3)$ and the radius is $\sqrt{(1/9) + (100/9) - (17/3)} = 5\sqrt{2}/3$.

EXAMPLE 2. *Find the equation to the circumcircle of the triangle whose sides are* $x + y = 1$, $x - 2y + 8 = 0$ *and* $2x - y + 1 = 0$.

SOLUTION. Consider the locus $(x + y - 1)(x - 2y + 8) + k(x - 2y + 8)(2x - y + 1) + l(2x - y + 1)(x + y - 1) = 0$. Clearly, this curve passes through the three vertices of the given triangle, for all values of k and l. We now choose k and l such that the above curve is a circle. This forces coefficient of $x^2 = 1 + 2k + 2l = -2 + 2k - l =$ coefficent of y^2; and

coefficient of $xy = (-2 + 1) + (-1 - 4)k + (2 - 1)l = -1 - 5k + l = 0$

Thus k, l should satisfy $l = -1$ and $k = -2/5$. This gives the circumcircle as

$$(x + y - 1)(x - 2y + 8) - 2/5(x - 2y + 8)(2x - y + 1)$$
$$- (2x - y + 1)(x + y - 1) = 0$$

Simplifying we get

$$3x^2 + 3y^2 - 2x - 20y + 17 = 0.$$

EXAMPLE 3. *Find the equation of the nine-point circle of the triangle whose vertices are* $A(2,4)$, $B(4, 6)$ *and* $C(6, 6)$.

Let A', B', C' be the midpoints of BC, CA, AB respectively. Then A' is $(5, 6)$, B' is $(4, 5)$ and C' is $(3, 5)$. Let the nine-point circle of ΔABC be $x^2 + y^2 + 2gx + 2fy + c = 0$ Then A', B', C' lie on this circle gives

$$10g + 12f + c = 61 \qquad (1)$$

$$8g + 10f + c = 41 \qquad (2)$$

$$6g + 10f + c = 34 \qquad (3)$$

Solving (1), (2) and (3) we get $g = -7/2, f = -10$ and $c = 94$.

Therefore the nine-point circle of ΔABC is $x^2 + y^2 - 7x - 20y + 94 = 0$.

EXAMPLE 4. *A circle is drawn with its centre at* $(-1, 1)$ *touching* $x^2 + y^2 - 4x + 6y - 3 = 0$ *externally. Prove that it touches both the axes.*

SOLUTION. Let r be the radius of the circle drawn with centre $(-1, 1)$. Since this circle touches $x^2 + y^2 - 4x - 6y - 3 = 0$, whose centre is $(2, -3)$ and whose radius is $\sqrt{(4 + 9 + 3)} = 4$ externally we must have distance betweent $(2, -3)$ and $(-1, 1)$ $= r + 4$.

This gives $\sqrt{(9 + 16)} = 5 = r + 4$ or $r = 1$. Hence the second circle drawn with centre $(-1, 1)$ is $(x + 1)^2 + (y - 1)^2 = 1$ which touches both the axes (see Fig. 7.50)

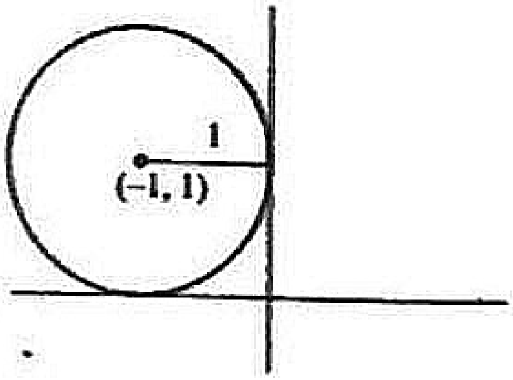


Fig. 7.50

Note. Any circle touching both the axes is of the form

$$(x \pm a)^2 + (y \pm a)^2 = a^2$$

or

$$(x \mp a)^2 + (y \pm a)^2 = a^2$$

EXAMPLE 5. *A and B are two fixed points. P is a moving point such that PA = nPB. Find the locus of P.*

SOLUTION. Let us take A to be (0, 0) and B $(a, 0)$. Then

$$\frac{PA^2}{PB^2} = n^2 \text{ gives}$$

$$\frac{x^2 + y^2}{(x-a)^2 + y^2} = n^2 \text{ where } P \text{ is } (x, y),$$

Therefore the locus of P is

$$x^2 + y^2 = n^2[(x-a)^2 + y^2]$$

or

$$(n^2 - 1)x^2 + (n^2 - 1)\, y^2 - 2n^2ax + n^2a^2 = 0$$

which is a circle.

EXAMPLE 6. *Show that the circle on the intercept of the line $lx + my = 1$ with $ax^2 + 2hxy + by^2 = 0$ as diameter is*

$$(am^2 - 2hlm + bl^2)(x^2 + y^2) + 2x(hm - bl) + 2y(hl - am) + a + b = 0$$

SOLUTION. $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines $y = m_1x$, $y = m_2x$ through the origin got by factorising

$$\frac{y^2}{x^2} + \frac{2h}{b}\frac{y}{x} + \frac{a}{b} \text{ as } \left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right).$$

Therefore we have $m_1 + m_2 = \dfrac{-2h}{b}$ and $m_1m_2 = \dfrac{a}{b}$ (1)

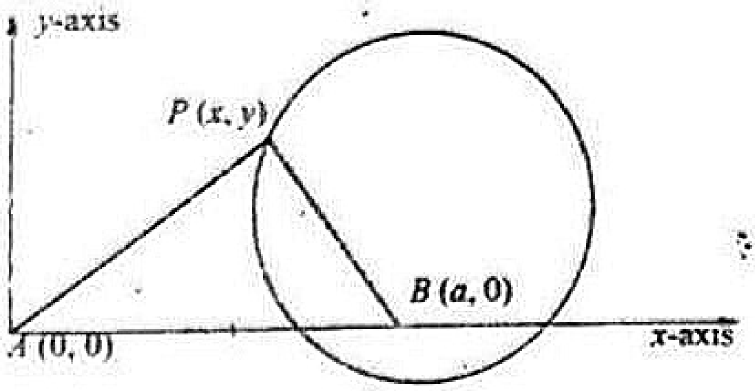


Fig. 7.51

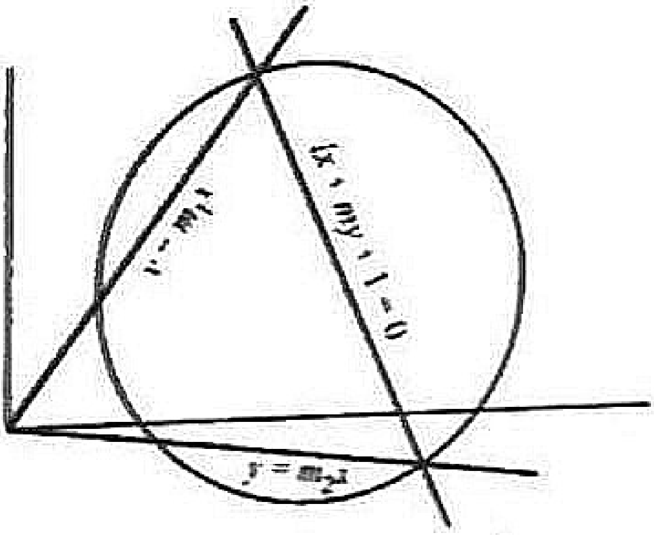


Fig. 7.52

Let OA be $y = m_1x$ and OB be $y = m_2x$. Then $A(x_1; y_1)$ and $B(x_2, y_2)$ satisfy

$$lx_1 + mm_1x_1 = 1 \text{ or } x_1 = \frac{1}{l + mm_1}$$

$$y_1 = \frac{m_1}{l + mm_1}.$$

Similarly $$x_2 = \frac{1}{l + mm_2} \text{ and } y_2 = \frac{m_2}{l + mm_2}$$

The equation to the circle on AB as diameter is

$$(x - x_1)(x_1 - x_2) + (y - y_1)(y - y_2) = 0 \text{ or}$$

$$x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0 \qquad (*)$$

We have $$x_1 + x_2 = \frac{1}{l + mm_1} + \frac{1}{(l + mm_2)} = \frac{2l + m(m_1 + m_2)}{l^2 + lm(m_1 + m_2) + m^2m_1m_2}$$

$$= \frac{2l + m(-2h/b)}{l^2 + lm(-2h/b) + m^2a/b} = \frac{2bl - 2hm}{bl^2 - 2hlm + am^2} \qquad (**)$$

Similarly $$y_1 + y_2 = \frac{-2hl + 2ma}{bl^2 - 2hlm + am^2}$$

Also $$x_1x_2 = \frac{1}{(l + mm_1)(l + mm_2)}$$

$$= \frac{b}{am^2 - 2hlm + bl^2} \qquad \text{from } (**)$$

Similarly. $$y_1y_2 = \frac{m_1m_2}{(l + mm_1)(l + mm_2)} = \frac{a}{am^2 - 2hlm + bl^2}$$

Substituting in (*) we get the required circle as

$$x^2 + y^2 - \frac{2(bl - hm)}{bl^2 - 2hlm + am^2}x - \frac{2(am - hl)}{bl^2 - 2hlm + am^2}y + \frac{a + b}{bl^2 - 2hlm + am^2} = 0$$

or $$(am^2 - 2\,hlm + bl^2)(x^2 + y^2) + 2\,(hm - bl)\,x + 2(hl - am)y + a + b = 0.$$

Note. If $ax^2 + 2hxy + by^2 = 0$ represents a pair of perpendicular lines, then the above circle passes through their point of intersection, namely (0, 0) (See Fig. 7.52 and extrapolate); and hence $a + b = 0$. This is the condition that $ax^2 + 2hxy + by^2 = 0$ may be two perpendicular lines.

EXAMPLE 7. *Find the circumcentre of the triangle formed by* $x + y = 0$, $x - y = 0$ *and* $lx + my - 1$. *If* l *and* m *vary such that* $l^2 + m^2 = 1$, *show that the locus of its circumcentre is the curve* $(x^2 - y^2)^2 = x^2 + y^2$.

SOLUTION. Solving the straight lines $x - y = 0, x + y = 0, lx + my = 1$ two by two we get the vertices as

$$A(0, 0),\ B\left(\frac{1}{l+m}, \frac{1}{l+m}\right) \text{and } C\left(\frac{1}{l-m}, \frac{1}{m-l}\right)$$

If (h, k) is the circumcentre then $SA^2 = SB^2 = SC^2$ and hence

$$h^2 + k^2 = \left(h - \frac{1}{l+m}\right)^2 + \left(k - \frac{1}{l+m}\right)^2$$

$$= \left(h - \frac{1}{l-m}\right)^2 + \left(k + \frac{1}{l-m}\right)^2$$

$$\frac{-2h}{l+m} - \frac{2k}{l+m} + \frac{2}{(l+m)^2} = 0 \text{ or } h + k = \frac{1}{l+m} \tag{1}$$

$$\frac{-2h}{l-m} + \frac{2k}{l-m} + \frac{2}{(l-m)^2} = 0 \text{ or } h - k = \frac{1}{l-m} \tag{2}$$

If l, m vary such that $l^2 + m^2 = 1$, then the locus of the circumcentre is got from

$$h^2 + k^2 = \frac{l^2 + m^2}{(l^2 - m^2)^2} = \frac{1}{(l^2 - m^2)^2} = (h^2 - k^2)^2.$$

Hence the locus of the circumcentre is $(x^2 - y^2) = (x^2 + y^2)$.

EXAMPLE 8. *Find the locus of the midpoints of chords of* $x^2 + y^2 = a^2$ *subtending a right angle at the point* (h, k).

SOLUTION. Let PQ be a chord of $x^2 + y^2 = a^2$ subtending 90° at $A(h, k)$. Let $R(x, y)$ be the midpoint of PQ. We are interested in finding the locus of R. The equation to PQ may be written as

$$xx_1 + yy_1 = x_1^2 + y_1^2 \text{ (Proposition 7)}$$

Suppose P is (x_2, y_2) and Q is (x_3, y_3). The circle on PQ as diameter has the equation $(x - x_2)(x - x_3) + (y - y_2)(y - y_3) = 0$ Now (h, k) lies on it gives $(h - x_2)(h - x_3) + (k - y_2)(k - y_3) = 0$

i.e., $$h^2 - (x_2 + x_3)h + x_2x_3 + k^2 - (y_2 + y_3)k + y_2y_3 = 0 \tag{1}$$

P and Q are the points of intersection of $xx_1 + yy_1 = x_1^2 + y_1^2 = r^2$ (say) with $x^2 + y^2 = a^2$. This gives $x^2 - 2xx_1 + (r^2 - a^2y_1^2/r^2) = 0$ as a quadratic having (x_2, x_3) as roots. Therefore $x_2 + x_3 = -2x_1$ and $x_2x_3 = r^2 - a^2 y_1^2$. By symmetry $y_2 + y_3 = -2y_1$, $y_2y_3 = r^2 - a^2x_1/r^2$. Substituting in (1) we get $h^2 - 2x_1 h + (r^2 - a^2 y_1^2/r^2) + k^2 - 2y_1k + r^2 - a^2 x_1^2/r^2) = 0$. Simplifying.

$$r^2(h^2+k^2) - 2r^2(x_1h + y_1k) + 2r^2 - a^2r^2 = 0 \text{ (since } x_1^2 + y_1^2 = r^2).$$

$$\therefore \quad 2(x_1^2 + y_1^2) - 2(hx_1 + ky_1) + h^2 + k^2 - a^2 = 0.$$

or the locus of (x_1, y_1) is $2(x^2 + y^2) - 2(hx + ky) + h^2 + k^2 - a^2 = 0$.

EXAMPLE 9. *Find the equation of the common tangents to the circles $x^2 + y^2 + 4x + 2y - 4 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$.*

SOLUTION. The circle $S_1 = x^2 + y^2 + 4x + 2y - 4 = 0$ has $A(-2, -1)$ as its centre and has radius 3. The other circle S_2 has $B(2,1)$ as its centre and has radius 1. The distance between the centres $= AB = 2\sqrt{5}$ > the sum of their radii which is 4. Hence we have four common tangents.

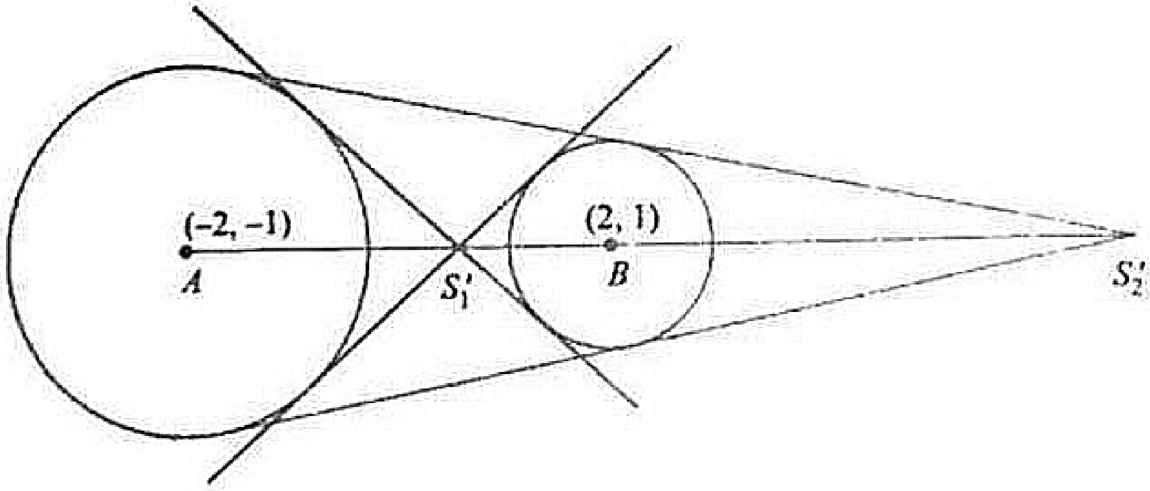


Fig. 7.53

The centre of similitude S'_1, S'_2 divide the line segment in the ratio of their radii namely 3 : 1 internally and externally. Hence

$$S'_1 \text{ is } (1, 1/2) \text{ and } S'_2 \text{ is } (4, 2)$$

Any straight line through S'_1 is of the form $y - 1/2 = m(x - 1)$ or $2mx - 2y - 2m + 1 = 0$. If this were to be a tangent to the circle S_1

then
$$\frac{2m(-2) - 2(-1) - 2m + 1}{\sqrt{(4m^2 + 4)}} = \frac{-6m + 3}{2\sqrt{(m^2 + 1)}} = 3$$

or $4(m^2 + 1) = (1 - 2m^2)$ or $0m^2 + 4m + 3 = 0$. Therefore $m_1 = \infty$ and $m^2 = -3/4$ are the roots. Hence the transverse common tangents are $x = 1$ and $3x + 4y - 5 = 0$.

Again any straight line through S'_2 (4, 2) is of the form

$$y - 2 = m(x - 4) \text{ or } mx - y - 4m + 2 = 0$$

If this were to be a tangent to S_1 then

$$\frac{m(-2) - (-1) - 4m + 2}{\sqrt{(m^2 + 1)}} = \frac{-6m + 3}{\sqrt{(1 + m^2)}} = \frac{2m + 1}{\sqrt{1 + m^2}} = 1$$

or $1 + m^2 = (-2m + 1)^2$ which gives $3m^2 - 4m = 0$. Its roots are $m = 0$, $m = 4/3$. Hence the direct common tangents are $y - 2 = 0$ and $3x - 4y + 10 = 0$.

EXAMPLE 10. *The circle $x^2 + y^2 = a^2$ is given by the parametric equation $x = a\cos\theta$, $y = a\sin\theta$. Find the equation to the chord joining 'θ' and 'ϕ' on the circle $x^2 + y^2 = a^2$.*

SOLUTION. The point 'θ' is $(a\cos\theta, a\sin\theta)$ and 'ϕ' is $(a\cos\phi, a\sin\phi)$. Therefore the equation to the chord joining 'θ' and 'ϕ' is

$$\frac{x - a\cos\theta}{a\cos\theta - a\cos\phi} = \frac{y - a\sin\theta}{a\sin\theta - a\sin\phi}$$

or
$$\frac{x - a\cos\theta}{2\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}} = \frac{y - a\sin\theta}{2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}} \quad \text{(Note that } \theta \neq \phi + 2k\pi\text{)}$$

Therefore
$$\frac{x - a\cos\theta}{\sin\frac{\theta+\phi}{2}} = -\frac{y - a\sin\theta}{\cos\frac{\theta+\phi}{2}}$$

or
$$x\cos\frac{\theta+\phi}{2} + y\sin\frac{\theta+\phi}{2} = a\left(\cos\theta\cos\frac{\theta+\phi}{2} + \sin\theta\sin\frac{\theta+\phi}{2}\right)$$

or
$$x\cos\frac{\theta+\phi}{2} + y\sin\frac{\theta+\phi}{2} = a\cos\frac{\theta-\phi}{2}$$

As a corollary we note that the tangent at 'θ' to the circle $x^2 + y^2 = a^2$ is $x\cos\theta + y\sin\theta = a$ (obtained by putting $\theta = \phi$ in the chord equation).

EXERCISE 7.3

1. Find the equation of the circle passing through (0, 1), (2, 3) and (–2, 5).
2. Find the equation of the circle with centre (2, 3) and touching the line $3x + 4y = 5$.
3. Two rods whose lengths are a and b slide along two perpendicular axes in such a way that their extremities are always concyclic. Find the locus of the centre of the circle.
4. Show that the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ touches the coordinate axes.
5. Show that $x^2 + y^2 = 400$ and $x^2 + y^2 - 10x - 24y + 120 = 0$ touch one another internally. Find the coordinates of the point of contact.
6. Find the locus of the centre of a circle which touches $x\cos\alpha + y\sin\alpha = p$ and the circle $(x - a)^2 + (y - b)^2 = c^2$.
7. The lines $lx + my + n = 0$ intersects the curve $ax^2 + 2hxy + by^2 = l$ at P, Q which lie at finite distances from (0, 0). The circle on PQ as diameter passes through (0, 0). Show that $n^2(a + b) = l^2 + m^2$.
8. Find the points on $x - y + 1 = 0$, the tangents from which to the circle $x^2 + y^2 - 3x = 0$ are of length 2.
9. Find the locus of a point the tangents from which to the circle $4x^2 + 4y^2 - 9 = 0$ and $9x^2 + 9y^2 - 16 = 0$ are in the ratio 3:4.
10. Find the equation of a line inclined at 45° to the axis of x, such that $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y - 15 = 0$ cut off equal lengths on it.
11. If $x\cos\alpha + y\sin\alpha = p$ touches $(x - a)^2 + (y - b)^2 = c^2$, then prove that $a\cos\alpha + b\sin\alpha - p = \pm c$.
12. Show that the locus of the feet of the perpendiculars drawn from the point $(a, 0)$ on tangents to the circle $x^2 + y^2 = a^2$ is $(x^2 + y^2 - ax)^2 = y^2 + (x - a)^2$.
13. Show that the locus of the midpoints of the chords of contact of tangents drawn to a given circle from points on another given circle is a third circle.
14. Find the equation of the common tangents to the circles
$$x^2 + y^2 - 22x + 4y + 100 = 0 \text{ and } x^2 + y^2 + 22x - 4y - 100 = 0.$$
15. Show that the tangents to the circle $x^2 + y^2 = 25$ which pass through (– 1, 7) are at right angles.

35. Let f be the family of circles passing through $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ cuts members of the given family are concurrent. Find the point of concurrence.

36. OX, OY are two coordinate axes inclined at an angle ω. If a straight line l cuts OY at $(0, c)$ and makes angle θ with the positive OX axis, prove that the equation to l with respect to the axes OX, OY is $y = mx + c$ where $m = \sin\theta/\sin((\omega - \theta) = \tan\theta/(\sin\omega - \cos\omega\tan\theta))$.

37. With notations as in problem 36, prove that the equation to the line at a perpendicular distance p from the origin and making angles α, β with OX, OY respectively is $x\cos\alpha + y\cos\beta = p$.

38. With notations as in problems 36 and 37 prove that the angle between $y = m_1x + c_1$ and $y = m_2x + c_2$ is

$$\tan^{-1}\frac{(m_1 - m_2)\sin\omega}{1 + (m_1 + m_2)\cos\omega + m_1m_2}$$

39. With axes as in the previous problem, if A, B are the points (x_1, y_1), (x_2, y_2) respectively prove that the equation to AB is $y - y_1 = \{(y_1 - y_2)/(x_1 - x_2)\}\,(x - x_1)$.

40. $ABCD$ is a quadrilateral such that the sides AB, DC meet at E and BC, AD meet at F. If L, M, N are the midpoints of AC, BD, EF prove that L, M, N are collinear. (Hint: take AB, AD as the oblique coordinate axes).

41. Prove that the area of the triangle formed by the straight lines $x\cos\alpha_i + y\sin\alpha_i = p_i$ $i = 1, 2, 3$ is $1/2\ \{\Sigma p_1 \sin(\alpha_3 - \alpha_2)/(\sin(\alpha_3 - \alpha_2)\sin(\alpha_1 - \alpha_3)\sin(\alpha_2 - \alpha_1))\}$.

42. Show that the orthocentre of the triangle formed by the lines $y = m_ix + a/m_i$ is $(-a, a(1/m_1 + 1/m_2 + 1/m_3 + 1/m_1m_2m_3))$.

43. In ΔOAB a straight line parallel to AB meets OB, OA at X, Y respectively; find the locus of the point of intersection of AX and BY.

CHAPTER

8

SYSTEMS OF LINEAR EQUATIONS

In Chapter 6 Section 11, we discussed the topic of Elimination. This was the process by which, from a system of equations, we eliminated one, two or three parameters and obtained a relation (equation) among the remaining parameters or variables. The process stopped there. In this Chapter we shall take a system of equations (all *linear* in the sense to be explained later) but now arrive at a **complete** solution for the variables involved.

8.1 TWO AND THREE UNKNOWNS

Suppose we had a simple linear equation in one unknown, such as

$$7x - 22 = 0 \tag{1}$$

We can easily solve this for x. We have only to keep the term containing x on the left hand side of the equation and take the other terms to the right side. Thus (1) leads to

$$7x = 22$$

i.e., $$x = 22/7.$$

Let us take another example. Consider

$$3(x + 5) - 2a = 0 \tag{2}$$

where x is unknown and a is a known quantity. We are asked to solve for x. Again we keep the term containing x on the left side and take all the other terms to the right side, including the term in a. We get

$$3x = 2a - 15$$

$$x = 2/3a - 5$$

thus giving the unknown x in terms of the known a. Thus the problem of a simple linear equation in one unknown is completely solved.

In this chapter we shall consider linear equations in more than one unknown-particularly, in two and three unknowns and learn how to solve them. Consider the equation

$$2x + 3y = 8 \tag{3}$$

where x and y are both unknowns and have to be solved for. The best that we can do now is to imitate what we did with (2) earlier. Keep either x or y on the left side and take the other variable to the right side. Let us keep x on the left hand side and take y to the right, thus:

$$2x = -3y + 8$$

35. Let f be the family of circles passing through $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ cuts members of the given family are concurrent. Find the point of concurrence.

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where x and y are both unknowns and have to be solved for. The best that we can do now is to imitate what we did with (2) earlier. Keep either x or y on the left side and take the other variable to the right side. Let us keep x on the left hand side and take y to the right, thus:

$$2x = -3y + 8$$

which gives
$$x = -3/2\, y + 4 \tag{4}$$
thus giving x in terms of y.

In the same way if we had kept y on the left side and taken x to the right, we would get
$$3y = -2x + 8$$
i.e.,
$$y = -2/3\, x + 8/3 \tag{5}$$
which gives y in terms of x.

Note that neither (4) nor (5) gives a complete solution for the unknowns. They only express one unknown in terms of the other unknown. But in the circumstances of the problem, this is the best that we can do. In other words, whenever there are two unknowns and only one linear equation is given relating them, the best that we can do is to express one in terms of the other.

Now let us consider two equations in two unknowns. Let us start with
$$2x + 3y = 8 \tag{6}$$
$$5x - y = 3 \tag{7}$$
We want to solve these equations for x and y. These two equations are said to constitute a Simultaneous Linear System of equations. Solving a linear system implies the simultaneity of the validity of the solution for all the equations of the system. It is called a *Linear System* because there is no term involving x or y in the second or higher degrees. Recall the definition of 'linear function' from section 5.6.

Let us now imitate what we did in the case of one equation, viz., (3) with two unknowns. There we kept one of the unknowns on one side of the equation and took the other to the other side. Let us do the same here with each of the two equations.

From (6) we get
$$2x = -3y + 8$$
$$x = -3/2\, y + 4 \tag{8}$$
From (7) we get
$$5x = y + 3$$
$$x = 1/5\, y + 3/5 \tag{9}$$
Since stimultaneity means that both the equations (6) and (7) are to be satisfied by the same values of x and y, the two expressions for, x, *viz.*, (8) and (9) one of which comes from (6) and the other comes from (7) should be the same. Thus we should have
$$-3/2\, y + 4 = 1/5\, y + 3/5 \tag{10}$$
Now this is an equation in a single unknown. So we should be able to solve it by collecting on one side, all terms involving the unknown. Accordingly we have.
$$-3/2\, y - 1/5\, y = -4 + 3/5$$
we may multiply by 10 throughout, in order to get rid of all fractions. This gives us
$$-15y - 2y = -40 + 6$$
i.e.,
$$-17y = -34$$
i.e.,
$$y = 2$$
Thus one of the unknowns is resolved and more than half the battle is over. We have now only to substitute this value of y in one of the given equations, either (6), or (7). Doing so in (7) we get

$$5x - 2 = 3$$

i.e., $$5x = 5$$

i.e., $$x = 1$$

The complete solution of the system comprising of (6) and (7) is $x = 1$ and $y = 2$.

Going back over the method we see that there are two stages in the working. The first stage is to manipulate with the given equations and arrive at an equation involving only one of the unknowns. This process is usually called the 'elimination of one of the unknowns'. This was what we did when we arrived at (10) from (6) and (7). The second stage is to solve this single equation for its only one unknown and then use that value (by back-substitution in the original equation) for the solution of the second unknown. This second stage is comparatively easy. It is the first stage that requires some ingenuity. In order for the student to get the real hang of this ingenuity, we shall take the same problem as the one solved above and exhibit the steps of the 'elimination' process in a simpler manner. Start with (6) and (7) once again.

$$2x + 3y = 8 \tag{6}$$

$$5x - y = 3 \tag{7}$$

Let us work for the 'elimination' of x. The first equation has $2x$ in it and the second has $5x$. These coefficients 2 and 5 have an l.c.m. of 10. If we multiply all the terms in the first equation by 5, the term in x will become $10x$. If we mulitiply all the terms in the second equation by 2, the term in x will become $10x$. Then a subtraction of one equation from the other eliminates x. We record this as follows:

(6) × 5 gives $$10x + 15y = 40 \tag{11}$$

(7) × 2 gives $$10x - 2y = 6 \tag{12}$$

(11) – (12) gives $$0x + 17y = 34$$

i.e., $$y = 2$$

Note that (11) and (12) are nothing but the original (6) and (7) except that we have multiplied the equations by 'suitable' constants such that the x term appears in both with the same coefficient. Now it only remains to subtract one equation from the other in order to get rid of the x term. Having done this and having got the value of y, a back-substitution in one of the original equations resolves the other unknown.

Looking back we see that the only difference between this and the earlier method is that we have a more conveniently streamlined procedure. It is in fact a standard strategy which can be expressed as follows:

"Multiplying each of the equations by suitable nonzero constants and by subtraction or addition eliminate one of the unknowns".

We can test our understanding of this strategy by going back to the same equations (6) and (7) once again and this time eliminating y instead of x. We start again with (6) and (7).

$$2x + 3y = 8 \tag{6}$$

$$5x - y = 3$$

The coefficients of y in the two equations are 3 and –1. Their l.c.m. is –3. So we should attempt to get $-3y$ in both the equations. This means we multiply the first equation by (–1) and the second by 3. This and the further subtraction process is what is meant by the symbolism:

$$-1 \times (6) - 3 \times (7)$$

This symbolism means:

Multiply equation (6) by – 1, multiply equation (7) by – 3 and add the resulting two equations.

This process gives,

$$(2x) \times (-1) - (5x \times 3) = (8) \times (-1) - (3 \times 3)$$

i.e., $$-17x = -17$$

i.e., $$x = 1$$

And now, a back-substitution of this in either (6) or (7) gives the answer

$$y = 2$$

This completes the solution.

Whether we eliminate x first or y first, it is immaterial. Whatever strikes as convenient may be done. We shall now illustrate the entire procedure by a new example, with minimum commentary on the working.

EXAMPLE 1.

$$3x - 4y = 7 \qquad (1)$$

$$2x + 5y = 10 \qquad (2)$$

(1) × 2: $$6x - 8y = 14 \qquad (3)$$

(2) × 3: $$6x + 15y = 30 \qquad (4)$$

(4) – (3): $$23y = 16$$

i.e., $$y = 16/23$$

Substituting in (1) we get

$$3x - 64/23 = 7$$

i.e., $$3x = 7 + 64/23 = 225/23$$

$$x = 75/23$$

Thus the solution is:

$$x = \frac{75}{23} \text{ and } y = \frac{16}{23}$$

In the above working, we have eliminated x first and obtained the solution for y, which, in turn, led to the solution for x. Alternatively we could have eliminated y first, obtained the solution for x which, in turn, would have led to the solution for y. The following working shows this alternative:

(1) × 5: $$15x - 20y = 35 \qquad (3')$$

(2) × 4: $$8x + 20y = 40 \qquad (4')$$

(3′) – (4′): $$23x = 75$$

i.e., $$x = 75/23$$

Substituting in (1) we get

$$225/23 - 4y = 7$$

i.e., $$4y = 64/23$$

i.e., $$y = 16/23$$

thus giving the same set of solutions for x and y as before.

The beauty of the strategy and of the working is that it has been so streamlined that we can extend the same procedure to solve a system of three equations in three unknowns an illustration of which we shall take up now.

EXAMPLE 2. Solve the system:

$$2x + 3y - 7z = 1 \qquad (1)$$

$$x - y - z = 0 \qquad (2)$$

$$3x + 2y + 2z = 5 \qquad (3)$$

The strategy will now be to eliminate one of the unknowns and arrive at two equations in the remaining two unknowns. Thereafter the procedure will be as in Example 1.

At this point we shall give the student an important advice. It is that we should follow a certain discipline in the recording of the steps in the working. This discipline will be appreciated by the student when he moves to the higher classes and begins to understand the deep mathematics that lies underneath such problems of solution of simmultaneous linear equations.

The three equations (1), (2) and (3) form a single system. Each time we meddle with one of them, say (1), multiply it by a constant, we arrive at a new equation [that may be called (1′)]. But actually what we have is a new system of three equations *viz.*, (1′) and the old (2) and (3). This 'new' system is not really new. It is said to be 'equivalent' to the old. Any solution of the new system

(1′), (2), (3)

is also a solution of the old system

(1), (2), (3) and vice versa.

The mathematics which proves this is not difficult, but we shall skip it now. **We only note that every time we get a new (equivalent) system in such a manner, we shall discipline ourselves to write all the equations of the new system together in one bunch — even though some of the equations are the same as the original.**

Secondly **we shall call the equations E1, E2 and E3 instead of (1), (2) and (3).** This is again, in anticipation of a convenience which we will need at a higher level.

Thirdly, since we are going to keep on meddling with the equations of the system several times, we shall not add to the numbering by writing *E*4, *E*5 and so on. **Every time the new system is written in full, we shall call the equations of the system again by the same names E1, E2 and E3.** The student will understand this when he sees the working below:

Now for Example 2.

$$2x + 3y - 7z = 1 \qquad E\,1$$

$$x - y - z = 0 \qquad E\,2$$

$$3x + 2y + 2z = 5 \qquad E\,3$$

Interchange of *E*2 with *E*1 gives an equivalent new system: (There is a convenience in keeping 1*x* at the top left corner)

$$x - y - z = 0 \qquad E\,1$$

$$2x + 3y - 7z = 1 \qquad E\,2$$

$$3x + 2y + 2z = 5 \qquad E\,3$$

*E*2 − 2 × *E*1 keeps the first equation unaltered and changes *E*2 as:

$$0x + 5y - 5z = 1$$

*E*3 − 3 × *E*1 keeps the first equation unaltered and changes *E*3 as

$$0x + 5y + 5z = 5$$

$E2 - 2E1$ gives

$$2x - 3y = 8 \qquad E1$$
$$0x + 0y = -1 \qquad E2$$

The last one contains an equation ($E2$) saying $0 = -1$ which is false. Whenever our process produces a system which contains a false equation like this, we conclude that the given system of equations is inconsistent. This means there cannot be any solution to the problem. Or in other words the two equations of the system cannot simultaneously hold.

EXAMPLE 6.

$$2x - 3y = 8 \qquad E1$$
$$4x - 6y = 16 \qquad E2$$

$E2 - 2E1$ gives

$$2x - 3y = 8 \qquad E1$$
$$0x + 0y = 0 \qquad E2$$

Here the 2nd equation is actually an identity. $0 = 0$ so it does not contribute anything to the solution of the problem. So we are left with the only equation

$$2x - 3y = 8$$

In this case we repeat what we did earlier (see page 316) in a similar situation; viz., we keep one unknown on one side and express it in terms of the other. Thus we get

$$2x = 3y + 8$$

i.e., $$x = 3/2y + 4 \qquad (*)$$

Whatever value we give to y, we get a corresponding value of x from (*). For instance, we get

$$y = 1 \text{ gives } x = \frac{11}{2};$$
$$y = 0 \text{ gives } x = 4;$$
$$y = -1 \text{ gives } x = 5/2.$$

Each pair of values thus obtained for x and y becomes a solution to the given problem. In other words there exists an infinity of solutions for the problem.

The special situations that arose in Examples 5 and 6 with two equations in two unknowns may also arise in the case of three equations with three unknowns. But before we go to these examples we shall illustrate the situation by a very simple example, namely that of one equation with one unknown. The general type of such an equation is

$$ax = b \qquad (*)$$

where a and b are constants and x is the unknown. Three cases may arise:

Case 1. $a \neq 0$. In this case, the solution is $x = b/a$. This is the unique solution to the problem in this case.

Case 2. $a = 0$ and $b \neq 0$. Now $0 \times x = b \neq 0$. This is impossible. So x has no value satisfying this case. In other words, the equation is inconsistent; *i.e.*, it cannot hold for any value of x.

Case 3. $a = 0 = b$. In this case, we have to find x such that

$$0x = 0$$

But any value of x will satisfy this. In other words, this case has infinite number of solutions.

Thus in general, the solution to a system of equations.

(1) could exist and be unique;

(2) could exist, but there may be infinite number of solutions;

(3) could be non-existent; *i.e.*, the equations are inconsistent.

For a general system of linear equations one can predict beforehand, *i.e.*, without actually trying to obtain a solution, whether the problem falls under case 1 or 2 or 3. This is part of the objective of the branch of mathematics known as LINEAR ALGEBRA.

Now let us go to the situations of case 2 and case 3, in the problem of three equations in three unknowns.

EXAMPLE 7.

$$\begin{aligned} x + 2y - 3z &= 0 && E1 \\ 5x - 4y + 7z &= 1 && E2 \\ 2x - 3y + 5z &= 1 && E3 \end{aligned}$$

$E2 - 5E1$ and $E3 - 2E1$ give

$$\begin{aligned} x + 2y - 3z &= 0 && E1 \\ 0x - 14y + 22z &= 1 && E2 \\ 0x - 7y + 11z &= 1 && E3 \end{aligned}$$

$E2 - 2E3$ gives

$$\begin{aligned} x + 2y - 3z &= 0 && E1 \\ 0x + 0y + 0z &= -1 && E2 \\ 0x - 7y + 11z &= 1 && E3 \end{aligned}$$

The second equation of this last system is obviously false for any values of x, y, z. Hence the equations are inconsistent.

EXAMPLE 8.

$$\begin{aligned} 2x - y + 3z &= 5 && E1 \\ x + 2y + z &= 1 && E2 \\ 3x + y + 4z &= 6 && E3 \end{aligned}$$

Interchanging $E1$ and $E2$ we get

$$\begin{aligned} x + 2y + z &= 1 && E1 \\ 2x - y + 3z &= 5 && E2 \\ 3x + y + 4z &= 6 && E3 \end{aligned}$$

Note that a term like $1x$ at the top left hand corner enables u's to multiply $E1$ suitably and subtract the result from $E2$ and $E3$ and thus arrive at $0x$ in both $E2$ and $E3$. Thus $E2 - 2E1$ and $E3 - 3E1$ give

$$\begin{aligned} x + 2y + z &= 1 && E1 \\ 0x - 5y + z &= 3 && E2 \\ 0x - 5y + z &= 3 && E3 \end{aligned}$$

$E3 - E2$ gives

$$x + 2y + z = 1 \qquad E1$$
$$0x - 5y + z = 3 \qquad E2$$
$$0x + 0y + 0z = 0 \qquad E3$$

This shows that there are essentially only two equations in the system.

Now $-\frac{1}{5} \times E2$ gives

$$x + 2y + z = 1 \qquad E1$$
$$0x + y - 1/5z = -3/5 \qquad E2$$

$E1 - 2E2$ gives

$$x + 0y + \frac{7}{5}z = \frac{11}{5} \qquad E1$$

$$0x + y - \frac{1}{5}z = -3/5 \qquad E2$$

Rewriting this we have

$$x = -\frac{7}{5}z + \frac{11}{5}$$

$$y = \frac{1}{5}z - \frac{3}{5}.$$

Thus x and y are expressed in terms of z. Since essentially there are only two equations in the system, each particularised value of z, gives certain values to x and y and together these three form one solution. We can give an infinite number of values to z. So there are an infinite number of solutions to the system. For example, giving $z = 1$, we get

$$x = -7/5 + 11/5 = 4/5$$
$$y = 1/5 - 3/5 = -2/5$$

So $x = \frac{4}{5}$, $y = -\frac{2}{5}$, $z = 1$ is a solution. Again, giving $z = 0$,

$$x = \frac{11}{5}, y = -\frac{3}{5}$$

and this forms another solution. Thus the system admits an infinite number of solutions.

Cross-multiplication rule. This is a rule which gives the solution to the system of two equations in two unknowns whenever a unique solution exists. Let us consider the system

$$ax + by + c = 0 \qquad E1$$
$$a'x + b'y + c' = 0 \qquad E2$$

where a, b, c, a', b', c' are constants such that

$$ab' - a'b \neq 0$$

$b' \times E1 - b \times E2$ eliminates y, thus:

$(ab' - a'b)x + 0y + (cb' - c'b) = 0$ so that

$$x = \frac{bc' - b'c}{ab' - a'b}$$

Note that the denominator is not zero and that is why we are able to divide the earlier equation by $ab' - a'b$ throughout.

In the same way, to eliminate x we do the operation $a' \times E1 - a \times E2$.

This gives

$$0 \times x + (a'b - ab')y + (ca' - c'a) = 0$$

so that,

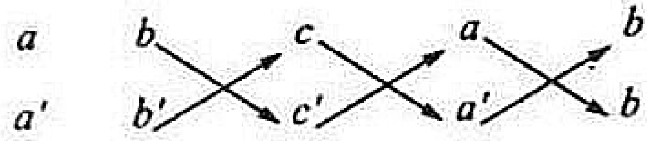
$$y = \frac{ca' - c'a}{ab' - a'b}$$

Thus the complete solution is

$$x = \frac{bc' - b'c}{ab' - a'b}, \qquad y = \frac{ca' - c'a}{ab' - a'b}$$

on the assumption $ab' - a'b \neq 0$.

As a mnemonic to remember this solution, one writes the coefficeints of the system as follows:



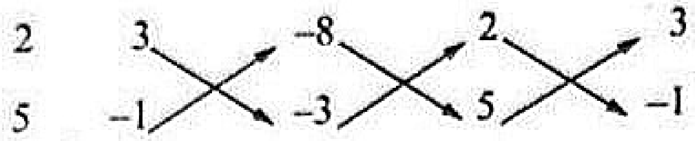
$$bc' - b'c \quad ca' - c'a \quad ab' - a'b \quad .$$

The first two expressions

$bc' - b'c$ and $ca' - c'a$ are numerators for the values of x and y respectively and the last one is the denominator for both. Thus

$$x = \frac{bc' - b'c}{ab' - a'b}; \ y = \frac{ca' - c'a}{ab' - a'b}$$

This formula, called the cross-multiplication rule, can be immediately applied to specific problems. For instance, applying it to equations (6) and (7) at the beginning of this Chapter, we get



$$x = \frac{-9-(8)}{-2-15}, \qquad y = \frac{-40+6}{-2-15}$$

$$= \frac{-17}{-17} = 1 \qquad = \frac{-34}{-17} = 2$$

$x = 1$, $y = 2$ is the solution, as was known already.

EXERCISE 8.1

Solve the following systems of linear equations in each case, by two methods viz., (*a*) elimination method of variables; (*b*) row reduction process:

1. $x - y = 5$
   $2x + y = 4$

2. $x - y = 5$
   $2x - 2y = 4$

3.
$$\begin{aligned} x-y&=5\\ 2x-2y&=10 \end{aligned}$$

4.
$$\begin{aligned} 2x+3y&=7\\ x+2y&=3 \end{aligned}$$

5.
$$\begin{aligned} x+y+z&=3\\ 2x+z&=4\\ 2y+z&=2 \end{aligned}$$

6.
$$\begin{aligned} x+2y-z-3w+u&=4\\ 2x+3y+z+2u&=10\\ y+2w+u&=5 \end{aligned}$$

7.
$$\begin{aligned} x+y-z&=2\\ 2x+z&=3\\ x-y+4z&=-1 \end{aligned}$$

8.
$$\begin{aligned} 3x+y+z&=1\\ 6x-y&=7\\ 6y+z&=5 \end{aligned}$$

9.
$$\begin{aligned} x+y+z&=1\\ 3x+3y+z&=5 \end{aligned}$$

10.
$$\begin{aligned} x+y+z&=1\\ y+z&=-1 \end{aligned}$$

11.
$$\begin{aligned} x+y-z&=2\\ -x+y+z&=4\\ x-y+z&=6 \end{aligned}$$

12.
$$\begin{aligned} x+y&=9\\ 2x+y&=3\\ x-y&=4 \end{aligned}$$

13.
$$\begin{aligned} x+y+z&=-4\\ 2x+2y+2z&=-8 \end{aligned}$$

14.
$$\begin{aligned} x+y+3z-w&=2\\ y+w&=5 \end{aligned}$$

15.
$$\begin{aligned} x-y+w+z&=10\\ y-z&=4\\ x+w&=14 \end{aligned}$$

16.
$$\begin{aligned} x-y+3z&=6\\ x+3y-3z&=-4\\ 5x+3y+3z&=10 \end{aligned}$$

17.
$$\begin{aligned} x+y+z+w&=2\\ x-2y+2z+2w&=6\\ 2x+y-2z+2w&=-5\\ 3x-y+3z-3w&=-3 \end{aligned}$$

8.2 INTRODUCTION TO DETERMINANTS AND MATRICES

Let us now consider the geometrical implications of the previous section. Let us go back to equations (6) and (7).

$$2x+3y=8 \tag{1}$$

$$5x-y=3 \tag{2}$$

On the two dimensional plane, these two equations each represent a line. To solve the two equations algebraically is just to find the point of intersection of these two lines. If the two lines are parallel the point of intersection will not exist as a finite point. If the two lines coincide, every point on one line may be taken as one of the several points of 'intersection' and so there are actually an infinite number of common points. This is what happens in the case of Example 6 (Sec 1)

$$\begin{aligned} 2x-3y&=8\\ 4x-6y&=16 \end{aligned}$$

Here there is only one line. So there are infinite number of solutions. In the case of Example 5,

$$\begin{aligned} 2x-3y&=8\\ 4x-6y&=15, \end{aligned}$$

it is easy to see that the two lines are parallel and distinct. So there is no point of intersection.

From the cross-multiplication rule that we enunciated for the case of two equations in two unknowns we see that, for a general system

$$ax + by + c = 0$$
$$a'x + b'y + c' = 0$$

three cases could arise.

Case 1. The two lines are distinct and not parallel. So there must exist a point of intersection. Being 'not parallel' is equivalent to saying

$$\frac{a}{a'} \neq \frac{b}{b'}$$

i.e., $ab' - a'b \neq 0$. We also saw earlier that a solution exists and is unique iff $ab' - a'b \neq 0$.

Case 2. The two lines are distinct and parallel. In this case there is no point of intersection. This case happens iff

$$\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$$

Actually this is part of the case complementary to case 1. In case 1 we had $ab' - a'b \neq 0$. In the present case we have $ab' - a'b = 0$ and in addition, we have $\frac{a}{a'} \neq \frac{c}{c'}$.

Case 3. This is the remaining part of case 2. We have

$ab' - a'b = 0$ and $\frac{a}{a'} = \frac{c}{c'}$. In fact we have $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

Now the two lines are coincident and so there are an infinite number of solutions.

We can tabulate the three cases as follows, for the system:

$$ax + by + c = 0$$
$$a'x + b'y + c' = 0$$

$ab' - a'b$	$bc' - b'c$	$ca' - c'a$	*Behaviour of Solution*
Non-zero	–	–	Solution exists and is unique
Zero	Non-zero	Non-zero	No solution (Equations are inconsistent)
Zero	Zero	Zero	Infinite number of solutions

The reader should convince himself that no other cases can arise. It is clear that the three quantities that decide the behaviour of the solution are

$ab' - a'b$, $bc' - b'c$ and $ca' - c'a$. Of these three, the behaviour of $ab' - a'b$ *i.e.*, whether it is zero or otherwise, is important for the existence or non-existence of a unique solution.

In fact from the above table we note the following. The system

$$ax + by + c = 0$$
$$a'x + b'y + c' = 0$$

has a unique solution iff

$$ab' - a'b \neq 0$$

The quantity $ab' - a'b$ is called the *Determinant* of the system. It has a special symbol for itself, *viz*.,

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$$

To arrive at this symbol, one looks at the system of equations, takes x, y terms in their proper positions and simply writes the coefficients

$$\begin{matrix} a & b \\ a' & b' \end{matrix}$$

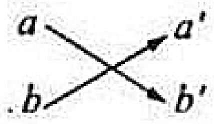
as they appear in the equations and draw two vertical lines on either side. The determinant is actually the symbol

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$$

and $ab' - a'b$ is usually, called the 'value of the determinant'. It is also customary to write

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = ab' - a'b.$$

One remembers the value $ab' - a'b$ by referring to the mnemonical diagram below, in particular, to the arrows in the diagram:



Thus we may state the following theroem.

Theorem The system

$$\begin{aligned} ax + by &= c \\ a'x + b'y &= c' \end{aligned} \qquad (*)$$

has a unique solution iff the determinant

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$$

is non-zero. If the determinant is zero, then the system

(*i*) has no solution if

$$\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}; \text{ and}$$

(*ii*) has an infinite number of solutions if

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$ □

Thus the determinant

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$$

plays a crucial role in predicting the behaviour of the system (*). In the case of three equations in three unknowns; say,

$$ax + by + cz = d$$
$$a'x + b'y + c'z = d'$$
$$a''x + b''y + c''z = d''$$

the behaviour of the solution similarly depends upon a quantity called the 'determinant' of the system, which is a function depending on the coefficients $a, b, c, a', b', c', a'', b'', c''$. But now this 'determinant' has to be defined. What we defined earlier, *viz.*,

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = ab' - a'b$$

is called a *second order determinant*. Before we define third order (and higher order) determinants, we prefer to make a useful digression by taking up the subject of Matrices.

A *matrix* is an array of symbols (which could be real or complex numbers) into rows and columns such that each row has the same number, n, of symbols and each column has the same number, m, of symbols. Thus it has m rows and n columns. Such a matrix is said to be of *size* $m \times n$. When $m = n$, the matrix is said to be a *square matrix*, of order n. Here are some examples of matrices.

$$\begin{pmatrix} a & a' \\ b & b' \end{pmatrix}:$$ This is a 2 × 2 matrix (also called a square matrix of order 2)

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$ This is a 2 × 3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ This is a $m \times n$ matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 3 & -4 \end{pmatrix}:$$ This is a square matrix of order 3.

With each square matrix of numbers we associate a 'determinant of the matrix'. We do this inductively.

With the 1 × 1 matrix (a), we associate the determinant of order 1 and with the only entry a. The value of the determinant is a.

With the 2 × 2 matrix

$$\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$$

we associate the determinant

$$\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$$

whose value we have already defined as $ab' - a'b$.

With the 3 × 3 matrix A:

The evaluation of determinants, however, in the abòve manner becomes too complicated, even with determinants of order four, not to speak of higher orders. But there are several properties of determinants which enable the evaluation to be done more elegantly and quickly. We shall state these properties below, some of them without any idea of a proof, because these proofs are rather involved and must be postponed to a higher level of maturity in the student's mathematical carreer.

Before we state these properties we need one more concept and notation with respect to matrices. Given an $m \times n$ matrix A (with m rows and n columns) we obtain a new matrix

A^T, called A-*transpose*,

whose rows are just the columns of A. In fact the rows of A are the columns of A^T and columns of A are the rows of A^T. A^T has n rows and m columns; so its size is $n \times m$. In the literature it is customary to say that one interchanges rows and columns of A and obtains A^T. More precisely, the first row of A is the first column of A^T, the 2nd row of A is the 2nd column of A^T and so on.

Now we are ready to list the properties of determinants, referred to above. Let A be a $n \times n$ matrix.

1. $\det A = \det A^T$. In other words, every square matrix and its transpose have the same determinant. Or, again, if the rows and columns are interchanged in a determinant, the value of the determinant remains the same.
2. If two rows of A are interchanged to produce a new matrix B, $\det B = -\det A$. In other words, if, in a determinant, two rows are interchanged, the value of the determinant changes in sign (and not in magnitude).
3. If every element of a given row of matrix A is multiplied by a number α, the matrix thus obtained has determinant equal to $\alpha \det A$. As a consequence, if every element in a row of a determinant has the same factor, this common factor can be taken outside the determinant.
4. If one row of a determinant has its elements of the form

   $$\alpha_1 + \beta_1, \alpha_2 + \beta_2 \ldots$$

   then the determinant itself is the sum of two determinants, one of which has

   $$\alpha_1 \alpha_2 \ldots$$

   in that particular row, and the rest of the rows same as in the original; and the other of which has

   $$\beta_1 \beta_2 \ldots$$

   in that particular row and the rest of the rows same as in the original.
5. If two rows of a determinant are identical, the value of the determinant is zero.
6. (Stated for 3rd order determinants. For the other orders, the statement and proof are analogous)

   Let $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta$$

   and let $A_1, B_1, C_1, A_2, B_2, \ldots$, be respectively, the cofactors of $a_1, b_1, c_1, a_2, b_2, \ldots$

Then

$$a_1A_2 + b_1B_2 + c_1C_2 = 0$$

$$a_1A_3 + b_1B_3 + c_1C_3 = 0 \text{ and so on.}$$

In other words, if the elements of any row are multiplied by the cofactors of the corresponding elements of a parallel row, the result is zero.

7. If a fixed multiple of the elements of one row of A are added to the corresponding elements of another row of A the resulting matrix has the same determinant as A.
8. Properties 2 to 7 are true if we write 'columns' instead of 'rows'.

Of the above, as stated earlier we skip the proofs of Nos. 1 and 2 without even giving an indication of the proof. But here are two examples illustrating the properties.

EXAMPLE 1.

Let
$$A = \begin{pmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & 2 & 4 \\ -3 & 2 & 3 & 1 \\ 1 & 4 & -1 & 2 \end{pmatrix}$$

Then
$$A^T = \begin{pmatrix} 1 & 0 & -3 & 1 \\ 3 & 1 & 2 & 4 \\ -5 & 2 & 3 & -1 \\ 0 & 4 & 1 & 2 \end{pmatrix}$$

The theorem (Property 1) says that these two matrices have the same determinant.

EXAMPLE 2.

$$\begin{vmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & 2 & 4 \\ -3 & 2 & 3 & 1 \\ 1 & 4 & -1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -1 & 2 \\ -3 & 2 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{vmatrix}$$

Note the Row 2 and Row 4 of the determinant on the LHS have been interchanged to obtain the determinant on the RHS.

Sketch of a proof of (3). (The proof is given for a 4th order determinant but is clearly indicative of the proof in the general case)

Let
$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

Let B be the matrix obtained from A by multiplying the 2nd row of A by α.

Then
$$B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \alpha b_1 & \alpha b_2 & \alpha b_3 & \alpha b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

Expanding det B in terms of the 2nd row, we get det B

$$= (\alpha b_1) \times \text{cofactor of } b_1 \text{ in } A$$
$$+ (\alpha b_2) \times \text{cofactor of } b_2 \text{ in } A$$

$$+ (\alpha b_3) \times \text{cofactor of } b_3 \text{ in } A$$

$$+ (\alpha b_4) \times \text{cofactor of } b_4 \text{ in } A$$

$$= \alpha[b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4]$$

where the capital letters stand for the corresponding cofactors

$$= \alpha \det A.,$$

thus proving (3) for determinants of order four.

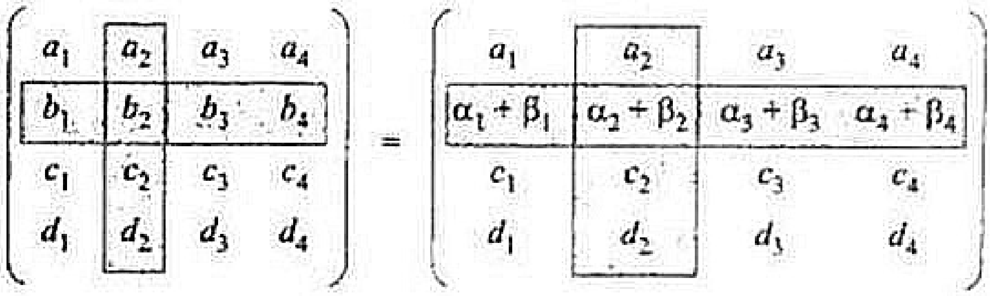
Sketch of a proof of (4). We give the proof for 4th order determinants.

Let $$\Delta = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \alpha_1+\beta_1 & \alpha_2+\beta_2 & \alpha_3+\beta_3 & \alpha_4+\beta_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

Let $$\Delta' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

The crucial point of the proof is the following fact.

Cofactor of b_i in Δ' is the same as cofactor of $\alpha_i + \beta_i$ in Δ. First note that the minors in the respective determinants are the same, by having a look at the following pictorial representation of the minors, say, of b_2 and of $\alpha_2 + \beta_2$.



Secondly check that the same signs get prefixed to the minors of b_2 and $\alpha_2 + \beta_2$ because they occupy the same position in the respective determinants.

Therefore, expanding Δ in terms of its 2nd row, we get

$$\Delta = (\alpha_1 + \beta_1) \times \text{cofactor of } b_1 \text{ in } \Delta'$$

$$+ (\alpha_2 + \beta_2) \times \text{cofactor of } b_2 \text{ in } \Delta'$$

$$+ (\alpha_3 + \beta_3) \times \text{cofactor of } b_3 \text{ in } \Delta'$$

$$+ (\alpha_4 + \beta_4) \times \text{cofactor of } b_4 \text{ in } \Delta'$$

$$= \sum_{i=1}^{4} \alpha_i \times \text{cofactor of } b_i \text{ in } \Delta'$$

$$+ \sum_{i=1}^{4} \beta_i \times \text{cofactor of } b_i \text{ in } \Delta'$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

this last step being true because one can expand these determinants in terms of their 2nd rows and also the cofactors of the α_i's and β_i's in these determinants are just those of b_i in Δ'.

Reader, note the arguments of this proof well. The beauty of the theory of determinants begins to present itself here!

Sketch of a proof of (5). Let det A be

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}$$

Note that the 2nd and 4th rows are identical. Interchanging 2nd and 4th row we get the same determinant. But property 2 says that the value of the determinant changes sign. So

$$\det A = -\det A$$

This means $\det A = 0$.

Sketch of a proof of (6)

$$a_1A_2 + b_1B_2 + c_1C_2$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (*)$$

for, expand the determinant on RHS in terms of the 2nd row and note that cofactors of a_1, b_1, c_1 in the 2nd row are the same as cofactors of a_2, b_2, c_2 in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

But the determinant on the RHS of (*) has two rows identical; so it is zero! The same argument will prove

$$\begin{aligned} & a_1A_3 + b_1B_3 + c_1C_3 \\ &= 0 \\ &= a_2A_1 + b_2B_1 + c_2C_1 \\ &= a_2A_3 + b_2B_3 + c_2C_3 \\ &= a_3A_1 + b_3B_1 + c_3C_1 \\ &= a_3A_2 + b_3B_2 + c_3C_2. \end{aligned}$$

Consolidating the results of (6) we may note that, in any determinant, if the elements of any row are multiplied by their respective cofactors and the results added we get the value of the determinant. On the other hand, if these elements are multiplied by the cofactors of the corresponding elements in a parallel row and the results added, we will get zero.

- **Sketch of a proof of (7)**

Let
$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

and
$$B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 + \alpha d_1 & b_2 + \alpha d_2 & b_3 + \alpha d_3 & b_4 + \alpha d_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

The theorem says: det B = det A.

We have det B

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \alpha d_1 & \alpha d_2 & \alpha d_3 & \alpha d_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \quad \text{by property (4)}$$

$$= \det A + \alpha \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \quad \text{by property (3)}$$

$$= \det A + \text{zero} \quad \text{by property (5)}$$

$$= \det A.$$

Proof of (8) This follows from property (1). In other words whatever is true of rows in a determinant is also true of columns.

We may now use all these 8 properties for manipulating with determinants. In doing so we keep the following in mind which is nothing but a summary of the lessons of experience gained by application of the above eight properties.

(*a*) We can expand a determinant in terms of any row or column,

(*b*) If we have to expand, it is desirable to expand in terms of a row or column which has many zeros in it,

(*c*) One of the strategies of manipulation with determinants is to obtain zeros in the same row or column,

(*d*) We can take out a common factor from any row or column,

(*e*) We can add to any row or column a constant multiple of a parallel row or column, and

(*f*) We can interchange any two rows (columns) provided we balance it by prefixing a minus sign outside the determinant.

EXAMPLE 3. *Evaluate*

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

SOLUTION. We start by trying to get zeros in the 2nd and 3rd entries of the 1st column. For this, we subtract 2 times the first row from the 2nd row and again add one time the first row to the 3rd row. Symbolically we represent this process by

$$R_2 - 2R_1 \text{ and } R_3 + R_1.$$

Note that only R_2 and R_3 change by these processes; R_1 remains as it is. So $R_2 - 2R_1$ gives

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ -1 & 0 & 1 \end{vmatrix}$$

On this, $R_3 + R_1$ gives

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & 2 & 4 \end{vmatrix}$$

This, on expansion in terms of the 1st column, gives $1 \times (-20 + 12) = -8$.

EXAMPLE 4.

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \qquad \text{(Do } R_3 - R_1 \text{ and } R_2 - R_1 \text{ on this)}$$

$$= \begin{vmatrix} 1 & a & bc \\ 1 & b-a & c(a-b) \\ 1 & c-a & b(a-c) \end{vmatrix} \qquad \text{(Take the common factor } a-b \text{ from 2nd row and } c-a \text{ from 3rd row of this)}$$

$$= (a-b)(c-a) \times \begin{vmatrix} 1 & a & bc \\ 0 & -1 & c \\ 0 & 1 & -b \end{vmatrix} \qquad \text{(Now expand in terms of 1st column)}$$

$$= (a-b)(c-a)(b-c)$$

$$= (a-b)(b-c)(c-a)$$

Now let us go back to the solution of linear systems of equations from which all this started. Take Example 2 of Sec.1. It is a system of three equations E_1, E_2 & E_3 with three unknowns. Three operations were repreatedly performed on this system and finally we had the solution of the system. If we carefully look at the way the equations have been manipulated, we may note that the variables x, y, z do not play any role. Only their coefficients play the crucial role and are subject to all the manipulations. We shall therefore reproduce the working of that problem, but now without the x, y, z. We shall exhibit only the coefficients. We shall consider the array of coefficients as a matrix (remembering that x stands for $1x$ and so 1 is the coefficient and similarly, no term in x stands for $0x$ and so 0 is the coefficient; and so on for y and z). So equations $E1$, $E2$, $E3$ will correspond to three rows of a matrix, which we shall denote as $R1$, $R2$, $R3$ (R standing for row). Thus we end up with the following presentation (see next page) of the manipulation of the matrix of coefficients which led to the solution of the system of equations. For a reading convenience we present the old working with the equations on the left side of the page and the matrix of coefficients on the right side of the page.

Now look at only the matrices on the right half of the page. The only operations we have used are of the three types:

Type 1: Interchange of two rows

Type 2 : Multiplying one row by a number

Type 3 : Adding to a row a non-zero constant times another row.

These are the three elementary row operations to which we have already referred earlier. These row operations acting on the original matrix of coefficients of the system give the final solution. This solution is explicit from the final stage of the matrix reduction. Note that throughout the matrix of coefficients keeps the alignment of the x terms, y terms and z-terms properly — so that, finally from the last matrix, we may read the solution as

$$x = 1, \quad y = 3/5 \text{ and } z = 2/5.$$

The process of reduction of a matix by these row operations is called Row reduction.

Now let us apply the row reduction method of solving a simultaneous linear system to the following four equations in four unknowns.

$E1 : 2x + 3y - 7z$	=	1	2	3	−7	1	$R1$
$E2 : x - y - z$	=	0	−1	−1	1	0	$R2$
$E3 : 3x + 2y + 2z$	=	5	3	2	2	5	$R3$
$E1 \leftrightarrow E2$ (Interchange):			$R1 \leftrightarrow R2$ (Interchange):				
$x - y - z$	=	0	1	−1	−1	0	
$2x + 3y - 7z$	=	1	2	3	−7	1	
$3x + 2y + 2z$	=	5	3	2	2	5	
$E2 - 2E1$ & $E3 - 3E1$:			$R2 - 2R1$ & $R3 - 3R1$:				
$x - y - z$	=	0	1	−1	−1	0	
$0x + 5y - 5z$	=	1	0	5	−5	1	
$0x + 5y + 5z$	=	5	0	5	5	5	
$E2 - E3$:			$R2 - R3$:				
$x - y - z$	=	0	1	−1	−1	0	
$0x + 0y - 10z$	=	−4	0	0	−10	−4	
$0x + 5y + 5z$	=	5	0	5	5	5	
$E2 \div (-10)$ & $E3 \div 5$:			$R2 \div (-10)$ & $R3 \div 5$:				
$x - y - z$	=	0	1	−1	−1	0	
$0x + 0y + z$	=	2/5	0	0	1	2/5	
$10x + y + z$	|=|	1	10	11	11	11	
$E3 - E2$:			$R3 - R2$:				
$x - y - z$	=	0	1	−1	−1	0	
$0x + 0y + z$	=	2/5	0	0	1	2/5	
$0x + y + 0z$	=	3/5	0	1	0	3/5	
$E2 \leftrightarrow E3$: (Interchange):			$R2 \leftrightarrow R3$ (Interchange):				
$x - y - z$	=	0	1	−1	−1	0	
y	=	3/5	0	1	0	3/5	
z	=	2/5	0	0	1	2/5	
$E1 + (E2 + E3)$:			$R1 + (R2 + R3)$				
x	=	1	1	0	0	1	
y	=	3/5	0	1	0	3/5	
z	=	2/5	0	0	1	2/5	

EXAMPLE 5.

$$x + 5y + z - 4w = 6$$
$$5x + y + 5z + 4w = 6$$
$$x - 9y + z + 10w = -8$$
$$x - 4y + z + 5w = -3$$

We present the row reduction of the matrix below:

$$\begin{matrix} 1 & 5 & 1 & -4 & 6 \\ 5 & 1 & 5 & 4 & 6 \\ 1 & -9 & 1 & 10 & -8 \\ 1 & -4 & 1 & 5 & -3 \end{matrix}$$

Apply $-5R1 + R2$; $-1R1 + R3$; $-1R1 + R4$

$$\begin{matrix} 1 & 5 & 1 & -4 & 6 \\ 0 & -24 & 0 & 24 & -24 \\ 0 & -14 & 0 & 14 & -14 \\ 0 & -9 & 0 & 9 & -9 \end{matrix}$$

Apply $-1/24\ R2$; $-1/14\ R3$; $-1/9\ R1$

$$\begin{matrix} 1 & 5 & 1 & -4 & 6 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \end{matrix}$$

Apply $-1R2 + R3$; $-1R2 + R4$

$$\begin{matrix} 1 & 5 & 1 & -4 & 6 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Apply $-5R2 + R1$

$$\begin{matrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Writing this in the equation form, we get

$$x + z + w = 1$$
$$y - w = 1$$

and there are only two equations. This means

$$x = 1 - z - w$$
$$y = 1 + w$$

Thus for every artitrary value given to z and w we get one pair of values for x and y. In other words there are infinite number of solutions for the system. For instance, taking $w = 1 = z$ we have

$$x = 1 - 2 = -1 \text{ and } y = 2$$

thus giving $x = -1$, $y = 2$, $z = 1$, $w = 1$ as one solution.

Taking $w = 1$ and $z = 0$ we would have $x = 0$, $y = 2$ thus giving the solution $x = 0$, $y = 2$, $z = 0$, $w = 1$. And so on we have an infinite number of solutions.

EXAMPLE 6.

$$\begin{aligned} 2x + 8y + 4z &= 1 \\ 5x - 6y + 10z &= -1 \\ x + 7y + 2z &= 0 \end{aligned}$$

$$\begin{array}{cccc} 2 & 8 & 4 & 1 \\ 5 & -6 & 10 & -1 \\ 1 & 7 & 2 & 0 \end{array}$$

$R_3 \leftrightarrow R_1$ gives

$$\begin{array}{cccc} 1 & 7 & 2 & 0 \\ 5 & -6 & 10 & -1 \\ 2 & 8 & 4 & 1 \end{array}$$

$-5R_1 + R_2$ & $-2R_1 + R_3$ gives

$$\begin{array}{cccc} 1 & 7 & 2 & 0 \\ 0 & -41 & 0 & -1 \\ 0 & -6 & 0 & 1 \end{array}$$

$-1/41\ R_2$ and $-1/6\ R_3$ give

$$\begin{array}{cccc} 1 & 7 & 2 & 0 \\ 0 & 1 & 0 & 1/41 \\ 0 & 1 & 0 & -3/6 \end{array}$$

$R_3 - R_2$ gives

$$\begin{array}{cccc} 1 & 7 & 2 & 0 \\ 0 & 1 & 0 & 1/41 \\ 0 & 0 & 0 & -1/6-1/41 \end{array}$$

The third equation here is

$$0x + 0y + 0z = -\frac{1}{6} - \frac{1}{41}$$

which is impossible. Therefore the system has no solution. The equations are inconsistent.

Incidentally the determinant of the last system is

$$\begin{vmatrix} 2 & 8 & 4 \\ 5 & -6 & 10 \\ 1 & 7 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -6 & 0 \\ 0 & -41 & 0 \\ 1 & 7 & 2 \end{vmatrix}$$

$$= 1 \times (-6 \times 0 + 41 \times 0) = 0$$

Also the determinant of the system of the four equations in Example 5 is

$$\begin{vmatrix} 1 & 5 & 1 & -4 \\ 5 & 1 & 5 & 4 \\ 1 & -9 & 1 & 10 \\ 1 & -4 & 1 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & 1 & -4 \\ 0 & -24 & 0 & 24 \\ 0 & -14 & 0 & 14 \\ 0 & -9 & 0 & 9 \end{vmatrix} \qquad \text{by } -5R_1 + R_2;$$

$-1R_1 + R_3$; and $-1R_1 + R_4$

$$= (-1/24) \times (-1/14) \times (-1/9) \begin{vmatrix} 0 & 5 & 1 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1 & 5 & 1 & -4 \\ 5 & 1 & 5 & 4 \\ 1 & -9 & 1 & 10 \\ 0 & -4 & 1 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & 1 & -4 \\ 0 & -24 & 0 & 24 \\ 0 & -14 & 0 & 14 \\ 0 & -9 & 0 & 9 \end{vmatrix} \qquad \text{by } -5R_1 + R_2; -1R_1 + R_3; \text{ and } -R_1 + 4$$

$$= (-1/24) \times (-1/14) \times (-1/9) \begin{vmatrix} 0 & 5 & 1 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{vmatrix} = 0$$

since three rows are identical.

Again, the determinant of the system of three equations of Example 2, Section 1 is

$$\begin{vmatrix} 2 & 3 & -7 \\ 1 & -1 & -1 \\ 3 & 2 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 5 & -5 \\ 1 & 0 & 0 \\ 3 & 5 & 5 \end{vmatrix} \qquad \text{by } C_2 + C_1 \text{ and } C_3 + C_1$$

$$= -1\ (25 + 25) = -50$$

and thus the determinant is non-zero. In fact it can be proved that **the solution of a system of n equations in n unknowns, with the RHS not all zero, exists and is unique, iff the determinant of the system is non-zero.** The three examples above corroborate this statement.

Our next task is to analyse the case when the determinant is zero, more deeply. But before we do that let us settle once for all the case of the non-zero determinant, by giving a precise formula for the solution of n equations in n unknowns with the R.H.S entries not all zero. We shall give the working only for the special case $n = 3$, but the reader can see that this generalises to any n. The formula we obtain is called Cramer's Rule.

CRAMER'S RULE

Consider

$$a_1x + b_1y + c_1z = d_1 \tag{1}$$
$$a_2x + b_2y + c_2z = d_2 \tag{2}$$
$$a_3x + b_3y + c_3z = d_3 \tag{3}$$

with the hypothesis,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & b_3 \end{vmatrix} \neq 0$$

PRELIMINARY REMARKS

Let the cofactors of Δ be $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$. Then we know that

$$a_iA_i + b_iB_i + c_iC_i = \Delta \quad \text{for } i = 1, 2, 3$$

This is almost the definition of the value of the determinant. Also we know

$$a_1A_1 + a_2A_2 + a_3A_3 = \Delta$$
$$b_1B_1 + b_2B_2 + b_3B_3 = \Delta$$
$$c_1C_1 + c_2C_2 + c_3C_3 = \Delta$$

since columns of a determinant behave exactly like rows.

Further we know that

$$b_1A_1 + b_2A_2 + b_3A_3 = 0$$
$$c_1A_1 + c_2A_2 + c_3A_3 = 0 \tag{*}$$

and similar results — where we multiply elements of a column by cofactors of corresponding elements in another column.

With this background, multiply equation (1) by A_1, equation (2) by A_2 and equation (3) by A_3; add the three results and see what happens. We get

$$\begin{aligned} x(a_1A_1 + a_2A_2 + a_3A_3) &+ y\,(b_1A_1 + b_2A_2 + b_3A_3) \\ &+ z(c_1A_1 + c_2A_2 + c_3A_3) \\ &= d_1A_1 + d_2A_2 + d_3A_3. \end{aligned}$$

On the LHS, the coefficients of y and z vanish, in virtue of (*). So we have

$$x\,\Delta = d_1A_1 + d_2A_2 + d_3A_3$$
$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Thus,

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\Delta}$$

Again, multiplying (1) by B_1, (2) by B_2 and (3) by B_3 and making a similar calculation, we get

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\Delta}$$

We obtain, similarly,

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\Delta}$$

Thus the system has been completely solved. This method of solution is called Cramer's Rule. Note that if $\Delta = 0$ the method fails.

EXAMPLE 7. *Solve, by Cramer's Rule, the following system of four equations with four unknowns.*

$$x + 3y + 2z = 1$$
$$2x + z + w = -1$$
$$x + 2y + 3z = 2$$
$$3x - y + w = 0$$

SOLUTION. Here, $\Delta = \begin{vmatrix} 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 3 & -1 & 0 & 1 \end{vmatrix}$ (Do $R4 - R2$ on this)

$$= \begin{vmatrix} 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & -1 & -1 & 0 \end{vmatrix} \quad \text{(expand in terms of the 4th Col.)}$$

$$= +1 \times \begin{vmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} \quad \text{(Do } R_2 - R_1 \text{ and } R_3 - R_1 \text{ on this)}$$

$$= \begin{vmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & -4 & -3 \end{vmatrix} = (3 + 4) = 7$$

Then $x = \frac{1}{7}\begin{vmatrix} 1 & 3 & 2 & 0 \\ -1 & 0 & 1 & 1 \\ 2 & 2 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$ (Do $R_4 - R_2$ on this)

$$= \frac{1}{7}\begin{vmatrix} 1 & 3 & 2 & 0 \\ -1 & 0 & 1 & 1 \\ 2 & 2 & 3 & 0 \\ 0 & -1 & -1 & 0 \end{vmatrix} \quad \text{(Expand this in terms of the 4th column)}$$

$$= \frac{1}{7}\begin{vmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} \qquad \text{(Do } R_2 - 2R_1 \text{ and } R_3 - R_1 \text{ on this)}$$

$$= \frac{1}{7}\begin{vmatrix} 1 & 3 & 2 \\ 0 & -4 & -1 \\ 0 & -4 & -3 \end{vmatrix} = 1/7\,(12 - 4) = 8/7$$

$$y = \frac{1}{7}\begin{vmatrix} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 3 & 0 & 0 & 1 \end{vmatrix} \qquad \text{(Do } R_4 - R_2 \text{ on this}$$

$$= \frac{1}{7}\begin{vmatrix} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 1 & -1 & 0 \end{vmatrix} \qquad \text{(Expand in terms of the 4th column)}$$

$$= \frac{1}{7} \times 1 \times \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{vmatrix} \qquad \text{(Do } R_2 - R_1 \text{ and } R_3 - R_1 \text{ on this)}$$

$$= \frac{1}{7}\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = -\,3/7$$

$$z = \frac{1}{7}\begin{vmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & -1 & 1 \\ 1 & 2 & 2 & 0 \\ 3 & -1 & 0 & 1 \end{vmatrix} \qquad \text{(Do } R_4 - R_2 \text{ on this)}$$

$$= \frac{1}{7}\begin{vmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & -1 & 1 \\ 1 & 2 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{vmatrix} \qquad \text{(Expand in terms of the 4th column)}$$

$$= \frac{1}{7} \times 1 \times \begin{vmatrix} 1 & 3 & 1 \\ 1 & 2 & 2 \\ 1 & -1 & 1 \end{vmatrix} \qquad \text{(Do } R_2 - R_1 \text{ and } R_3 - R_1 \text{ on this)}$$

$$= \frac{1}{7}\begin{vmatrix} 1 & 3 & 1 \\ 0 & -1 & 1 \\ 0 & -4 & 0 \end{vmatrix} = \frac{4}{7}$$

$$w = \frac{1}{7}\begin{vmatrix} 1 & 3 & 2 & 1 \\ 2 & 0 & 1 & -1 \\ 1 & 2 & 3 & 2 \\ 3 & -1 & 0 & 0 \end{vmatrix} \qquad \text{(Do } R_2 + R_1 \text{ and } R_3 - 2R_1 \text{ on this)}$$

$$= \frac{1}{7}\begin{vmatrix} 1 & 3 & 2 & 1 \\ 3 & 3 & 3 & 0 \\ -1 & -4 & -1 & 0 \\ 3 & -1 & 0 & 0 \end{vmatrix} \qquad \text{(Expand in terms of the 4th column)}$$

$$= \frac{1}{7} \times (-1) \times \begin{vmatrix} 3 & 3 & 3 \\ -1 & -4 & -1 \\ 3 & -1 & 0 \end{vmatrix}$$ (Take out the common factor 3)

$$= -\frac{3}{7} \times \begin{vmatrix} 1 & 1 & 1 \\ -1 & -4 & -1 \\ 3 & -1 & 0 \end{vmatrix}$$ (Do $R_2 + R_1$ on this)

$$= -\frac{3}{7} \times \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 3 & -1 & 0 \end{vmatrix} = -3/7 \times 9 = -27/7$$

Thus the complete solution is:

$$x = \frac{8}{7}, y = -\frac{3}{7}, z = \frac{4}{7}, w = -\frac{27}{7}.$$

Though Cramer's Rule gives a foolproof method of computation, the method of row reduction is more elegant and is applicable even when $\Delta = 0$. The efficiency of the method may be seen by a reworking of Example 7, now by row-reduction process.

$$\begin{array}{ccccc} 1 & 3 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 0 & 2 \\ 3 & -1 & 0 & 1 & 0 \\ \hline \end{array}$$

Apply $R_2 - 2R_1$
$R_3 - R_1$ & $R_4 - 3R_1$

$$\begin{array}{ccccc} 1 & 3 & 2 & 0 & 1 \\ 0 & -6 & -3 & 1 & -3 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & -10 & -6 & 1 & -3 \\ \hline \end{array}$$

$R_2 \leftrightarrow R_3$
Then apply $(-1) \times R_2$

$$\begin{array}{ccccc} 1 & 3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & -6 & -3 & 1 & -3 \\ 0 & -10 & -6 & 1 & -3 \\ \hline \end{array}$$

Apply $-3R_2 + R_1$; $6R_2 + R_3$;
$10R_2 + R_4$

$$\begin{array}{ccccc} 1 & 0 & 5 & 0 & 4 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & -9 & 1 & -9 \\ 0 & 0 & -16 & 1 & -13 \\ \hline \end{array}$$

Apply $-R_4 + R_3$

$$\begin{array}{ccccc} 1 & 0 & 5 & 0 & 4 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 7 & 0 & 4 \\ 0 & 0 & -16 & 1 & -13 \\ \hline \end{array}$$

Apply $\frac{1}{4} \times R_3$

$$\begin{array}{ccccc l}
1 & 0 & 5 & 0 & 4 & \\
0 & 1 & -1 & 0 & -1 & \\
0 & 0 & 1 & 0 & 4/7 & \text{Apply } -5R_3+R_1;\ R_3+R_2; \\
0 & 0 & -16 & 1 & -13 & 16R_3+R_4 \\
\hline
1 & 0 & 0 & 0 & 4-20/7=8/7 & \\
0 & 1 & 0 & 0 & -1+4/7=-3/7 & \\
0 & 0 & 1 & 0 & 4/7 & \\
0 & 0 & 0 & 1 & -13+64/7=-27/7 & \\
\hline
\end{array}$$

Thus $x=\frac{8}{7};\ y=-\frac{3}{7};\ z=\frac{4}{7};\ w=-\frac{27}{7}$.

In a similar working of Example 5 however, we find that there are an infinite number of solutions. There we also had the determinant of the system to have zero value. Wherever the determinant is zero, a unique solution does not exist. Either there is no solution (as in the case of Example 6) or there is an infinity of solutions (as in the case of Example 5). In this last case, scrutinising the answer for Example 5 we see that

$$x=1-z-w$$
$$y=1+w.$$

Every value of z and w gives a pair of values for x and y and there is a solution of the system. Thus there are as many solutions of the system as there are possible values for z and w. What value we give to z does not depend on the value we give to w; and vice versa. Thus z and w can be independently given values, each an infinity of values. We say that the solution has **two degrees of freedom.** Once z and w are fixed x and y depend on them and are uniquely got.

In all the above examples and illustrations of m equations with n unknowns, we always had $m=n$. *i.e.*, the number of equations and the number of unknowns were the same. When this is not so the determinant method is not applicable. We have to follow row reduction of the matrix or use a direct method of elimination of variables. The following Examples illustrate this.

EXAMPLE 8.

$$x+2y-z-3w+u=4$$
$$2x+3y+z+2u=10$$
$$y+2w+u=5.$$

SOLUTION. Here there are 3 equations and 5 unknowns. We follow the row reduction method.

$$\begin{array}{cccccc l}
1 & 2 & -1 & -3 & 1 & 4 & \\
2 & 3 & 1 & 0 & 2 & 10 & \\
0 & 1 & 0 & 2 & 1 & 5 & \text{Apply } -2R_1+R_2 \\
\hline
1 & 2 & -1 & -3 & 1 & 4 & \\
0 & -1 & 3 & 6 & 0 & 2 & \\
0 & 1 & 0 & 2 & 1 & 5 & \text{Interchange } R_2 \text{ and } R_3 \\
\hline
\end{array}$$

$$\begin{array}{cccccc} 1 & 2 & -1 & -3 & 1 & 4 \\ 0 & 1 & 0 & 2 & 1 & 5 \\ 0 & -1 & 3 & 6 & 0 & 2 \\ \hline \end{array}$$

Apply $-2R_2 + R_1$ and $R_3 + R_1$

$$\begin{array}{cccccc} 1 & 0 & -1 & -7 & -1 & -6 \\ 0 & 1 & 0 & 2 & 1 & 5 \\ 0 & 0 & 3 & 8 & 1 & 7 \\ \hline \end{array}$$

$\frac{1}{3} \times R_3$

$$\begin{array}{cccccc} 1 & 0 & -1 & -7 & -1 & -6 \\ 0 & 1 & 0 & 2 & 1 & 5 \\ 0 & 0 & 1 & 8/3 & 1/3 & 7/3 \\ \hline \end{array}$$

$R_3 + R_1$

$$\begin{array}{cccccc} 1 & 0 & 0 & -15/3 & -2/3 & -11/3 \\ 0 & 1 & 0 & 2 & 1 & 5 \\ 0 & 0 & 1 & 8/3 & 1/3 & 7/3 \\ \hline \end{array}$$

This gives

$$x = \frac{13}{3} w + \frac{2}{3} u - \frac{11}{3}$$
$$y = -2w - u + 5$$
$$z = -\frac{8}{3} w - \frac{1}{3} u + \frac{7}{3}$$

thus showing an infinite number of solutions, with two degrees of freedom, since w and u can be arbitrarily fixed.

We can do the same problem by direct elimination of variables:

$$x + 2y - z - 3w + u = 4$$
$$2x + 3y + z + 2u = 10$$
$$y + 2w + u = 5.$$

Since there are only three equations we keep there variables on the LHS and take the remaining to the RHS; thus.

$$x + 2y - z = 3w - u + 4 \qquad (1)$$
$$2x + 3y + z = -2u + 10 \qquad (2)$$
$$y = -2w - u - 6 \qquad (3)$$

(1) − 2 × (3) gives

$$x - z = 7w + u - 6 \qquad (4)$$

(2) − 3 × (3) gives

$$2x + z = 6w + u - 5 \qquad (5)$$

(4) + (5) gives

$$3x = 13w + 2u - 11$$

(5) − 2 × (4) gives

$$3z = -8w - u + 7$$

Thus
$$x = \frac{13}{3}w + \frac{2}{3}u - \frac{11}{3}$$
$$y = -2w - u + 5$$
$$z = -\frac{8}{3}w - \frac{1}{3}u + \frac{7}{3}$$

thus giving an infinite number of solutions with two degrees of freedom as before. Note that, in this last example, there is no sanctity attached to the variables x, y, z. Though the answer shows x, y, z in terms of the two variables w and u, we could have as well kept z, w, y on the LHS and obtained solutions for them in terms of x and y. The problem fixes only the number of free variables; it does not determine which variables are to be free. That would be our choice.

EXERCISE 8.2

1. Evaluate the following determinants:

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 0 & 2 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 4 & 1 \\ -1 & 1 & 2 & 0 \\ 0 & 3 & 1 & 3 \end{vmatrix}$

(c) $\begin{vmatrix} -1 & 1 & 0 & 3 \\ 3 & 0 & 1 & 1 \\ 2 & -1 & 2 & 2 \\ 2 & 3 & 0 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

2. Find all the cofactors of the matrix
$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$$
and find its determinant.

3. Find the cofactors A_{11}, A_{12}, A_{13}, and A_{14} of
$$A = \begin{pmatrix} 3 & 4 & 2 & 3 \\ 1 & 3 & -1 & 2 \\ 4 & 5 & -3 & 0 \\ 2 & -2 & 6 & 4 \end{pmatrix}$$
and find det A.

4. Evaluate the determinant of the matrix
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$$

5. Find the values of λ for which the system of equation
$$(\lambda - 1)x + 2y + z = 0$$
$$2x + (\lambda - 1)y - 2z = 0$$
$$x - 2y + (\lambda - 1)z = 0$$
has infinitely many solutions.

6. Solve, by Cramer's rule, those systems of linear equations in Ex 8.1 to which Cramer's rule is applicable.

7. $ax + by + c = 0$

$$a'x + b'y + c' = 0, \text{ with } ab' - a'b \neq 0.$$

Therefore $\dfrac{x}{bc' - b'c} = \dfrac{y}{ca' - c'a} = \dfrac{1}{ab' - a'b}$

What is the error, if any, in this argument?

PROBLEMS

1. Solve the following linear systems:

(*a*)
$$\begin{aligned} x + y - 8z + 3w &= 2 \\ -7x + y + 16z - 5w &= 2 \\ x - y - z + 2w &= -2 \\ 3x + 2y - 13z + w &= 4. \end{aligned}$$

(*b*)
$$\begin{aligned} x + 2y + 2z + 3w + u &= -1 \\ 3x + 6y + 7z + 10w + 4u &= 4 \\ 5x + 10y - z + 4w - 6u &= -16. \end{aligned}$$

2. For which values of k does

$kx + y = 1, \quad x + ky = 1$

have no solution, one solutions or infinitely many solutions?

3. If A_n is an 11th order determinant

$$\begin{vmatrix} 2 & -1 & 0 & .. & .. & 0 \\ -1 & 2 & -1 & 0 & ... & 0 \\ 0 & -1 & 2 & -1 & 0.. & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{vmatrix}$$

Prove that

$$A_n = 2A_{n-1} - A_{n-2}$$

Hence prove that $A_n = n + 1$.

4. Prove that a square determinant of the nth order which has zeros on the diagonal and ones elsewhere is equal to $(-1)^n n$.

5. The *row rank* of a matrix A is defined as follows. Perform row reduction on A such that the resulting matrix B satisfies

(*a*) The first non-zero entry in each non-zero row (*i.e.*, a row in which not all elements are zero) is 1; and

(*b*) If a column contains the first non-zero entry of any row, then every other entry in that column is zero.

The number of non-zero rows in B is called *the row rank of A*.

With each linear system of equations, two matrices may be associated. One is the coefficient matrix. The other is the augmented matrix which is nothing but the one obtained by appending the right hand side entries to the matrix of coefficients. The development of this subject in Higher Algebra leads to the following theorem:

A linear system of equations has a solution iff the coefficient matrix and the augmented matrix have the same row rank.

Verify this in all the Examples & Exercises of this chapter where we are given a linear system of equations.

6. If

$$\begin{vmatrix} 1 & \cos\theta & 0 & 0 \\ \cos\theta & 1 & \cos\alpha & \cos\beta \\ 0 & \cos\alpha & 1 & \cos\gamma \\ 0 & \cos\beta & \cos\gamma & 1 \end{vmatrix} = 0, \text{ then}$$

$$\theta = n\pi + (1)^n \sin^{-1}\frac{\sqrt{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\gamma}}{\sin\gamma}.$$

7. Under what condition is the following true?

$$\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$$

8. Without expanding the determinants, prove that

$$\begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$$

9. Compute the determinant

$$\begin{vmatrix} 25 & 24 & 23 & 22 & 21 \\ 20 & 19 & 18 & 17 & 16 \\ 15 & 14 & 13 & 12 & 11 \\ 10 & 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 & 1 \end{vmatrix}$$

10. Prove that

(i) $$\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$$

(ii) $$\begin{vmatrix} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 1 & 1 & 0 & c \\ a & b & c & d \end{vmatrix} = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca + 2d.$$

(iii) $$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2$$

11. Prove that

$$\begin{vmatrix} 1 & n & n & \cdot & \cdot & \cdot & n \\ n & 2 & n & \cdot & \cdot & \cdot & n \\ n & n & 3 & \cdot & \cdot & \cdot & n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n & n & n & \cdot & \cdot & \cdot & n \end{vmatrix} = (-1)^{n-1}\, n!$$

12. Compute the determinant

$$\begin{vmatrix} O_{m \times n} & I_m \\ I_n & O_{n \times m} \end{vmatrix}$$

where I_r is the identity matrix of order r and $O_{p \times q}$ is the zero matrix of the order $p \times q$.

CHAP

9

PERMUTATIONS AND COMBINATIONS

9.1 PERMUTATIONS

The main subject of this chapter is counting. Given a set of objects the problem is to arrange a subset according to some specification or to select a subset as per some specification. We shall actually be interested in the number of such possible arrangements or selections. First of all we shall make precise a fundamental principle of counting which we intuitively use in our everyday life.

Suppose there are two flights (Flight Nos. 1 and 2) in the morning from place *A* to place *B* and three flights (Flight Nos. 3, 4 & 5) in the evening from place-*B* to place-*C*. We ask the question. In how many ways can one fly from place *A* to place *C* via place *B*? No other restrictions are to be assumed.

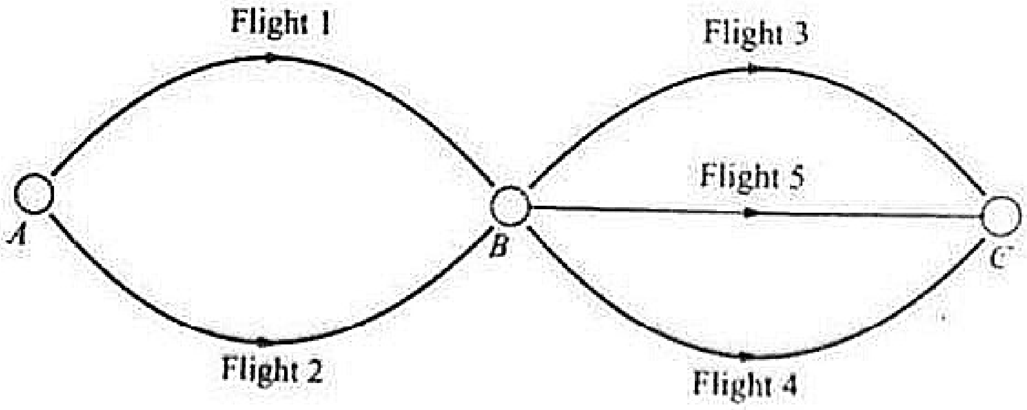


Fig. 9.1

We reason as follows. There are two ways of flying from *A* to *B*. For each such choice of flight, there are three ways of flying from *B* to *C*. Flight 1 can be followed up by any one of the three flights 3, 4 or 5 and similarly flight 2 can be followed up by any one of 3, 4 or 5. Thus there are $2 \times 3 = 6$ ways of flying from *A* to *C*. The following listing also confirms this finding:

Flight 1 followed by Flight 3
Flight 1 followed by Flight 4
Flight 1 followed by Flight 5
Flight 2 followed by Flight 3
Flight 2 followed by Flight 4
Flight 2 followed by Flight 5

The most important point to note here is the fact that what you did in the 2nd leg of the flight was totally *independent* of what you did in the 1st leg of the flight. In other words the choice of the flight from B to C had nothing to do with the choice of the flight from A to B. Thus the two choices for the first leg when paired off with each of the three choices for the 2nd leg gives us $2 \times 3 = 6$ possible ways for the flight from A to C.

This argument is the essence of what one calls the fundamental principle of counting. It can be abstracted as follows:

Suppose an event E can happen in any one of m mutually exclusive ways. '*Mutually exclusive*" means: if one way is chosen, the other way(s) are automatically not chosen. In the above example, Flight 1 and Flight 2 are mutually exclusive ways of flying from A to B. If one flies by flight 1 then he is not simultaneously flying by flight 2 and vice versa. Similarly flights 3, 4 and 5 are mutually exclusive ways in the 2nd leg of the flight. If one is flying by flight 4, say, he is not at the same time flying by flight 3 or 5.

To continue our abstraction, suppose event E can happen in any one of m mutually exclusive ways. Suppose also, independently of event E, another event F can happen in any one of n mutually exclusive ways. Then, the principle says, the two events E and F can together happen in mn ways.

We shall illustrate this principle by a few examples. A proper understanding of all these illustrative examples would go a long way in making the subject of counting fully comprehensible.

EXAMPLE 1. There are 25 mathematics books and 24 physics books on a library shelf. In how many ways can we choose one mathematics and one physics book?

Choosing a mathematics book is	Event E.
Choosing a physics book is	Event F.

Event E can happen in any one of 25 mutually exclusive ways. Event F can happen in any one of 24 mutually exclusive ways. What physics book we choose is independent of what maths book we choose and vice versa. So Events E and F are independent. Thus the fundamental counting principle applies and the two books can together be chosen in $25 \times 24 = 600$ ways.

EXAMPLE 2. *In how many ways can a family consisting of a mother, two sons and two daughters-in-law be arranged for a photograph satisfying the following conditions?*

(*i*) *There are only three chairs. So two persons have to stand behind.*

(*ii*) *The mother is to occupy the central chair, and*

(*iii*) *Either both the daughters-in-law are to sit in the chairs or both of them are to stand behind.*

SOLUTION. No other conditions are imposed. We reason as follows. There is no choice for the occupation of the central chair. The choice is only in the occupation of the end-chairs and in the decision of who stands where. Let event E be the seating in the chairs. There are four mutually exclusive ways: *viz.*, (from left to right)

Daughter-in-law 1 and Daughter-in-law 2;
or Daughter-in-law 2 and Daughter-in-law 1;
or Son 1 and Son 2;
or Son 2 and Son 1.

The event F will be the allocation of standing positions for the remainsing two persons (whoever they are). This can be done in 2 mutually exclusive ways, *viz.*,

Person 1 and Person 2;

or Person 2 and Person 1.

Thus there are four choices for event E and two for F. The actual handling of event F in terms of its two choices is independent of event E. So the two events together can happen in $4 \times 2 = 8$ ways. These 8 ways are pictured in the following diagram.

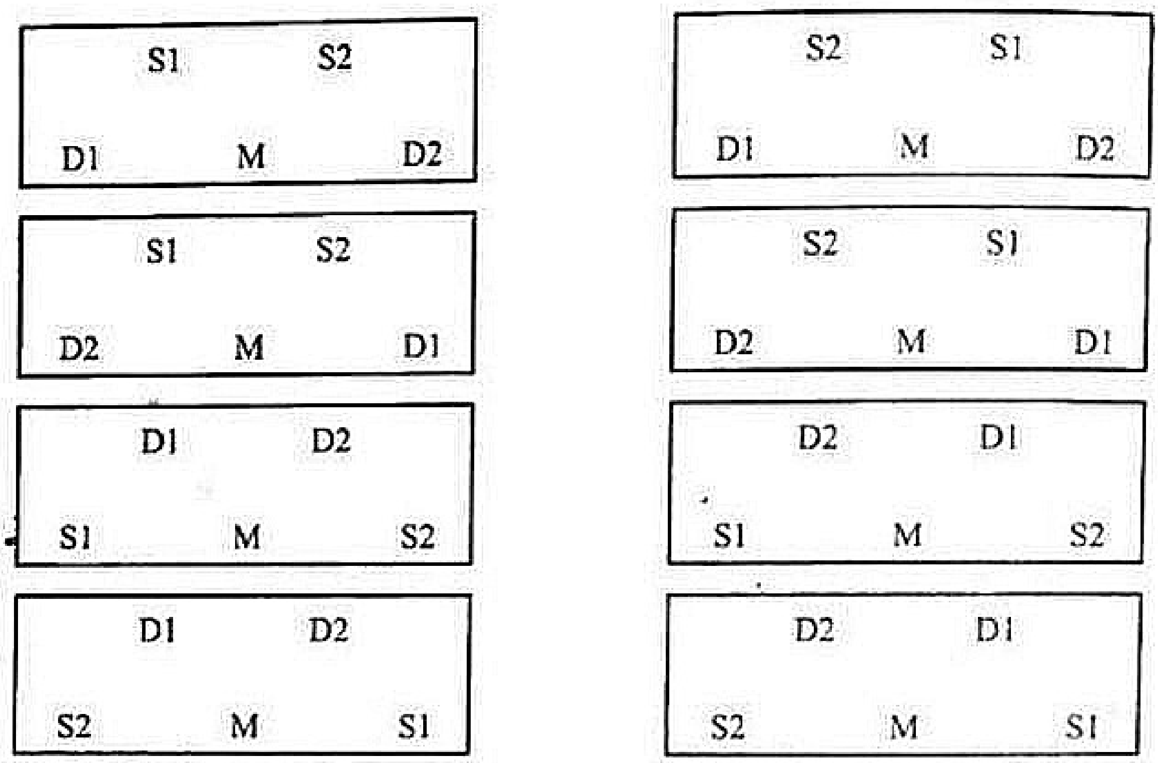


Fig. 9.2

EXAMPLE 3. *A die is a six-faced cube, with the faces reading 1, 2, 3, 4, 5 and 6. When two dice are thrown we add the digits they show on top and take that sum as the result of the throw. We ask the question. In how many different ways can the following situation happen, viz.,*

SOLUTION. First throw of the 2 dice shows a total of 5;

and second throw of the 2 dice shows a total of 4.

Event E (the first throw resulting in 5) can happen in one of four ways, *viz.*,

$3 + 2;\ 4 + 1;\ 2 + 3;\ 1 + 4.$

Event F (the second throw resulting in 4) can happen in one of three ways, *viz.*,

$2 + 2;\ 1 + 3;\ 3 + 1.$

The two events can together happen in $4 \times 3 = 12$ ways.

Note. We shall not each time say that the two events are independent. But we shall not hesitate to discuss the independence whenever there is likely to be a doubt.

EXAMPLE 4. *How many integers are there less than 1000, ending with 3, 6 or 9?*

SOLUTION. We shall consider three blank spaces (ordered from left to right) to represent numbers less

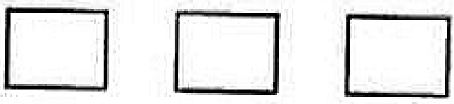


Fig. 9.3

than 1000. Two-digit and one-digit numbers will have zeros in the first place and first two places respectively. The number zero is 000. So the 1000 numbers less than 1000 (including zero) are obtained by filling up the three blank spaces in all possible ways (precisely, $10 \times 10 \times 10 = 10^3$ ways) by the digits 0, 1, 2, 3, ..., 9. Since we want only those numbers that are ending with 3, 6, or 9 we reason as follows. The first place has all the 10 choices: the second place also has 10 choices; whereas the third place has only 3 choices, *viz.*, 3, 6 and 9. Thus the three blank spaces can be filled with $10 \times 10 \times 3 = 300$ ways. So 300 is the answer to the problem.

EXAMPLE 5. A DNA chain is composed of basic building blocks in the form of four chemicals, known by the symbols *A*, *C*, *T* and *G*. Consider three-letter chains consisting of these symbols (with or without repetitions). How many such chains are there?

The argument is the same as in the previous example and is standard for all such situations. We imagine three blank spaces numbered 1, 2, 3 from left to right. It is left to us to fill the spaces with the symbols *A*, *C*, *T*, *G*. Space 1 can be filled up by any one of the 4 letters and so in four ways. Since repetitions are allowed, the second space can also be filled up in four ways. Thus the first two spaces can be filled up in $4 \times 4 = 16$ ways by the fundamental principle of counting. Now the 3rd space can be filled up in four ways. For each of the 16 ways of filling up the first two spaces there are four ways of filling up the third space. So the fundamental principle applies. Thus the three spaces can be filled up in $16 \times 4 = 64$ ways. Hence there are 64 three-letter DNA chains.

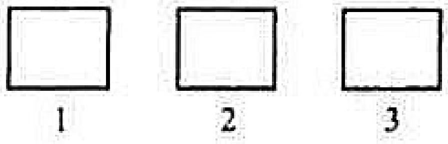


Fig. 9.4

Incidentally, in the last two examples, we have intuitively extended the fundamental principle of counting to more than 2 events. We can abstract this and record the principle as follows.

Full statement of the fundamental principle of counting: If $E_1, E_2, \ldots E_n$ are n independent events and E_i, $i = 1$ to n can happen in one of m_i mutually exclusive ways then all the events $E_1, E_2, \ldots, E_n$ can together happen in

$$m_1 \, m_2 \ldots m_n \text{ ways.} \tag{1}$$

Note that 'independent events' means (in this larger setup) the happening of one does not affect any of the others. In the above example of the three-letter DNA-chain the filling up of the 2nd space, for instance, has nothing to do with the filling up of either the first space or the third space.

We shall now continue with the last example, the DNA-chain, to talk about PERMUTATIONS. A permutation of a set $\{x_1, x_2, \ldots, x_n\}$ is a rearrangement of the symbols. In other words we consider the original collection as our ordered arrangement

$$x_1, x_2, \ldots, x_n$$

(as if people are sitting in a line for a photograph) and we now consider any possible rearrangement. Thus, in the case of the symbols *A*, *C*, *T*, *G* of the DNA-chain the arrangements (4–letter DNA-chains)

$$ACTG, CAGT, TGCA, GTCA, \ldots$$

The event F will be the allocation of standing positions for the remainsing two persons (whoever they are). This can be done in 2 mutually exclusive ways, *viz.*.

Person 1 and Person 2;
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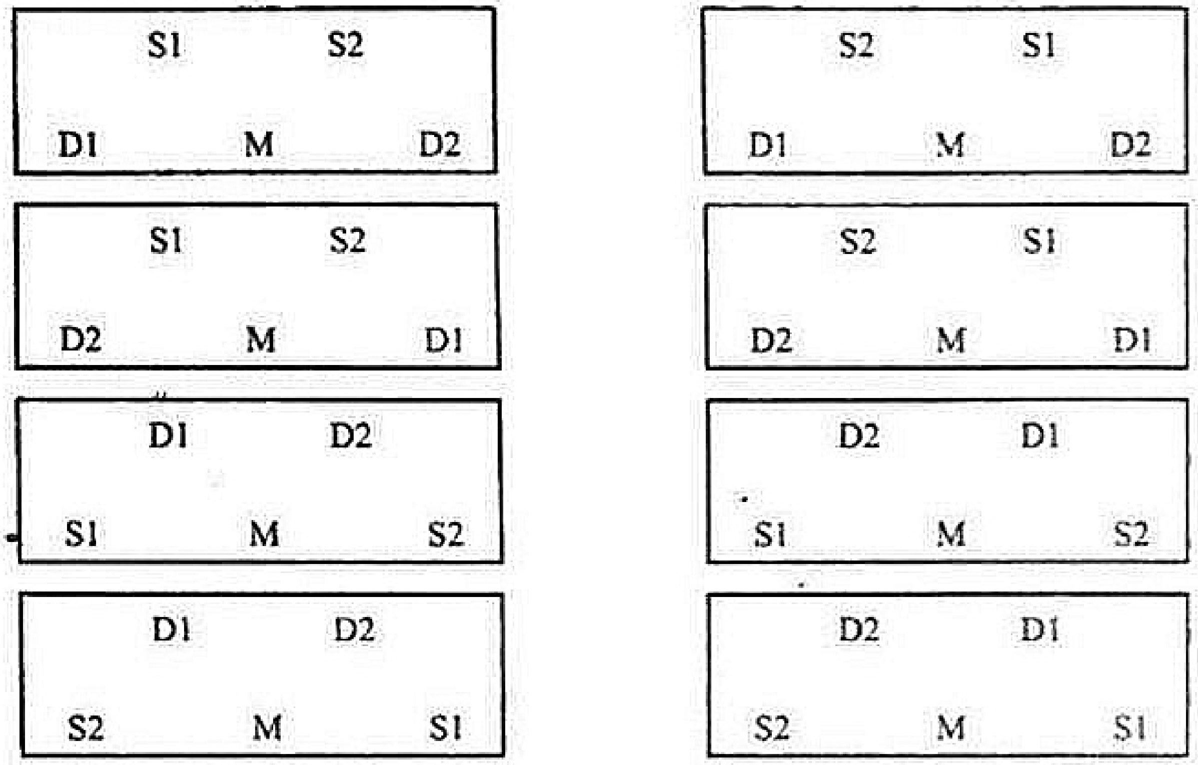

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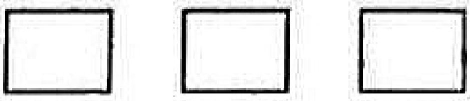

Fig. 9.3

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The argument is the same as in the previous example and is standard for all such situations. We imagine three blank spaces numbered 1, 2, 3 from left to right. It is left to us to fill the spaces with the symbols *A*, *C*, *T*, *G*. Space 1 can be filled up by any one of the 4 letters and so in four ways. Since repetitions are allowed, the second space can also be filled up in four ways. Thus the first two spaces can be filled up in $4 \times 4 = 16$ ways by the fundamental principle of counting. Now the 3rd space can be filled up in four ways. For each of the 16 ways of filling up the first two spaces there are four ways of filling up the third space. So the fundamental principle applies. Thus the three spaces can be filled up in $16 \times 4 = 64$ ways. Hence there are 64 three-letter DNA chains.

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$$m_1 \, m_2 \ldots m_n \text{ ways.} \tag{1}$$

Note that 'independent events' means (in this larger setup) the happening of one does not affect any of the others. In the above example of the three-letter DNA-chain the filling up of the 2nd space, for instance, has nothing to do with the filling up of either the first space or the third space.

We shall now continue with the last example, the DNA-chain, to talk about PERMUTATIONS. A permutation of a set $\{x_1, x_2, \ldots, x_n\}$ is a rearrangement of the symbols. In other words we consider the original collection as our ordered arrangement

$$x_1, x_2, \ldots, x_n$$

(as if people are sitting in a line for a photograph) and we now consider any possible rearrangement. Thus, in the case of the symbols *A*, *C*, *T*, *G* of the DNA-chain the arrangements (4–letter DNA-chains)

ACTG, *CAGT*, *TGCA*, *GTCA*, ...

and many more such rearrangements of all the symbols are just 'permutations' of the four symbols. Note that we are not permitting any repetitions now. (The example of the photograph situation will come in handy now). Given the four symbols we may consider 3–permutations of the four symbols, by taking only 3 symbols out of the four and 'permuting' them. In the case of the symbols A, C, T, G of the DNA-chain, we will now get 3-letter DNA-chains, but now without repetitions. Suppose we ask the question: How many such are there?

We reason as follows. Start with three blank spaces numbered 1, 2, 3 from left to right. Space 1 can be filled up by any one of the four symbols A, C, T, G, therefore there are four ways of filling up the first space. But having filled up the first space, there are only three symbols left — since we have used up one symbol (whatever that be). So the 2nd space has only three choices. In other words, after filling up the first space, there are only three ways of filling up the 2nd space. Note that we are here talking of two events, *viz.*,

Event E: Filling up of space 1 by any one of A, C, T, G (and therefore can happen in any one of four ways); and

Event F: After filling up the first space, the filling up of the 2nd space by any one of the remaining 3 symbols (and therefore can happen in any one of three ways).

These two events are independent; because the choice among the three ways of happening of Event F is not dependent on Event E.

Note. This subtle point in the argument has to be carefully understood. In the definition of 'Event F', the words 'after filling up the first space' are important. Which three symbols contribute to event F may be dependent on event E, but the event F itself is the choice of one of the three symbols constituting event F. This choice is not influenced by event E.

So the two events can together happen in any one of $4 \times 3 = 12$ ways. Having filled up the first two spaces by two of the symbols we have now to fill-up the third space by any one of the remaining two symbols. The third event, call it event G, is therefore the filling up (after the filling up of the first two spaces) of the 3rd space. This can happen in any one of two ways. And as before the third event G is independent of the other two events. So the three events can together happen in $4 \times 3 \times 2 = 24$ ways. In other words the three-letter DNA-chains, without repetitions, are 24 in number. Or, what is the same thing, there are 24, 3-permutations of four symbols A, T, C, G. These are listed below (in a systematic way):

ATC
ACT
ATG
AGT
ACG
AGC

TCG
TGC
TCA
TAC
TGA
TAG

CGA
CAG
CGT
CTG
CAT
CTA

GAT
GTA
GAC
GCA
GTC
GCT

The 3-permutations of four letters are also called **'permutations of four letters taken three at a time'.**

Generalising the above argument we are able to prove the following Theorem:

Theorem 1. The number of permutations of n objects taken r at a time is

$$n\,(n-1)\,(n-2)\,...\,(n-r+1). \tag{2}$$

Proof. We shall imagine r ordered blank spaces that have to be filled up by any of n letters (standing for the n objects) without repetitions.

The first space can be filled up by any one of the n letters and therefore in n ways. Having filled up the first space, we may fill-up the 2nd space by any one of the remaining $n-1$ letters, *i.e.*, in $(n-1)$ ways. Thus the first two spaces can together be filled up in $n(n-1)$ ways. Now there remain $(n-2)$ letters and so the filling up of the 3rd space has $(n-2)$ choices. Thus, after the filling up of the first two spaces by two letters, the third space can be filled up in $(n-2)$ ways. Hence the first three spaces can together be filled up in and so on. When it is a question of the first 3 spaces the product is made up

$$n(n-1)(n-2) \text{ ways}$$

of 3 factors: n, $n-1$ and $n-2$. So in the case of the number of ways of filling up the r spaces, the product is made up of r factors

$$n, n-1, n-2, \ldots, n-(r-1).$$

Thus the required number of ways is

$$n(n-1)(n-2)\ldots(n-r+1).$$ ☐

Note. The formalisation of the above argument in the form of a rigorous proof using mathematical induction is left as an exercise for the student.

NOTATION. The symbols

$$^nP_r;\ P(n, r);\ n_r$$

are all used by mathematicians to denote the (above) number of permutations of n things taken r at a time. The first of these is rather out of fashion. We shall therefore use either $P(n, r)$ or n_r in the sequel.

Thus

$$P(n, r) = n_r = n(n-1)(n-2)\ldots(n-r+1). \tag{3}$$

Illustration. Going back to the previous Example we see that the number of three-letter DNA-chains without repetitions is

$$P(4, 3) = 4_3 = 4(4-1)(4-2)$$
$$= 4 \times 3 \times 2 = 24.$$

Note that

$$P(n, 1) = n$$
$$P(n, 2) = n(n-1)$$

and so on. Also note that $P(n, 0)$ does not make sense if we mean by it the number of permutations of n things taking none at a time. But we interpret this as 1 since there are no different ways of taking none of the things at a time. So $P(n, 0)$ is stipulated as equal to 1.

The number $P(n, n)$ is a very important number and we take it in the next paragraph.

NUMBER OF ALL PERMUTATIONS OF n OBJECTS

If we are interested in 4-letter DNA-chains (without repetitions) we are actually looking for all permutations of four objects, taking all at a time.

This would be

$$P(4, 4) = 4_4 = 4(4-1)(4-2)(4-3)$$
$$= 4 \times 3 \times 2 \times 1 = 24$$

In general,

$$P(n, n) = n_n = n(n-1)(n-2) \ldots (n-(n-1))$$
$$= n(n-1)(n-2) \ldots 3.2.1.$$

Thus we note that the number of all permutations of n symbols is

$$n(n-1)(n-2) \ldots 3.2.1.$$

This number is so often in use not only in counting problems but in other parts of mathematics as well, that there is a separate symbol and nomenclature for it.

Definition 1. The number

$$n(n-1)(n-2) \ldots 3.2.1$$

is called **'Factorial n'** and is denoted by

$$\underline{|n} \text{ or } n!$$

The notation $\underline{|n}$ is rather old-fashioned. We shall consistently be using $n!$ for factorial n.

Thus we have

$$P(n, n) = n_n = n! \tag{4}$$

Note that

$$1! = 1; \qquad 2! = 2.1 = 2;$$
$$3! = 3.2.1 = 6; \qquad 4! = 4.3.2.1 = 24;$$
$$5! = 120; \qquad 6! = 720;$$
$$7! = 5040 \text{ and so on.}$$

Also $P(n, r)$

$$= n(n-1)(n-2) \ldots (n-r+1)$$
$$= \frac{n(n-1)(n-2) \ldots (n-(r-1))(n-r)(n-(r+1)) \ldots 3.2.1}{(n-r)(n-(r+1)) \ldots 3.2.1}$$
$$= \frac{n!}{(n-r)!} \tag{5}$$

Illustration. $P(7, 3) = 7.6.5$

$$= \frac{7.6.5.4.3.2.1}{4.3.2.1} = \frac{7!}{4!}.$$

We shall now give several examples of the use of the counting numbers $P(n, r)$.

EXAMPLE 6. *How many permutations are there of the letters of the word "ENGLISH"?*

SOLUTION. There are 7 letters. They can be permuted or rearranged in $P(7,7) = 7$ $= 7!$ ways. The answer is 5040.

EXAMPLE 7. *How many of the permutations of the word "ENGLISH" will (i) start with E ? (ii) end with H? (iii) start with E and end with H?*

SOLUTION. Though this is not a direct application of the formula for n_r as in the previous example, a little study of the problem will show the connection.

In part (*i*) we want the permutation to start with '*E*'. In other words, of the 7 blank spaces we have to fill-up with the letters of the word "ENGLISH", the first space allows no choice. It has to be filled up with '*E*' only. So let us place '*E*' in the first space. Having done this, we are left with only 6 letters and there are 6 places for them. This allows us 6! choices, since 6 letters can be permuted in 6! ways. Thus the answer to this part is 6! = 720.

In part (*ii*) we have to keep '*H*' in the last blank space. Again this leaves a remainder of 6 letters and 6 places for them to go into, this can be done in 6! = 720 ways. So the answer to this part is also 720.

In part (*iii*) we have to keep '*E*' in the first place and '*H*' in the last place. Having done this, we are left with 5 letters and 5 spaces only:

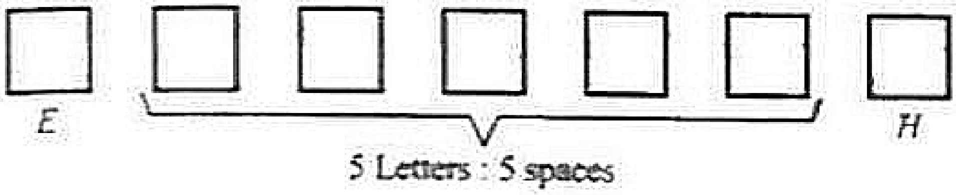

Fig. 9.5

this can be done in 5! ways. Each such way of arranging these 5 letters along with '*E*' in the first place and '*H*' in the last place gives us a required permutation for part (*iii*) of the problem. Hence the answer to this part is 5! = 120.

EXAMPLE 2. (A second look). Going back to Example 2 we note that we can now shorten the number of steps. The problem is a question of deciding on two mutually exclusive alternatives as follows:

Either Sons in the front row (and therefore the Daughters-in-law to occupy the 2nd row)

Or Daughters-in-law in the front row (and therefore the sons to occupy the 2nd row).

Taking alternative 1, we see that there are 2 chairs for the 2 sons. So they can be seated in 2! = 2 ways. For each such way of seating the sons, the two daughters-in-law can be positioned in the 2nd row, in the two standing places, in 2! = 2 ways. Thus the first alternative can be accomplished in 2 × 2 = 4 ways.

By a similar argument, the 2nd alternative can be accomplished in 2 × 2 = 4 ways.

The two alternatives are mutually exclusive. So the total number of ways of arranging them all for the photograph is

$$\begin{aligned} &= \text{the no. of ways for Alternative 1} \\ &\quad + \text{the no. of ways for Alternative 2} \\ &= 4 + 4 = 8. \end{aligned}$$

Note 1. Note the distinction between 'mutually exclusive events' and 'independent events'. The latter gives the product rule as per the fundamental principle of counting. The former gives tre addition rule as in the above example. In general the number of ways in which two mutually exclusive events can happen (not together, but severally) is the sum of the ways in which each of them can happen

Note 2. The working of Example 2 (Relook) above is simpler than that in the working of the same example earlier. This is because we have used the formula for the number of permutations. At the same time we had also to use the fundamental principle of counting. In the earlier working we had only the fundamental principle in our hands as a tool. As we develop the subject we will learn to use more and more formulae but learn also to combine them ingeniously and judiciously in order to reduce the actual burden of brute force calculation. It is therefore necessary in this parl of mathematics to keep working a large number of problems.

EXAMPLE 8. *How many ways are there to select an ordered set of 3 letters from the set {a, b, c, d, e, f}.*

SOLUTION. This is equivalent to the number of 3-permutations from 6 objects. So it equals $P(6, 3) = 6.5.4 = 120$.

EXAMPLE 9. *If all the permutations of the letters of the word "UNIVERSAL" are arranged (and numbered serially) in alphabetical order as in a dictionary,*

- *(i) What is the first word?*
- *(ii) What is the last word?*
- *(iii) How many words are there under each letter?*
- *(iv) What is the serial number of the word: RIVENSULA? UNIVERSAL?*

SOLUTION. (*i*) AEILNRSUV is the first word.

(*ii*) The reverse of the above; viz., VUSRNLIEA is the last word.

(*iii*) 8! = 40320; because we can keep one letter fixed (i.e., as the first letter of the word) and permute only the remaining 8 letters.

(*iv*) To calculate the serial number of the word RIVENSULA in the alphabetical order, we have to systematically exhaust the words that go before he specific word. This is done in the following table which is self explanatory.

TABLE OF COUNTING THE WORDS WHICH APPEAR BEFORE RIVENSULA

Words beginning with	*No. of such words*	*Count for totalling*
A	8!	40320
E	8!	40320
I	8!	40320
L	8!	40320
N	8!	40320
RA	7!	5040
RE	7!	5040
RIA	6!	720
RIE	6!	720
RIL	6!	720
RIN	6!	720
RIU	6!	720
RIVA	5!	120
RIVEA	4!	24
RIVEL	4!	24
RIVENA	3!	6
RIVENL	3!	6
RIVENSA	2!	2
RIVENSL	2!	2
RIVENSUA	1!	1
		2,16,185

The next word after the above list is RIVENSULA. So the serial number required is 2,16,816.

Calculation of the serial number of the word 'UNIVERSAL' is left as an exercise. The answer is: 3,04,481.

EXAMPLE 10. *Consider the set {a, b, c, d, e}. How many three-letter words can be made out of them, with or without meaning? How many of these will have at least one vowel in them? Answer these questions for both cases when repetitions of letters are allowed and when repetitions of letters are not allowed?*

SOLUTION. Case 1. Repetitions allowed.

Number of all three-letter words = 5 × 5 × 5 = 125; because each space in the three-letter word can be filled up by any of the five letters.

To count the number of such words with vowels in them, let us calculate the complementary number, *viz.*, the number of words without any vowel in them.

This number is 3 × 3 × 3 = 27, because each blank space can be filled up only by *b*, *c* or *d*. Hence the number of words which contain at least one vowel, is 125 –27 = 98.

Case 2. Repetitions not allowed.

Number of such 3-letter words

$$= \text{Number of 3-permutations of 5 objects}$$
$$= P\,(5,\,3) = 5.4.3 = 60.$$

Number of these words which have at least a vowel in them = $60 - P\,(3,\,3) = 60 - 6 = 54$.

EXAMPLE 11. *In how many ways can 3 objects be distributed in 5 boxes so that no two objects go to the same box? How will the answer change if there is no restrictive condition for the distribution? Generalise this problem to the situation of n objects and m boxes.*

SOLUTION. Each distribution of 3 objects into 5 boxes with the restrictive condition, may be considered as a 3-permutation of 5 symbols thus: Call the objects *a*, *b*, *c*. Call the boxes B_1, B_2, B_3, B_4, B_5.

Suppose

Object *a* goes into B_1
Object *b* goes into B_2
Object *c* goes into B_3.

Let us say that permutation $B_1\ B_2\ B_3$ represents this distribution. Conversely, let $B_3\ B_2\ B_5$ be a 3-permutation. The corresponding distribution shall be:

Object *a* goes into B_3
Object *b* goes into B_2
Object *c* goes into B_5.

Thus there is a 1 – 1 correspondence between distributions of 3 objects into 5 boxes with the condition stated in the problem and 3-permutations of 5 symbols. Therefore the required, number of such distributions is $P\,(5,\,3) = 5.4.3 = 60$.

In the case of *n* objects and *m* boxes the answer is

$$P\,(m,\,n) = m\,(m-1)\,...\,(m-n+1).$$

If the restrictive condition is relaxed, the problem becomes easy. For each object there are five choices and so the answer is $5 \times 5 \times 5 = 125$. In the general case the answer is $m \times m \times ... \times m$ (n times) *i.e.*, m^n.

EXAMPLE 12. *In how many ways can the letters of the word MOM be permuted among themselves ?*

SOLUTION. The number of letters to be permuted appears to be three. The temptation is to say that the answer is $3! = 6$. But note that the letters are M, M and O and so one letter is repeated. In other words there is one letter appearing twice and another letter appearing once. Essentially there are only 2 letters to be permuted. The answer however is not $2! = 2$.

In order to have a proper grasp of the nature of the problem, let us temporarily distinguish the two M's in the problem by M_1 and M_2. Then the letters M_1, M_2 and O can be permuted in the following six ways.

$$M_1M_2O \qquad OM_1M_2 \qquad M_1OM_2$$
$$M_2M_1O \qquad OM_2M_1 \qquad M_2OM_1$$

If we identify M_1 and M_2 and call it M we have only the following:

$$MMO \quad ; \qquad OMM \quad ; \qquad MOM$$

Thus when M_1 and M_2 are distinct, there are six permutations; when M_1 and M_2 are not distinct there are only 3 permutations. There is a reduction of the number by a factor of 2. This reduction happens as follows.

M_1 and M_2 occupy	No. of permutations when M_1 and M_2 are distinct	M_1 and M_2 are identical
the first two positions	2	1
the first and third positions	2	[illegible]
the second and third positions	2	1

Whenever M_1 and M_2 occupy two fixed positions, they themselves can be permuted among themselves to give $2! = 2$ permutations. This number $2!$ is what gives rise to the number 2 in the 2nd column of the Table above. Since each such pair of permutations coalesce into a single permutation as M_1 coincides with M_2, thus giving rise to 1 in the 3rd column, the factor of 1/2 appears as the factor of reduction from the 'distinct' case to the 'nondistinct' case. In the general case, the situation is going to be similar as we shall see presently.

EXAMPLE 13. *In how many ways can the letters of the word INDIA be permuted among themselves ?*

SOLUTION. Note that the letter I appears twice among the five letters. So if we consider the two I's as I_1 and I_2 the total number of permutations would be $5! = 120$. Of these every pair of permutations where I_1 and I_2 have interchanged their positions, would become the same permunation when we write I for both I_1 and I_2. Thus there is a reduction by a factor of 1/2 from 120. The answer is $1/2 \times (120) = 60$

Note that the factor 1/2 is actually $1/2!$. The $2!$ comes from the fact that once the positions of I_1 and I_2 are fixed they can be permuted among themselves in $2!$ ways.

EXAMPLE 14. *In how many ways can the letters of the word DADDY be permuted among themselves ?*

SOLUTION. Here three letters are alike: viz. D, D and D. Thus of the 120 permutations which would otherwise have been there with D_1, D_2, D_3, Y and A every set of permutations which only permute D_1, D_2 and D_3 among themselves will coalesce into a single permutation when D_1, D_2, D_3 become identical.

Thus for instance, the six permutations

$$\begin{gathered} D_1 D_2 A D_3 Y \\ D_1 D_3 A D_2 Y \\ D_2 D_3 A D_1 Y \\ D_2 D_1 A D_3 Y \\ D_3 D_1 A D_2 Y \\ D_3 D_2 A D_1 Y \end{gathered}$$

coalesce into a single permutation $DDADY$. The reduction therefore is by a factor of $1/3!$. Hence the answer is $1/6 \times (120) = 20$.

We can now generalise the argument of the last three examples and prove the following Theorem:

Theorem 2. If n things are to be permuted and of these n things, if n_1 things are alike of one kind and the remaining n_2 things are alike of a different kind, then the number of distinct permutations of the n things is

$$\frac{n!}{n_1!\,n_2!} \tag{6}$$

Proof. Taking any one such required permutation and without altering the positions of the n_1 things except by a permutation among themselves and similarly permuting the n_2 things among themselves, we can produce $n_1!\,n_2!$ permutations. These latter are counted as distinct in the count of the total on $n!$ permutations of n distinct things. They are counted as the same when we identify the n_1 things among themselves and the n_2 things among themselves. This is like identifying the D_1, D_2, D_3 among themselves in the last example. Hence the total number of required type of permutations is

$$\frac{n!}{n_1!\,n_2!}$$ ❑

and this completes the proof of the Theorem.

Theorem 2 can be extended to accommodate

n_1 things are alike of one kind
n_2 things are alike of another kind
....
n_k things are alike of another kind.

Here the final number would be

$$\frac{n!}{n_1!\,n_2!\dots n_k!}, \text{ (where } n_1 + n_2 + \dots + n_k = n\text{)} \tag{7}$$

Remark. This is called the **multinomial coefficient.**

In Ex. 14, $n_1 = 3$, $n_2 = 1$, $n_3 = 1$ and $n = 5$.
Ex. 13, $n_1 = 2$, $n_2 = 1$, $n_3 = 1$, $n_4 = 1$ and $n = 5$.

EXAMPLE 15. In two-dimensional space the point (a, b) is said to be a lattice point if a, b are integers. Two lattice points (a, b) and (c, d) are neighbours if they agree in one of the coordinates and differ in the other coordinate by 1. Every pair of neighbours is connected by a directed line with the direction coinciding with the positive direction of the coordinate axis to which it is parallel. In how many ways can one move from $(0, 0)$ to (a, b) along the directed paths ?

To move from $(0, 0)$ to (a, b) one has to move through a steps of 1 unit each along the x-axis and b steps of 1 unit each along the y-axis. Each such move can be written as a permutation

$$x\,x\,y\,x\,y\,y\,x\,\ldots\,x\,x\,y$$

where there are a 'x's and b y's. Here each x represents a 1–unit move along the x–axis and y represents a 1–unit move along the y–axis. Thus there are as many moves from $(0, 0)$ to (a, b) as there are permutations of the above kind. These permutations are nothing put permutations of $(a + b)$ things, 'a' of which are of one kind (*viz.*, they are x) and b of which are of a different kine (*viz.*, they are y). So the required number is the number of all such permutations.

This is $$\frac{(a+b)!}{a!\,b!}$$

EXAMPLE 16. *Show that* $(6!)^{5!}$ *is a divisor of* $(6!)!$ *(Recall Problem No. 27 at the end of Chapter 2).*

SOLUTION. Consider (6!) objects which are grouped into 5! groups, each group containing 6 objects. Note that $6 \times 5! = 6!$. Let each group be considered a separate kind, but the members within the group as identical kind. Then if we look at permutation of the 6! objects, since it is divided into 5! groups of different kinds, each containing 6 objects of the same kind the total number of permutations is

$$\frac{(6!)!}{6!6!\ldots6!}$$

there being 5! factors in the denominator, each equal to 6!. This number is therefore

$$\frac{(6!)!}{(6!)^{5!}}$$

Since this counts the number of permutations of the 6! objects, it is an integer. This incidentally means $(6!)^{5!}$ is a divisor of $(6!)!$

EXAMPLE 17. *In how many ways can you permute the letters of the word VIVEKANANDA?*

SOLUTION. Such permutations of the word are called anagrams of the word.

There are 2 V's, 3 A's 2 N's and 1I, 1E, 1K, 1D.

So the answer is $$\frac{11!}{2!3!2!}$$

$$= 11.10.9.8.7.6.5.$$

Important Note Regarding Exercise 9.1

At this point we close this section on permutations. In the manner of the precedence set up in the earlier chapters a set of problems under the heading Exercise 9.1 would have appeared here. But the subject of Combinations, which is the topic of the next section, is so much interwoven with the topic of Permutations that a student, when confronted with problems in this part of Mathematics has usually a hard time deciding whether he has to do Permutations or Combinations. Keeping this in mind, we have transferred all problems which would have normally appeared here to Exercise 9.2 at the end of Section 9.2. So the Exercise 9.1 for the student reader at this point is to move to Exercise 9.2 and pick up the problems which deal only with permutations and work them! This sorting out is itself an educative challenge.

9.2 COMBINATIONS

So far we have always concerned ourselves with permutations or arrangements, the order in which the symbols or the objects appeared in our selection mattered. Taking the example of the 3-letter DNA-chain (without repetition), we see that there are four bases:

$$A, T, C, \& G$$

from which we have to take three letters and arrange them. We know the answer is 24 (from page 363). But let us do a slow-motion experiment now. Take any three letters out of the four. Suppose they are A, T, C. These three letters can be permuted among themselves, in $3! = 6$ ways. The actual permutations are:

$$A\,T\,C, A\,C\,T, T\,C\,A, T\,A\,C, C\,A\,T, C\,T\,A.$$

These six are the only three-letter DNA chains (without repetitions) if the letters are restricted to A, T and C.

If we take another selsction of three bases, say A, T and G, we have similarly six permutations of the three letters:

$$A\,T\,G, A\,G\,T, T\,G\,A, T\,A\,G, G\,A\,T, G\,T\,A.$$

Again take still another selection: say, A, C and G. These three will give rise to six permutations:

$$A\,C\,G, A\,G\,C, C\,G\,A, C\,A\,G, G\,A\,C, G\,C\,A.$$

One more selection of three letters is possible: T, C, G. These three will give the following six permutations among themselves:

$$T\,C\,G, T\,G\,C, C\,G\,T, C\,T\,G, G\,T\,C, G\,C\,T.$$

A little thinking and experimentation will tell us that there are no more selections of three letters from A, T, C, G. In fact one way to have a confirmation of this is to argue as follows. If we have to select three letters out of four, each such selection will omit one letter; thus

The selection $\{A$, T and $C\}$ omits G;
the selection $\{A$, T and $G\}$ omits C;
the selection $\{A$, C and $G)$ omits T;
and the selection $\{T$, C and $G\}$ omits A.

There is no other way to omit a letter and so there are no other 3-letter selections. Thus there are four 3-letter selections *viz.*, $\{A, T, C\}$; $\{A, T, G\}$; $\{A, C, G\}$ and $\{T, C, G\}$.

Each gives rise to six 3-letter permutations and this adds up to 24 3-letter permutations. In other words the 24 3-letter permutations partitions into 4 groups. Each group contains 6 permutations of the same three letters.

Suppose the question was: Given four letters A, T, C and G how many 3-letter selections can be made? Here selection (or combination) means, the order in which the letters appear is irrelevant to the issue. Of course we know the answer is 4, but let us not jump too soon to the answer. We know there are 24 3-letter permutations because it is just $4_3 = 4.3.2 = 24$. We know also each 3-letter selection will give 6 3-permutations, because it is just $3_3 = 3! = 3.2.1 = 6$. So the set of 24 permutations will partition itself into a certain number of groups, each group containing 6 permutations (of the same 3 letters) and each group arises from a selection of three letters. This argument leads us to conclude that there are $24 \div 6 = 4$ such selections! Consult the following organisation of the 24 permutations in order to understand the above argument.

TABLE OF 24 3-LETTER PERMUTATIONS ORGANISED INTO GROUPS CONTAINING SAME THREE LETTERS

ATC	*ATG*	*ACG*	*TCG*
TAC	*TAG*	*CAG*	*CTG*
ACT	*AGT*	*AGC*	*TGC*
CAT	*GAT*	*GAC*	*GTC*
TCA	*TGA*	*CGA*	*CGT*
CTA	*GTA*	*GCA*	*GCT*

Thus we may say, the number of combinations (= selections) of 4 letters taken 3 at a time is

$$\frac{(4)_3}{(3)_3} = \frac{4.3.2}{3.2.1} = 4.$$

We shall quickly discuss one more illustration before we take up the general case of r-letter combinations out of n letters. Let $r = 2$ and $n = 5$. We are interested in selecting 2 objects from a set of 5 objects. Just as a concrete example, consider 5 Tennis players out of which we have to select two, say, for a demonstration match. In how many ways can this be done?

Let the five players be named A, B, C, D and E. If two players are to be selected and the order mattered, then the answer would be

$$5_2 = 5.4 = 20.$$

But in this problem the order does not matter. Each selection of a pair of players, say, A and C would have been counted in the 20 above, once as AC and once again as CA that is, two times. This number 2 is actually the number of permutations of the two letters (or players) among themselves; so it is actually $2_2 = 2! = 2$. So the number of 2-player selections would be $20 \div 2 = 10$. These are clearly:

$$AB, AC, AD, AE, BC, BD, BE, CD, CE \text{ and } DE.$$

The passage to the general case is clear now. We formalise this in the following theorem and its proof.

Theorem 3. The number of combinations of n symbols taken r at a time is

$$\frac{n_r}{r!} \tag{8}$$

Proof. Combination means a selection in which order does not matter. On the other hand, if order mattered, the resulting ordered selection is called a permutation.

The number of such permutations, namely r-permutations of n letters, is already known to be

$$n_r = \frac{n!}{(n-r)!}$$

In this count of the total number of r-permutations, each r-selection is counted $r!$ times, because each r-selection contributes $r!$ permutations among themselves. So the actual number of r-selections is the above number divided by $r!$; in other words, it equals

$$\frac{n_r}{r!} \text{ i.e., } \frac{n!}{r!(n-r)!}$$ ❑

Notation. This number is important for many calculations. It is denoted by the symbol

$$\binom{n}{r} \text{ or } {}^nC_r$$

We shall use the former symbol always. Thus,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{9}$$

Illustrations

$$\binom{5}{2} = \frac{5!}{2!3!} = \frac{5.4.3.2.1}{2.1.3.2.1} = 10.$$

EXAMPLE 1. *How many diagonals are there in a convex seven-sided polygon (called a heptagon)? Note that a diagonal is a line joining any two vertices which are not adjacent.*

SOLUTION. Two vertices can be selected from seven vertices in

$$= \binom{7}{2} = \frac{7!}{2!5!} = 21 \text{ ways}$$

Each such pair will give a line, but not always a diagonal since the pair could be a pair of adjacent vertices. But if they are adjacent vertices, the resulting line is actually a side of the polygon. There are clearly 7 such sides. The remaining lines must be diagonals. Thus there are 21 − 7 = 14 diagonals.

EXAMPLE 2. *In how many ways can we form a committee of three from a set of 10 men and 8 women, such that our committee consists of at least one woman?*

SOLUTION. Whenever the requirement is that there should be at least something happening, it is a piece of strategy to calculate the number associated with the complementary happening. Here, for example, we want at least one woman member in our committee. What if we calculated the number of ways in which we could form a committee with no woman member in it? This means all the 3 members have to come from the set of 10 men only. Out of 10 men we can select three in

$$\binom{10}{3} = \frac{10!}{3!7!} = \frac{10.9.8}{1.2.3} = 120$$

ways. So this is the complementary number. This has to be subtracted from the total number of ways in which a committee of three can be formed, without any man-woman restriction. There are 18 people. We want to select three. This can be done in

$$\binom{18}{3} = \frac{18.17.16}{1.2.3} = 816$$

ways. Thus the required number is 816 – 120 = 696. This is the number of ways in which a 3-member committee can be formed from 10 men and 8 women, with at least one woman member in the committee.

An alternative method: In this method there is no ingenuity required. It is a straightforward brute-force calculation. We want a three-member committee with at least one woman. So let us calculate

(*i*) the number of ways in which 3-member committees could be formed with precisely one woman member in it;

(*ii*) the case of 2 woman-members in the committee; and

(*iii*) the case of 3-woman-members (*i.e.*, all woman-committee) and then add the three numbers.

Answer to (*i*) is obtained as follows. To select one woman, we have to do it from the set of 8 women. This can be done in $\binom{8}{1} = 8$ ways. Having done this, to get the two remaining members of the committee we have to select 2 men from the 10 men. This can be done in $\binom{10}{2} = \frac{10.9}{1.2} = 45$ ways.

These two selections (of men on the one side and the woman on the other side) are independent. So they can together be done, that is, the 3-member commttee (with precisely one woman) can be formed in 8 × 45 = 360 ways.

Answer to (*ii*) is obtained by selecting 2 women out of the eight available — this can be done in $\binom{8}{2} = \frac{8.7}{1.2} = 28$ ways and then by selecting 1 man from the set of 10 men — this can be done in $\binom{10}{1} = 10$ ways — and then combining the two; and this can be done in 28 × 10 = 280 ways. So answer to (*ii*) is 280.

Answer to (*iii*) is easy. We want an all-woman three-member committee. There are 8 women available. So this can be done in $= \binom{8}{3} = \frac{8.7.6}{1.2.3} = 56$ ways.

Adding the three numbers, we get the required number as

$$360 + 280 + 56 = 696.$$

EXAMPLE 3. *Consider the set of five digits*

$$\{1, 3, 5, 7, 9\}$$

A 3-permutation of this set is said to be 'increasing' if for every digit in the permutation, the succeeding digit is bigger. How many 3-permutations are 'increasing'?

SOLUTION. For instance 3 5 9 is an increasing permutation; whereas 5 3 7 is not. Note that for every 3 digits out of the given set, there is only one permutation which is increasing. In other words there are precisely as many increasing permutations as there are subsets of 3 elements of the given set $\{1, 3, 5, 7, 9\}$. The number of such 5 subsets is just the number of combinations taken 3 at a time. So it is $= \binom{5}{3} = 10$.

These 10 increasing permutations are listed below:

135 ; 137 ; 139 ; 157 ; 159 ; 179 ; 359 ; 379.

Note. The method used in the above example is a fundamental characteristic of counting problems. Instead of counting the members of a set we count the members of a second set — which we know must have the same number of elements as the first set. The argument is more or less of the following type. Suppose we want to count the spokes of a wheel where each spoke is a radial line going from the centre of the wheel to a point in the circumference. Instead of counting the spokes we count the end-points (of the spokes) on the circumference. This works out because there is a one-one correspondence between the set of all the spokes and the set of all the end points. For each spoke there is an end point and for each end-point there is a spoke. See Fig. 9.6. This kind of matching between two sets is called a *one-one correspondence* between them. Whenever two sets are in one-one correspondence, they have the same number of elements. Instead of counting one of them we may count the other. This is what we did in the example. The set of increasing 3-permutations of $\{1, 3, 5, 7, 9\}$ and the set of 3-subsets of $\{1, 3, 5, 7, 9\}$ are in one-one correspondence. Therefore we counted the latter in order to count the former. In Mathematics counting problems very often use this strategy. In fact a good lot of ingenuity may be needed to construct a suitable set, which is in one-one correspondence with the set that we have to count, and which is comparatively easier to count. The next example illustrates such an ingenuity.

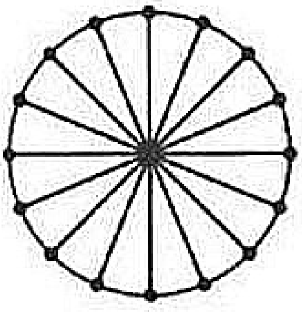

Fig. 9.6

EXAMPLE 4. *Ten books are arranged in a line on a bookshelf. In how many ways can we select four books such that no two consecutive books from the shelf are chosen?*

SOLUTION. Number the books as

1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

We are going to ingeniously construct a new set which matches in a one-one manner with the set of all required selections.

Suppose one such selection is:

$$A = \{1, 3, 7, 9\}.$$

We associate with this selection, the following binary sequence of 10 digits:

A' : 1 0 1 0 0 0 1 0 1 0

The construction of this sequence follows the simple rule: If the digit i appears in A, make the i^{th} digit in A' equal to 1, otherwise the i^{th} digit shall be zero. In the above selection, 1, 3, 7 and 9 appear in A. Therefore the 1st, 3rd, 7th and 9th digits in the 10-digit sequence A' will be 1's and all others, are zero. This rule produces a unique 10-digit sequence A' corresponding to A. Note that A' does not have two 1's consecutive. This is because the selection A does not have consecutive digits in it; and this is in pursuance of the very requirement of selection that no two consecutive books are chosen from the shelf. Thus, to each selection of books satisfying the requirement there is a 10-digit sequence of six 0's and four 1's with the property that two 1's do not appear together.

Conversely if we had a 10-digit sequence of six 0's and four 1's with the property that two 1's do not appear together, *e.g.*, the sequence: 0100100101

we can associate with this, uniquely, a selection of books of the required kind. *e.g*. The selection corresponding to the above sequence will be:

{2, 5, 8, 10}.

It is clear thus there is a one-one correspondence between the set of book selections required and the set of 10-digit binary sequences of six 0's and four 1's with the additional property that no two 1's appear together. So we now count the latter set!

Even for this we need to do a little artifice. (*Reader, notice that in this subject tricks and artifices abound; that is the beauty and challenge of this branch of Mathematics!*). We need to count 10-digit binary sequences with six 0's and four 1's. The zeros may appear in succession but the ones should not. So we first place the six zeros in a line, with an empty space between each successive pair of zeroes and a space each at the beginning and at the end, thus:

. 0 . 0 . 0 . 0 . 0 . 0 .

Here each dot represents a vacant space. We may fill up four of these vacant spaces by 1's. If we ignore the remaininig vacant spaces, this will give us a 10-digit sequence of the required variety: *e.g*.

1010000101.

How many such sequences can be formed?

Clearly as many as there are ways in which we can choose 4 vacant spaces from the 7 vacant spaces. This number we know is $\binom{7}{4}$; and this is the answer to the problem, *viz*.,

$$\binom{7}{4} = \frac{7.6.5.4}{1.2.3.4} = 35.$$

Just for the sake of completeness and for purposes of clarity, we present below the 35 10-digit sequences and the corresponding four-book selections.

(In this table, '*X*' denotes the number ten: '10')

1010101000	1357	1000100101	158X
1010100100	1358	1000010101	168X
1010100010	1359	0101010100	2468
1010100001	135X	0101010010	2469
1010010100	1368	0101010001	246X
1010010010	1369	0101001010	2479
1010010001	136X	0101001001	247X
1010001010	1379	0101000101	248X
1010001001	137X	0100101010	2579
1010000101	138X	0100101001	257X
1001010100	1468	0100100101	258X
1001010010	1469	0100010101	268X
1001010001	146X	0010101010	3579
1001001010	1479	0010101001	357X
1001001001	147X	0010100101	358X
1001000101	148X	0010010101	368X
1000101010	1579	0001010101	468X
1000101001	157X		

EXAMPLE 5. *How many distinct solutions are there in non-negative integers of*

$$x + y + z + w = 10$$

for the variables x, y, z, w?

SOLUTION. This is another problem where the same strategy as in the previous one is going to be useful. In order to motivate the working, let us take a simpler illustration of the same type of problem. Consider the equation

$$x + y = 5.$$

We can tabulate the distinct solutions of this equation as follows:

x	y	*Binary sequence*					
5	0	1	1	1	1	1	0
0	5	0	1	1	1	1	1
4	1	1	1	1	1	0	1
1	4	1	0	1	1	1	1
3	2	1	1	1	0	1	1
2	3	1	1	0	1	1	1

The third column of the above table lists a binary sequence of 5 1's and one zero, each corresponding to one of the solutions in a unique way. The interpretation (and therefore die construction) of the binary sequence is clear. The 'zero' separates the x value and y value. The number of 1's that appear to the left of the zero represents the x value and the number of 1's that appear to the right of the zero represents the y value. There are no more solutions as can be seen from the listing. The set of solutions is in 1 – 1 correspondence with the set of permutations of five 1's and one zero. This number is the multinomial coefficient

$$\frac{6!}{5!1!}$$

which simplifies to 6. There are precisely 6 solutions to the equation $x + y = 5$ in non-negative integers.

Now taking the cue from this illustration, to count the solutions in non-negative integers of

$$x + y + z + w = 10$$

we count the permutations of ten 1's and three zeros. The number three is because we have to separate x and y, y and z and finally z and w each by a zero; thus,

1 1 0 1 1 1 1 0 1 1 1 1 0

would mean

$$x = 2,\ y = 4,\ z = 4 \text{ and } w = 0$$

and so on. The number of such permutations is

$$\frac{13!}{10!3!}$$

which is equal to

$$\frac{13.12.11}{1.2.3} \text{ i.e. } 286.$$

This is therefore the number of distinct solutions in non-negative integers of

$$x + y + z + w = 10.$$

Generalising this to $x_1 + x_2 + ... + x_n = N$ we record the generalisation as:

The number of non-negative integer solutions of $x_1 + x_2 + ... + x_n = N$ is

$$\binom{N+n-1}{n-1}$$

See also Example 12.

EXAMPLE 6. *In how many ways can four non-distinct objects be distributed into six distinct boxes, so that no box may contain more than one object? Generalise the situation for n objects with m boxes.*

SOLUTION. First let us make clear what 'distinct' and 'non-distinct' mean. 'Distinct boxes' means the boxes can each be named distinctly from every other box. In other words there are six entities namely

Box A, Box B, Box C, Box D, Box E and Box F.

Non-distinct objects means we know only there are so many objects (in this case, four) but we cannot distinguish between them. In other words, the objects are indistinguishable — they carry no numbers, no names, no labels.

Now let us first attempt the simpler case when the four objects are distinct (1, 2, 3, 4) and the six boxes are distinct (say A, B, C, D, E and F). This is the case of Example 11 of the previous section. We know that each such distribution of 4 objects into 6 boxes can be matched with a 4 permutation of the 6 boxes A, B, C, D, E, F. So there are precisely $P(6, 4) = \frac{6!}{2!} = 360$ such distributions.

Here, for instance, a distribution, say,

$$\begin{aligned} 1 &\longrightarrow \text{B} \\ 2 &\longrightarrow \text{A} \\ 3 &\longrightarrow \text{F} \\ 4 &\longrightarrow \text{D} \end{aligned}$$

gives the 4-permutation $B\ A\ F\ D$ of the six symbols A, B, C, D, E, F.

Now when we change the problem from 'distinct objects' to 'non-distinct objects', then it is not relevant to know what object goes into a box; it is only relevant to know whether an object goes into the box or not. Thus, if B, A, F, D are the boxes in which the objects go, no more information is needed nor is possible since the objects are indistinguishable. The fact that B, A, F, D are the four boxes chosen is the only relevant information. Thus each distribution corresponds to a choice of four boxes out of the six. There are as many distributions therefore as there are choices of four boxes out of the six. So the number of distributions is $\binom{6}{4}$ which is 15.

In general, if n non-distinct objects are distributed into m boxes such that no box has more than one object, the number of distributions is $\binom{m}{n}$ (Caution: $m \geq n$).

EXAMPLE 7. *Show that*

$$\binom{n}{r} = \binom{n}{n-r} \tag{11}$$

SOLUTION. Method 1. $\binom{n}{r}$ = is the number of selections of r things out of n things.

To select r things is the same as discarding $n - r$ things. For each selection of r things there is a discarding of the remaining $n - r$ things. The number of ways in which we can discard $n - r$ things is $\binom{n}{n-r}$.

Since the set of selections of r things and the set of discardings of $n - r$ things are in $1 - 1$ correspondence, the two numbers above are equal.

Method 2.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{12}$$

and
$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

Hence the two numbers are equal.

EXAMPLE 8. *Show that*

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad (1 \leq r \leq n).$$

As a corollary, show that $\binom{2n}{n}$ is always even.

SOLUTION. Method 1. $\binom{n}{r}$ is the number of combinations of n objects taken r at a time.

Fix an object, say a, out of the n obects. All the $\binom{n}{r}$ combinations can be grouped into (*i*) those that contain the object a and (*ii*) those that do not contain a.

To count the former, we have object a and we need only to choose $(r - 1)$ from the remaining $(n - 1)$. This number is therefore $\binom{n-1}{r-1}$.

To count (*ii*), we omit object a, and now we need to choose r objects from the remaining $(n - 1)$. So this number is $\binom{n-1}{r}$.

Hence the equation follows. The corollary follows by replacing r by n and n by $2n$.

Method 2. R.H.S.

$$= \frac{(n-1)!}{(r-1)!(n-1-r+1)!} + \frac{(n-1)!}{r!(n-1-r)!}$$

$$= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right]$$

$$= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{(r+n-r)}{(n-r)(r)}\right]$$

$$= \frac{n(n-1)!}{r(r-1)!(n-r)(n-r-1)!}$$

$$= \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

EXAMPLE 9. *Show that*

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k} \qquad \ldots 1 \le k \le r \le n.$$

SOLUTION. Method 1. L.H.S.

$$= \frac{n!}{r!(n-r)!} \times \frac{r!}{k!(r-k)!}$$

$$= \frac{n!}{(n-r)!(r-k)!k!}$$

$$= \frac{n!}{k!(n-k)!} \frac{(n-k)!}{(n-r)!(r-k)!}$$

$$= \binom{n}{k} \times \frac{(n-k)!}{(r-k)!(n-k-(r-k))!}$$

$$= \binom{n}{k}\binom{n-k}{r-k}.$$

Method 2. The proof will be clear if we first illustrate with, say, $n = 6$, $r = 5$ and $k = 3$. Let $\{a, b, c, d, e, f\}$ be the 6-element set. How many 5-subsets are there? There are $\binom{6}{5} = 6$ of them. From each of this six 5-subsets, suppose we form 3-subsets. Each 5-subset will give rise to $\binom{5}{3} = \binom{5}{2} = 10$ 3-subsets.

If we write them all down we thus have $6 \times 10 = 60$ 3-subsets. But the same 3-subset will come from different 5-subsets. *viz.*, $\{bcd\}$, for instance, will come, once from $\{b, c, d, e, f\}$ once from $\{a, b, c, d, e\}$ and once from $\{a, b, c, d, f\}$.

To make this calculation exact, we see that each 3-subset comes from as many 5-subsets as there are ways of choosing the remaining 2 elements from the remaining

$6-3=3$ elements. This is $\binom{3}{2}=3$. Thus each 3-subset occurs 3 times in this count.

There are $\binom{6}{3}=20$ 3-subsets. So the total count of all 3-subsets written down (with repetitions) in the above listing is $20 \times 3 = 60$.

Now let us take up the general case. Choose an r-subset. This can be done in $\binom{n}{r}$ ways. From each r-subset choose as many k-set as possible.

This can be done in $\binom{r}{k}$ ways. Thus a k-set can be arrived at by any one of $\binom{n}{r}\binom{r}{k}$ ways.

The same count can be had by looking at a single k-set which can be obtained in one of $\binom{n}{k}$ ways. But each of these may have come from an r-set, which can be obtained in $\binom{n-k}{r-k}$ ways of choosing the remaining $r-k$ elements from the remaining $n-k$ elements. So the above count of all k-sets written down is $\binom{n-k}{r-k}$. Hence the result.

EXMAPLE 10. (Vandermonde's Identity)

Prove $$\binom{n+m}{r}=\binom{n}{0}\binom{m}{r}+\binom{n}{1}\binom{m}{r-1}+\binom{n}{2}\binom{m}{r-2}+\dots+\binom{n}{r}\binom{m}{0}$$

SOLUTION. Let there be n boys and m girls. We want to choose a team of r persons, the boy-girl proportion allowed to be all possible cases. The r persons can be chosen from $n+m$ persons in $\binom{n+m}{r}$ ways. But we can look at it from the boy-girl proportion angle. The following table exhibits all possible boy-girl distributions and the corresponding number of ways in which a choice can be made according to that distribution.

Boy	*Girl*	*No. of ways*
0	r	$\binom{n}{0}\binom{m}{r}$
1	$r-1$	$\binom{n}{1}\binom{m}{r-1}$
2	$r-2$	$\binom{n}{2}\binom{m}{r-2}$
.	.	.
.	.	.
.	.	.
r	0	$\binom{n}{r}\binom{m}{0}$

Hence the equality required!

EXAMPLE 11. *Show that*

$$\binom{n}{0}+\binom{n+1}{1}+\ldots+\binom{n+r}{r}=\binom{n+r+1}{r}$$

SOLUTION. Method 1. From Example 8 we have

$$\binom{n+r+1}{r}=\binom{n+r}{r}+\binom{n+r}{r-1}$$

By the same rule, again,

$$\binom{n+r}{r-1}=\binom{n+r-1}{r-1}+\binom{n+r-1}{r-2}$$

$$\binom{n+r-1}{r-2}=\binom{n+r-2}{r-2}+\binom{n+r-3}{r-3}$$

$$\begin{matrix}\cdot\ \cdot & \cdot\ \cdot & \cdot\\ \cdot\ \cdot & \cdot\ \cdot & \cdot\end{matrix}$$

$$\binom{n+3}{2}=\binom{n+2}{2}+\binom{n+2}{1}$$

$$\binom{n+2}{1}=\binom{n+1}{1}+\binom{n+1}{0}$$

Adding, we get

$$\binom{n+r+1}{r}=\binom{n+r}{r}+\binom{n+r-1}{r-1}+\binom{n+r-2}{r-2}$$

$$+\ldots+\binom{n+1}{1}+\binom{n}{0}, \qquad \text{since } \binom{n}{0}=\binom{n+1}{0}=1.$$

Method 2 Fix any r of the $n+r+1$ objects given. Call them $A_1, A_2, \ldots, A_r$. Now our choice of r objects from the $n+r+1$ objects may or may not contain any or all of the set $\{A_1, A_2, \ldots, A_r\}$. We are going to exhaust all possibilities.

Case 1. It does not contain A_1.

This will happen in $\binom{n+r}{r}$ ways for the r things have to be chosen from the remaining $n+r$ things.

Case 2. It contains A_1 but does not contain A_2.

This will happen in $\binom{n+r-1}{r-1}$ ways because, having chosen A_1 and rejected A_2 we have only $n+r-1$ things to choose from and we need only $r-1$.

Case 3. It contains A_1, A_2 but does not contain A_3.

This will happen in $\binom{n+r-2}{r-2}$ ways.

Case 4 ..etc.

Case r It contains $A_1, A_2, \ldots A_{r-1}$ but does not contain A_r.

This will happen in $\binom{n+1}{1}$ ways.

Case (r + 1) It contains $A_1, A_2 \ldots A_r$.

This will happen in $\binom{n}{0}$ ways.

Thus $$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\ldots+\binom{n+r}{r}=\binom{n+r+1}{r}$$

EXAMPLE 12. *In how many ways can we choose 6 candies from 8 brands that are available? **(It is assumed here that you can choose the same brand repeatedly).** Generalise.*

SOLUTION. This is a problem of **combinations with repetitions allowed**–also called **redundant combinations.** We choose 6 brands from 8 brands, but now repetitions are allowed. So the answer is not $\binom{8}{6}$. For example you can choose all the 6 candidates from the same brand. Let us call the brands

$$B_1\ B_2\ B_3\ B_4\ B_5\ B_6\ B_7\ B_8$$

so now it is a question of how many we choose of brand B_1, how many of brand B_2 and so on. Let us represent this symbolically as

$$\underset{(1)}{x\,x \ldots x} \mid \underset{(2)}{x \ldots x} \mid \underset{(3)}{x \ldots x} \mid \ \ldots \ \mid \underset{(8)}{x \ldots x}$$

Here the eight brands are denoted as 8 boxes. If for instance, our choice is

$$B_2\ B_2\ B_4\ B_7\ B_7\ B_7$$

we write this as

$$\mid x\,x \mid \mid x \mid \mid \mid x\,x\,x \mid \qquad (*)$$

Each vertical separator stands for the separation between one brand and the next brand. The no. of x's says how many we have taken of that brand. An arrangement of 6 x's and 7 vertical separators as in (*) says precisely what our choice is. In this case it says that B_1 is not chosen, B_2 is chosen twice, B_3 is not chosen, B_4 is chosen once, B_5, B_6 are not chosen, B_7 is chosen three times and B_8 is not chosen.

So it is $B_2\ B_2\ B_4\ B_7\ B_7\ B_7$. So now it is up to us to count in how many ways we can have 6 x's and 7 separators arranged in a line. This means we have 13 spaces in which we can put the seven vertical separators anywhere we like and fill up the rest of the spaces with x's. This can be done in $\binom{13}{7}$ ways.

The answer is therfore $\binom{13}{6}$, which is the same as $\binom{13}{7}$.

Generalising this to r-combinations of n things with repetitions, we have spaces for the r things chosen and the $n - 1$ vertical separators between the n objects. Thus we have $n - 1 + r$ entities. We have to select $n - 1$ entities as spaces for the separators.

So the answer is $\binom{n+r-1}{n-1}$ which is the same as $\binom{n+r-1}{r}$. (Why?). This is the same as the number of non-negative integer solutions of $x_1 + x_2 + \ldots + x_n = r$. (See (10) at the end of Example 5).

EXAMPLE 13. *Show that* $\binom{n+5}{5}$ *distinct throws are possible with a throw of n dice which are indistinguishable among themselves.*

SOLUTION. The dice are indistinguishable. Let us understand the problem with $n = 2$. The different possible throws are:

1, 1					
1, 2	2, 2				
1, 3	2, 3	3, 3			
1, 4	2, 4	3, 4	4, 4		
1, 5	2, 5	3, 5	4, 5	5, 5	
1, 6	2, 6	3, 6	4, 6	5, 6	6, 6

The total number is 21 which is equal to $\binom{2+5}{5}$. The fact that the dice are indistinguistable expresses itself thus. It is not a question of which the shows up what number. It is only a question of what numbers are shown up on top. The numbers on the dice are 1, 2, 3, 4, 5, 6. These numbers show up in the n dice in varous combinations with repetitions. So we are now looking at combinations of 6 things taken n at a time with repetitions permitted, up to a maximum of n times. The answer to Example 12 shows this number to be $\binom{6+n-1}{n}$ which is the same as $\binom{n+5}{5}$.

EXAMPLE 14. *How many distributions are possible of 5 indistinguishable (= non-distinct) objects into 7 distinct boxes if there is no restriction on how many each box may contain. Generalise to n indistinguishable objects into m distinct boxes.*

SOLUTION. Since the boxes are distinct, we may name them

$$B_1, B_2, B_3, B_4, B_5, B_6, B_7.$$

Now the objects are indistinguishable. So it is only a question of how many go into each box. So we may proceed exactly as in Example 12. In fact we are going to discover that this problem is precisely the same as that one, except for the change in the numbers.

So let there be six vertical separators (between the successive pairs of the 7 boxes) and 5x's. In how many can we fill up $5 + 6 = 11$ empty spaces with these six separators and 5x's. It can be done in $\binom{11}{5}$ ways. This is therefore the answer to the problem. In fact it is the same as the answer to the question: How many combinations are there of 7 things taken 5 at a time with repetitions and we know the answer is $\binom{7+5-1}{5} = \binom{11}{5}$.

The distribution of n indistinguishable objects into m distinct boxes, if there is no restriction on how many each box may contain, is therefore the same problem as the number of combinations of m things taken n at a time with repetitions permitted.

And the answer is $\binom{m+n-1}{n}$.

EXERCISE 9.1

1. How many permutations of 1, 2, 3 ... 7 begin with an even number? How many begin and end with an even number? How many of the latter also have an even number in the middle?
2. Show that the number of one-one mappings of a set A with n objects into a set B with m objects $(n \leq m)$ is m_n.
3. The Reserve Bank of India prints currency notes in denominations of Two Rupees, Five Rupees, Ten Rupees, Twenty Rupees, Fifty Rupees, One hundred Rupees and Five hundred Rupees. In how many ways can it display ten currency notes, not necessarily of different denominations? How many of these will have all denominations?
4. In how many ways can an employer distribute ₹ 1000/- as Festival bonus to his five employees? No fraction of a Rupee is allowed. The only condition to be followed is: Each employee should get at least ₹ 50/-.
5. How many 6-letter words of binary digits are there?
6. For what values of n would the following be true? (*i*) $11_n = 12_{n-1}$ (*ii*) $9_n = 10_{n-1}$.
7. The results of 20 chess games (win, lose or draw) have to be predicted. How many different forecasts can contain exactly 15 correct results?
8. A person has n friends. How large must n be, so that the person can invite a different pair of friends every day for four weeks in a row.
9. How many integers between 1000 and 9999 (both inclusive) hae distinct digits? Of these, how many are even numbers? How many consist entirely of odd digits?
10. In how many permutations of the word AUROBIND do the vowels appear in the alphabetical order?
11. Show that the following give the same number: (*a*) The number of selections of r objects from n different objects, with repetitions permitted; (*b*) The number of distributions of r non-distinct objects into n distinct boxes; (*c*) The number of non-negative integer solutions to $x_1 + x_2 + \ldots + x_n = r$.
12. In how many ways can you permute the letters of the word CONSTITUTION?
13. Construct another problem in initation of Example 16 of section 1.
14. How many arrangements of five 0's and six 1's are there with no consecutive 1's?
15. A rectangular city is divided by streets into squares. There are n such squares from north to south and k squares from east to west. Find the number of shortest walks from the north-eastern end of the city to the south-western end.
16. In how many ways can the number n be presented as an ordered sum of k non-negative components?
17. In how many ways can n people stand to form a ring?
18. Prove that

(*a*) $\binom{n}{k-r} \Big/ \binom{n}{k} = (k)_r \,/\, (n-k+r)_r$ (*b*) $\binom{n-r}{k-r} \Big/ \binom{n}{k} = (k)_r \,/\, (n)_r$

19. The set $\{AC, GC, C, C, GC, T, AC, T\}$ is the set of fragments which together makeup a DNA-chain. But we do not know in what order they are to be put together. In all how many ways can they be put together?

20. How many increasing permutations of m symbols are there from the n-set of numbers $\{a_1, a_2, \ldots, a_n\}$ where the order among the numbers is given by $a_1 < a_2 < a_3 < \ldots < a_n$?

21. Prove, by logical reasoning from the definition of the numbers, rather than by use of formulae, the following:

(*a*) $n_r = (n-1)_r + r(n-1)_{r-1}$ (*b*) $n_n = n_{n-1}$

(c) $n_n = n \times (n-1)_{n-1}$ (*d*) $\binom{n}{r} r_r = n_r$.

(*e*) $r\binom{n}{r} = n\binom{n-1}{r-1}$.

22. How many functions defined on a set of n points are possible with values 0 or 1? How many of these functions have precisely m 1's in their range?

23. A lift automatically operated has a further computer facility of recording how many people leave the lift at each floor. It starts at floor 1 and goes up to floor 6. How many different records are possible of the people leaving the lift? What if the 8 people consisted of 3 men and 5 women and the computer can distinguish a man from a woman?

24. Delegates from 9 countries including countries A, B, C, D are to be seated in a row. How many different seating arrangements are possible if the delegates of countries A and B are to be seated next to each other and the delegates of countries C and D are not to be seated next to each other.? How will the answer change if the seating is done at a round table?

25. Prove that there are $\binom{n-1}{n-r}$ positive integer valued solutions of $x_1 + x_2 + \ldots + x_r = n$.

26. Prove that

$$\tan nA = \frac{\binom{n}{1}t - \binom{n}{3}t^3 + \binom{n}{5}t^5 - \ldots}{1 - \binom{n}{2}t^2 + \binom{n}{4}t^4 - \ldots.}$$

where $t = \tan A$.

9.3 BINOMIAL THEOREM

The numbers

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{1.2}$$

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}$$

. . . . .

$$\binom{n}{n} = 1$$

are called BINOMIAL COEFFICIENTS because they occur in the expansions of the powers of the binomial expression $(a + b)^n$.

Thus, $\quad (a + b)^n = (a + b).\,(a + b) \ldots (a + b)$ n times

One term in this expansion is, naturally, a^n, obtained by taking the a's from all the parentheses. Similarly, another term is b^n. A typical term will be

$$a^r\, b^{n-r}$$

obtained by taking a's from r of the parentheses and the b's from the remaining $(n - r)$ parentheses. In how many ways can you choose r parentheses from n that are available? Clearly there are $\binom{n}{r}$ ways. Having chosen the a's from r parentheses, there is no more choice for the b's, because they have to come from all the remaining parentheses. Thus the term $a^r\, b^{n-r}$ occurs in the expansion, $\binom{n}{r}$ times. So the coefficient of $a^r\, b^{n-r}$ is $\binom{n}{r}$.

Any difficulty in understanding the above can be sorted out by looking at, say $(a + b)^5$. This is

$$(a + b)\,(a + b)\,(a + b)(a + b)\,(a + b)$$

A term like a^2b^3 will occur by taking a's from two of the parentheses and b's from the remaining. But 2 parentheses can be chosen from five in $\binom{5}{2} = 10$ ways; so the coefficient of a^2b^3 is 10. Note that once we have decided which parentheses contribute to the a's, there is no more choice for b. The 10 ways of obtaining a^2b^3 are illustrated below:

(**a** + .)	(**a** + .)	(. + *b*)	(. + *b*)	(. + *b*)
(**a** + .)	(. + *b*)	(**a** + .)	(. + *b*)	(. + *b*)
(**a** + .)	(. + *b*)	(. + *b*)	(**a** + .)	(. + *b*)
(**a** + .)	(. + *b*)	(. + *b*)	(. + *b*)	(**a** + .)
(. + *b*)	(**a** + .)	(**a** + .)	(. + *b*)	(. + *b*)
(. + *b*)	(**a** + .)	(. + *b*)	(**a** + .)	(. + *b*)
(. + *b*)	(**a** + .)	(. + *b*)	(. + *b*)	(**a** + .)
(. + *b*)	(. + *b*)	(**a** + .)	(**a** + .)	(. + *b*)
(. + *b*)	(. + *b*)	(**a** + .)	(. + *b*)	(**a** + .)
(. + *b*)	(. + *b*)	(. + *b*)	(**a** + .)	(**a** + .)

In each case, the choice of the a's is shown by showing them in boldface. Thus the coefficient $a^r\, b^{n-r}$ in $(a + b)^n$ is $\binom{n}{r}$. This is true for every value of r, every $r = 0, 1, 2, \ldots, n$. Hence we have

$$(a+b)^n = a^n + \binom{n}{n-1}a^{n-1}b^1 + \binom{n}{n-2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^r b^{n-r} + \ldots + \binom{n}{0}b^n$$

Since $\binom{n}{r} = \binom{n}{n-r}$ for every $r = 0., 1, 2, \ldots., n$ the above can be rewritten as

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b_2 + \ldots + \binom{n}{n-r}a^r b^{n-r} + \ldots + \binom{n}{n}b^n$$

and also as

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

Writing $a = 1$, $b = x$, this becomes

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n.$$

Any one of the four identities above may be called "**The Binomial theorem for a positive integral index**". Note that, in all the above, n is a positive integer.

Note For another proof of the Binomial Theorem see Chapter 15. Section 6.

EXAMPLE 1. *Find the coefficient of a, b in the expansion of* $(2a + 3b)^5$.

SOLUTION. The term containing a^4b occurs in

$$\binom{5}{4}(2a)^4(3b)^1$$

that is, $5 \times 2^4 \times 3 \times a^4b$

So the required coefficient is 240.

EXAMPLE 2. *Expand* $(a + 1/a).^6$

SOLUTION. This is

$$= a^6 + \binom{6}{1}a^5 \times 1/a + \binom{6}{2}a^4(1/a)^2 + \binom{6}{3}a^3(1/a)^3 + \binom{6}{4}a^2(1/a)^4 + \binom{6}{5}a(1/a)^5 + \binom{6}{6}(1/a)^6$$

$$= a^6 + 6a^4 + 15a^2 + 20 + 15/a^2 + 6/a^4 + 1/a^6.$$

EXAMPLE 3. *Find the constant term in*

(*i*) $\left(2x^2 + \frac{1}{x}\right)^5$ (*ii*) $\left(2x^2 + \frac{1}{x}\right)^9$

SOLUTION. The typical term in the expansion of (*i*) is

$$\binom{5}{r}(2x^2)^r\,(1/x)^{5-r}$$

If this should reduce to a constant, the term $(x^{2r})\ x\ 1/x^{5-r}$ should reduce to x°. This means $2r = 5 - r$ *i.e.* $3r = 5$ which is impossible for any positive integer r. Thus there is no constant term in the above expansion. This may be confirmed by an actual expansion.

$$(2x^2 + 1/x)^5 = (2x^2)^5 + \binom{5}{1}(2x^2)^4\,(1/x) + \binom{5}{2}(2x^2)^3(1/x)^2$$

$$+ \binom{5}{3}(2x^2)^2\,(1/x)^3 + \binom{5}{4}(2x^2)\,(1/x)^4 + \binom{5}{5}(1/x)^5$$

$$= 32x^{10} + 80x^7 + 80x^4 + 40x + 10 \times 1/x^2 + 1/x^5.$$

(*ii*) In this case, the typical term is

$$\binom{9}{r}(2x^2)^r\,(1/x)^{9-r}$$

If this should reduce to a constant, we should have $2r = 9 - r$ *i.e.* $3r = 9$ so $r = 3$.

Giving $r = 3$ to the typical them, we not $\binom{9}{3}(2x^2)^3\,(1/x)^6$

$$= \frac{9.8.7}{1.2.3.} \times 2^3 \times x^6/x^6 = 672.$$

EXAMPLE 4. *Prove that*

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$$

and explain this identity combinatorially.

SOLUTION. We have, $(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$

Writing $x = 1$ both sides, we get

$$(1 + 1)^n = 1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

which is the required identity.

Now 2^n is the total number of subsets (including the empty subset) of a set of n distinct objects. But this number is also equal to the

number of empty subsets + the no. of 1-subsets + the no. of 2-subsets + ...
+ the no. of n-subsets

which is equal to

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}.$$

EXAMPLE 5. *Prove that*

$$\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+\ldots+2^n\binom{n}{n}=3^n$$

Write $x = 1$ in $(1 + 2x)^n$.

EXAMPLE 6. *Show that, given an n-set A the number of subsets of A that contain an even number is equal to the number of subsets that contain an odd number.*

SOLUTION. We are required to show that

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots$$

In other words we have to show

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\ldots-\ldots=0$$

This is true because

$$(1-(-x))^n = 1+\binom{n}{1}(-x)+\binom{n}{2}(-x)^2+\binom{n}{3}(-x)^2+\ldots$$

and on substituting $x = 1$, we get

$$0 = 1-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\ldots$$

which is the required identity.

EXAMPLE 7. (Illustrative) Here is a pictorial proof of the Binomial Theorem. (This was the way the ancient Hindu Mathematicians approached the problem).

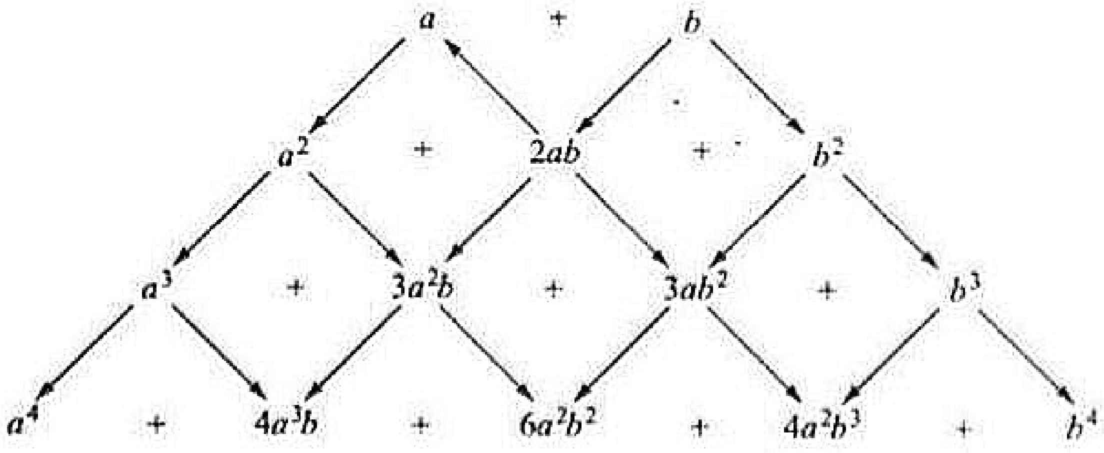

Fig. 9.7

The nth row in this diagram is $(a + b)^n$. Each arrow moving towards the left (right) is equivalent to multiplication by a (respectively, b). Whenever two arrows converge to the same position, we add the terms so obtained and write it there.

Since by the very construction of the diagram, a multiplies each term of a given row, and b multiplies each term of the same row, we find that each successive row is the multiplication of the previous row by $(a + b)$. Hence the result.

Now isolate only the coefficients from the diagram. We obtain the following diagram for the binomial coefficients, called **Pascal's Triangle.**

$$
\begin{array}{lccccccccccccc}
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & \cdot & & \cdot & & \cdot & \vdots & & \cdot & & \cdot & & \\
 & & \binom{n}{0} & & \binom{n}{1} & & \binom{n}{2} & \binom{n}{3} & & \binom{n}{4} \cdots & \binom{n}{n-1} & \binom{n}{n} & &
\end{array}
$$

In the kth row, each number is obtained by adding the two numbers immediately above it. See the arrows leading up to 15. In fact this confirms the result

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

already proved by us.

EXERCISE 9.3

1. Expand

$$\left(y^2 - \frac{1}{y^2}\right)^8$$

2. Find the cofficient of
   (a) x^7y^2 in $(4x-3y)^9$
   (b) x^9 in $(x+2)^4(4x+7)^6$
3. Find the term independent of x in
   (a) $\left(x-\frac{1}{x}\right)^5\left(x+\frac{1}{x}\right)^3$ (b) $\left(4x^2+\frac{2}{x}\right)^7$ (c) $\left(\frac{3}{x}-2x^2\right)^9$
4. Prove that

$$5^n - 1 = 4\left\{\binom{n}{1} + 4\binom{n}{2} + \dots + 4^{n-1}\binom{n}{n}\right\}.$$

5. Find the number of rational terms in the expansion of $(\sqrt{3}+\sqrt[6]{2})^{16}$.

PROBLEMS

1. In how many ways can n persons shake hands?
2. In how many ways can 6 speakers A, B, C, D, E, F address a gathering if
   (a) A speaks after B?
   (b) A speaks immediately after B?
3. How many subsets of the set $\{1, 2, \dots, 10\}$ contain at least one odd integer?
4. Prove that the number of isosceles triangles with integer sides, no side exceeding n is $(1/4)(3n^2+1)$ or $(3/4)n^2$ according as n is odd or even.

5. What is the number of distinct terms in the expansion of $(x_1 + x_2 + ... + x_m)^n$?
6. How many non-decreasing sequences of length r can be formed from $\{1, 2, ..., n\}$. How many of these are strictly increasing?
7. Show that there is no bijective (one-one, onto) function from a set to its power set. (The power-set is the set of all mappings from the set to the two-element set $\{0, 1\}$).
8. If three distinct integers are chosen at random show that there will exist two among them, say a and b such that 30 divides $(a^3b - ab^3)$.
9. Consider a finite set S of points in a plane which are not all collinear. Show that there is a line in the plane which passes through exactly two points of S.
10. How many different rectangles can be drawn on an 8 × 8 Chess board?
11. n objects are arranged in a row. A subset of these objects is called *unfriendly* if no two of its elements are consecutive. Show that the number of unfriendly subsets of a k-elements set is

$$\binom{n-k+1}{k}$$

12. Prove that

$$\binom{n}{0}\binom{m}{n}+\binom{n}{1}\binom{m+1}{n}+\binom{n}{2}\binom{m+2}{n}+\dots \text{ to } (n+1) \text{ terms}$$

$$=\binom{n}{0}\binom{m}{0}+\binom{n}{1}\binom{m}{1}2+\binom{n}{2}\binom{m}{2}2^2+\dots \text{ to } (n+1) \text{ terms.}$$

13. How many ways are there to seat six different boys and six different girls around a circular table? How many if boys and girls alternate?
14. Four numbers are chosen from 1 to 20. If $1 \le k \le 17$, in how many ways is the difference between the smallest and largest equal to k?
15. How many positive integers are three with distinct digits?
16. What is the maximum number of terms (monomials) in a homogeneous expression of degree m in n variables? What is the answer if the expression is not necessarily homogeneous? What is the connection with Problem No. 5 above?
17. What is the greatest number of acute angles that can occur in a convex n-gon?
18. Into how many regions do the diagonals of a convex 10-gon divide the interior if no three diagonals are concurrent inside the 10-gon? Also in how many points do they intersect in the interior?
19. There are five points in a plane. From each point, perpendiculars are drawn to the lines joining the other points. What is the *maximum* number of points of intersection of these perpendiculars?
20. In a party people shake hands with one another (*not necessarily every one with every one else*). (*a*) Show that two persons shake hands the same number of times. (*b*) Show that the number of people who shake hands an odd number of times is even.

CHAPT

10

FACTORIZATION OF POLYNOMIALS

10.1 INTRODUCTION

An expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0, \quad a_n \neq 0 \tag{1}$$

is called a *polynomial* of degree n. Here a_n is called the *leading coefficient* of $p(x)$. If all the coefficients $a_0, a_1, \dots, a_n$ are integers, then $p(x)$ is called a polynomial with integer coefficients, or briefly a polynomial over **Z**. Similarly we say $p(x)$ is a polynomial over **Q**, if all $a_j, j = 0,1,2,\dots, n$ are rational numbers; $p(x)$ is a polynomial over **R** if a_j, $j = 0,1,2,\dots, n$ are real numbers; $p(x)$ is a polynomial over **C**, if all $a_j, j = 1,2,3,\dots,n$, are complex numbers. We adopt the following notations;

Z[x]– the set of all polynomials over **Z**;
Q[x]– the set of all polynomials over **Q**;
R[x]– the set of all polynomials over **R**;
C[x]– the set of all polynomials over **C**.

The plus signs in the expression (1) as such have no meaning because, we have not given any meaning to x, – which could be anything in the world. But if $p(x)$ is in **Z**(x) and if we define $p(k)$ for any integer k by

$$p(k) = a_n k^n + a_{n-1} k^{n-1} + \dots + a_n$$

then the addition signs automatically become meaningful; it is addition in **Z**. Similar considerations hold for polynomials in **Q**[x], **R**[x] and **C**[x].

If $p(x)$ is given by (1), then n is called the *degree* of $p(x)$. Thus the degree of a polynomial $p(x)$ is the highest power of x occuring in $p(x)$ with a nonvanishing coefficient. If $n = 0$ in (1), then $p(x) = a_o$. Thus if $p(x)$ is in **Z**[x], and degree of $p(x)$ is zero, then regarding $p(x)$ as a function from **Z** into **Z** we get a constant function taking the constant value a_o for all values of x in **Z**. We can reason similarly for polynomials in **Q**[x], **R**[x]. Thus a polynomial of degree zero is a constant polynomial.

If all a_j are zero in (1), such a polynomial is called the *zero polynomial*. We don't define the degree of a zero polynomial. If $p(x)$ and $q(x)$ are two polynomials,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_o, \; a_n \neq 0$$
$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_o, \; b_m \neq 0,$$

then we say

$$p(x) = q(x)$$

5. What is the number of distinct terms in the expansion of $(x_1 + x_2 + ... + x_m)^n$?

6. How many non-decreasing sequences of length r can be formed from $\{1, 2, ..., n\}$. How many of these are strictly increasing?

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CHA

10

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$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_o, \; a_n \neq 0$$
$$q(x) = b_m x^m + b_{m-1} x^{m-1} + ... + b_o, \; b_m \neq 0,$$

then we say

$$p(x) = q(x)$$

if $m = n$ and $a_j = b_j$ for $j = 0, 1, 2, \dots, n$.

In chapter 5, we have studied many properties of quadratic polynomials over **R**, *i.e.*, polynomials of degree 2 with real coefficients. In particular, therein we have derived conditions for the existence of real roots of a quadratic polynomial and its irreducibility over **R**. In fact, given a polynomial

$$p(x) = ax^2 + bx + c,$$

where a, b, c are real numbers, we have proved that the equation

$$P(x) = 0 \tag{2}$$

has real roots iff the discriminant

$$D = b^2 - 4ac$$

is nonnegative. In the case $D \geq 0$, the polynomial $p(x)$ can be factored as

$$p(x) = a(x - \alpha)(x - \beta) \tag{3}$$

where α and β are the real roots of the equation (2). If $D < 0$, then (2) has only complex roots α and β, and once again we have a factorization

$$p(x) = a(x - \alpha)(x - \beta). \tag{4}$$

Thus whenever α is a zero of a quadratic polynomial $p(x)$, $(x - \alpha)$ is a factor of $p(x)$. We shall see that this holds true for any polynomial $p(x)$.

If we are given two integers, then we know (see Chapter 2) what we mean by their greatest common divisor (g.c.d) and their least common multiple (l.c.m). These ideas can also be extended to polynomials. There is also the counterpart of Euclid's algorithm for finding g.c.d of two integers in the set of all real polynomials **R**$[x]$. It is the purpose of the present chapter to study these ideas.

10.2 ADDITION AND MULTIPLICATION OF POLYNOMIALS

Given any two integers, addition and multiplication can be performed with them. We shall now introduce a concept of addition and multiplication of polynomials.

Given any two polynomials in $R[x]$, we add them by adding the coefficient of like powers of x.

EXAMPLE 1. *Find the sum of*

$$p(x) = \sqrt{5}\,x^4 + 3x^2 + 4x + 2,$$

$$q(x) = 6x^3 + \sqrt{7}x + \sqrt{3}.$$

SOLUTION. We first observe that the maximum power of x that appears in $p(x)$ or $q(x)$ is x^4. We can arrange the coefficients of like powes of x as in table 10.1.

TABLE 10.1

Polynomial	*Coefficient of the power*				
	x^4	x^5	x^2	x	*Constant*
$p(x)$	$\sqrt{5}$	0	3	4	2
$q(x)$	0	6	0	$\sqrt{7}$	$\sqrt{3}$
$p(x) + q(x)$	$\sqrt{5}$	6	3	$4 + \sqrt{7}$	$2 + \sqrt{3}$

We have implicity assumed here that if a certain power x^k is missing in a polynomial then the coefficient of x^k in that polynomial is equal to zero. This enables us to add two polynomials of different degrees. So we have

$$p(x) + q(x) = \sqrt{5}\,x^4 + 6x^3 + 3x^2 + (4 + \sqrt{7})x + (2 + \sqrt{3}).$$

Given a polynomial $p(x)$, the polynomial $-p(x)$ is obtained by changing each coefficient of $p(x)$ to its negative.

Thus if

$$p(x) = 3x^3 - 5x^2 - \sqrt{3}x + 4,$$

then $-p(x)$ is given by

$$-p(x) = -3x^3 + 5x^2 + \sqrt{3}x - 4.$$

EXAMPLE 2. *Find $p(x) - q(x)$ where*

$$p(x) = x^3 - 5x^2 + 2x + 9,$$
$$q(x) = x^4 + 8x^2 - 5.$$

SOLUTION. By $p(x) - q(x)$ we mean $p(x) + (-q(x))$. So we have

$$p(x) - q(x) = -x^4 + x^3 - 13x^2 + 2x + 14.$$

We define the product of two polynomials by multiplying them term by term and then adding the coefficients of like powers of x using the law of indices $x^k x^l = x^{(k+l)}$.

EXAMPLE 3. *Consider the polynomials*

$$p(x) = 3x^2 + 2x + 1,$$
$$q(x) = x + 3.$$

SOLUTION. Then their product is

$$\begin{aligned} p(x)\,q(x) &= 3x^3 + 9x^2 + 2x^2 + 6x + x + 3 \\ &= 3x^3 + 11x^2 + 7x + 3. \end{aligned}$$

EXAMPLE 4. *Multiply $p(x)$ and $q(x)$, where*

$$p(x) = \sqrt{5}\,x^4 + 3x^2 + 4x + 2,$$
$$q(x) = 6x^3 + \sqrt{7}x^2 + \sqrt{3}.$$

SOLUTION. Multiplying term by term, we have

$$\begin{aligned} p(x)\,q(x) &= 6\sqrt{5}x^7 + \sqrt{35}x^6 + \sqrt{15}x^4 + 18x^5 + 3\sqrt{7}x^4 + 3\sqrt{3}x^2 \\ &\qquad + 24x^4 + 4\sqrt{7}\,x^3 + 4\sqrt{3}\,x + 12x^3 + 2\sqrt{7}\,x^2 + 2\sqrt{3} \\ &= 6\sqrt{5}x^7 + \sqrt{35}x^6 + 18x^5 + (\sqrt{15} + 3\sqrt{7} + 24)x^4 \\ &\qquad + (4\sqrt{7} + 12)x^3 + (3\sqrt{3} + 2\sqrt{7})x^2 + 4\sqrt{3}x + 2\sqrt{3}. \end{aligned}$$

EXAMPLE 5. *Find the sum and product of $p(x)$ and $q(x)$, where*

$$p(x) = x^7 + 9x^3 + 3x + 1,$$
$$q(x) = x^5 + 6x^2 + 4x.$$

SOLUTION. Adding like powers of x, we get

$$p(x) + q(x) = x^7 + x^5 + 9x^3 + 6x^2 + 7x + 1.$$

Similarly,

$$\begin{aligned} p(x)\,q(x) &= x^{12} + 6x^9 + 4x^8 + 9x^8 + 54x^5 + 36x^4 \\ &\qquad + 3x^6 + 18x^3 + 12x^2 + x^5 + 6x^2 + 4x \\ &= x^{12} + 6x^9 + 13x^8 + 3x^6 + 55x^5 + 36x^4 + 18x^3 + 18x^2 + 4x. \end{aligned}$$

The given examples reveal how the degrees of sum and product are related to the degrees of individual polynomials. If $p(x)$ and $q(x)$ are two polynomials, then

$$\deg(p(x) + q(x)) \leq \max \{ \deg p(x), \deg q(x) \} \tag{1}$$

and $$\deg(p(x)\, q(x)) = \deg p(x) + \deg q(x) \tag{2}$$

Here we have used $\deg p(x)$ to denote the degree of $p(x)$. If we take

$$p(x) = x^2 + 1, \qquad q(x) = -x^2 + 1,$$

then $\deg(p(x) + q(x)) = \deg (2) = 0$

whereas

$\max \{\deg p(x), \deg q(x)\} = 2$. Thus, there may be strict inequality in (1).

EXERCISE 10.1

Find the sum and product in the following:

1. $p(x) = 8x^4 + 9x^3 + 12x + 4,$ $\quad q(x) = x^6 + 10x^2 + 9$
2. $p(x) = \sqrt{2}\,x^5 + 7x^2 + \sqrt{5}x + 6,$ $\quad q(x) = x^{10} + 10x^2 - 9$
3. $p(x) = x - 1,$ $\quad q(x) = x^9 + x^8 + x^7 + \ldots + x + 1$
4. $p(x) = x^2 + \sqrt{2}x + 1,$ $\quad q(x) = x^2 - \sqrt{2}x + 1$
5. $p(x) = \sqrt{7}x^4 + 8x^3 + 4,$ $\quad q(x) = \sqrt{7}x^3 - 6x^2 + 2x$
6. $p(x) = 0.5x^3 + 2x^2 + \sqrt{2}x + 1,$ $\quad q(x) = 9x^5 + 3x + 6$
7. $p(x) = \dfrac{1}{2}x^4 + \dfrac{3}{4}x^2 + \dfrac{5}{6},$ $\quad q(x) = 7x^6 + 3x^3 + 5$
8. $p(x) = x^{10} + x^7 + x^4 + x + 1,$ $\quad q(x) = x^3 - 1$
9. $p(x) = x^5 - x^4 + x^3 - x^2 + x - 1, q(x) = x + 1$
10. $p(x) = 10x^9 + 9x^8 + 8x^7 + \ldots + 2x + 1,$ $\quad q(x) = x^2 - x + 1$
11. $p(x) = \sqrt{3}x^3 + \sqrt{2}x^2 + x,$ $\quad q(x) = -\sqrt{3}x^3 + \sqrt{2}x - x$
12. $p(x) = 10x^6 + 6x^2 + 2,$ $\quad q(x) = -10x^6 - 6x^2 + 2x$
13. $p(x) = 10x^3 + 9x^2 + 2x + 1,$ $\quad q(x) = x^2 + 2, r(x) = x + 1$
14. $p(x) = 4x^2 + x + 1,$ $\quad q(x) = 3x^2 - 5x + 2, r(x) = x - 1$

10.3 DIVISION OF POLYNOMIALS

Given two integers m and n with $n > 0$, we know that we can divide m by n to get a quotient q and a remainder r and express it as

$$m = nq + r \tag{1}$$

where $0 \leq r < n$. Here q and r are uniquely determined by m and n. A relation of the form (1) is also true for polynomials. We shall begin with the division of a polynomial by another.

EXAMPLE 1. *Let us divide $a(x) = x^3 + 8x^2 + 21x + 8$ by $b(x) = x + 2$. The direct division table is given here*

$$
\begin{array}{rl}
x+2) & x^3+8x^2+21x+18 \quad (x^2+6x+9 \\
 & \underline{x^3+2x^2\qquad\qquad\quad} \\
 & \quad 6x^2+21x \\
 & \quad \underline{6x^2+12x\qquad} \\
 & \qquad\qquad 9x+18 \\
 & \qquad\qquad \underline{9x+18} \\
 & \qquad\qquad\quad 0+0
\end{array}
$$

This gives the quotient $x^2 + 6x + 9$ and the remainder zero.

EXAMPLE 2. *Divide $a(x)$ by $b(x)$, where*

$$a(x) = x^5 + 3x^2 + 9,$$
$$b(x) = x^2 + 4x + 1$$

SOLUTION. The division process is as follows.

$$
\begin{array}{rl}
x^2+4x+1) & x^5+0x^4+0x^3+3x^2+0x+9 \quad (x^3-4x^2+15x-53 \\
 & \underline{x^5+4x^4+x^3\qquad\qquad\qquad} \\
 & \quad -4x^4-\ \ x^3+3x^2 \\
 & \quad \underline{-4x^4-16x^3-4x^2} \\
 & \qquad\qquad 15x^3+\ \ 7x^2+0x \\
 & \qquad\qquad \underline{15x^3+60x^2+15x} \\
 & \qquad\qquad\qquad -53x^2-\ \ 15x+\ \ 9 \\
 & \qquad\qquad\qquad \underline{-53x^2-212x-53} \\
 & \qquad\qquad\qquad\qquad\quad 197x+62
\end{array}
$$

Thus the quotient after division is

$$q(x) = x^3 - 4x^2 + 15x - 53,$$

and the remainder is

$$r(x) = 197x + 62.$$

The result of the division can be written in the form

$$(x^5 + 3x^2 + 9) = (x^2 + 4x + 1)(x^3 - 4x^2 + 15x - 53) + (197x + 62).$$

In example 1, we can write the division in the form

$$a(x) = b(x)q(x) + r(x) \qquad (2)$$

where $\quad q(x) = x^2 + 6x + 9,$

and $\quad r(x) = 0.$

Similarly, we can write the division in example 2 in the form (2) with

$$q(x) = x^3 - 4x^2 + 15x - 53,$$
$$r(x) = 197x + 62.$$

If $a(x)$ and $b(x)$ are such that

$$\deg a(x) < \deg b(x),$$

then again we have a relation of the form (2) since, using the symbol 0 for the zero polynomial, we have,

$$a(x) = b(x)\,.\,0 + a(x)$$

so that $q(x) = 0$ and $r(x) = a(x)$. In all these cases, we observe that $r(x)$ is either the zero polynomial or

$$\deg r(x) < \deg b(x).$$

These observations lead to what is called the *Division Algorithm.*

DIVISION ALGORITHM

If $a(x)$ and $b(x)$ are polynomials with real coefficients, then there are unique polynomials $q(x)$ and $r(x)$ with real coefficients such that

$$a(x) = b(x)\, q(x) + r(x) \tag{3}$$

where either $r(x)$ is the zero polynomial or

$$\deg r(x) < \deg b(x)$$

A proof of this can be given using the principle of induction and we omit it here.

The process of division can also be carried using a method called *Horner's method of synthetic division* or synthetic division, for short. We describe this method in the next few examples.

EXAMPLE 3. *Consider the division of* $a(x) = 4x^4 - 6x^2 - 2x + 1$ *by* $b(x) = x - 2$.

SOLUTION. The process of synthetic division is described on the left side of the following page, whereas the actual division is shown as the right side.

The coefficient of x^4 is 4. Hence the first term in the quotient must be $4x^3$ and its coefficient is 4. This appears as the first term in the last row of the left side. When $4x^3$ is multiplied by -2, we get $-8x^3$ and this has to be subtracted from the x^3 term in the given polynomial. This amounts to add the coefficient of x^3 to 8; *i.e.* adding 0 to 8 and we get 8. This leaves $8x^3$ and hence $8x^2$ must appear in the quotient. When we multiply $8x^2$ by -2, we get $-16x^2$ and this has to be subtracted from the x^2 term in the given polynomial. This process is same as adding 16 to the coefficient of x^2 in $a(x)$ and this leads to $10x^2$. Thus the number 10 appearing on the L.H.S. is the coefficient of x^2. We can continue the process till we account for all the coefficients of $a(x)$. We get the quotient

$$q(x) = 4x^3 + 8x^2 + 10x + 18,$$

and the remainder

$$r(x) = 37.$$

Let us consider a general polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

Suppose we have to divide $p(x)$ by $(x - \alpha)$. We can find unique polynomials $q(x)$ and $r(x)$ such that

$$p(x) = (x - \alpha)\, q(x) + r(x)$$

$$\begin{array}{r|rrrrrl}
1-(+2) & 4 & 0 & -6 & -2 & +1 & \Big(4+8+10+18 \\
 & 4 & -8 & & & & \\ \cline{2-3}
 & & 8 & -\ 6 & & & \\
 & & 8 & -16 & & & \\ \cline{3-4}
 & & & +10 & -2 & & \\
 & & & 10 & -20 & & \\ \cline{4-5}
 & & & & +18 & +\ 1 & \\
 & & & & 18 & -36 & \\ \cline{5-6}
 & & & & & 37 &
\end{array}$$

$$\begin{array}{r|rrrrrl}
x-2 & 4x^4 & +0x^3 & -6x^2 & -2x & +1 & \big(4x^3+8x^2+10x+18 \\
 & 4x^4 & -8x^3 & & & & \\
\cline{2-3}
 & & +8x^3 & -6x^2 & & & \\
 & & 8x^3 & -16x^2 & & & \\
\cline{3-4}
 & & & 10x^2 & -2x & & \\
 & & & 10x^2 & -20x & & \\
\cline{4-5}
 & & & & +18x & +1 & \\
 & & & & 18x & -36 & \\
\cline{5-6}
 & & & & & 37 &
\end{array}$$

where $\deg r(x) < \deg (x - \alpha) = 1$.

Thus $r(x)$ is a constant say $r(x) = r$. We also note that $\deg q(x) = n - 1$, so that

$$q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_o, \quad b_{n-1} \neq 0.$$

Hence we have the identity

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_o = (x - \alpha)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_o) + r.$$

The R.H.S. can also be written as

$$b_{n-1}x^n + (b_{n-2} - \alpha b_{n-1})x^{n-1} + \ldots + (b_o - \alpha b_1)x + (r - \alpha b_o).$$

Now comparing the like powers of x, we get

$$\begin{aligned}
b_{n-1} &= a_n \\
b_{n-2} &= a_{n-1} + \alpha b_{n-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
b_1 &= a_2 + \alpha b_2 \\
b_o &= a_1 + \alpha b_1 \\
r &= a_o + \alpha b_o.
\end{aligned}$$

This can be written as follows.

$$\begin{array}{r|ccccccccccccc}
(1-\alpha) & a_n & + & a_{n-1} & + & a_{n-2} & + & \ldots & + & a_1 & + & a_0 & & \\
 & & + & \alpha b_{n-1} & + & \alpha b_{n-2} & + & \ldots & + & \alpha b_1 & + & \alpha b_0 & & \\
\hline
 & b_{n-1} & + & b_{n-2} & + & b_{n-3} & + & \ldots & + & b_0 & + & r & &
\end{array}$$

EXAMPLE 4. *Use synthetic division to divide $5x^4 + 6x + 2$ by $x + 4$.*

SOLUTION. Since $x + 4 = x - (-4)$, we can take $\alpha = -4$ in the synthetic division we have performed earlier;

$$\begin{array}{r|ccccccccc}
1-(-4) & 5 & + & 0 & + & 0 & + & 6 & + & 2 \\
 & & - & 20 & + & 80 & - & 320 & + & 1256 \\
\hline
 & 5 & - & 20 & + & 80 & - & 314 & + & 1258
\end{array}$$

Thus the remainder after division is 1258 and the quotient is $5x^3 - 20x^2 + 80x - 314$.

We can use synthetic division even when the divisor is a polynomial of higher degree.

EXAMPLE 5. *Divide $3x^4 - 5x^3 - 11x^2 + x - 1$ by $x^2 - 2x - 2$.*

SOLUTION. We write $x^2 - 2x - 2 = x^2 - (2x + 2)$. The division process can be recorded as follows:

Table 10.1

$$\begin{array}{r|rrrrrl}
1-(2+2) & 3 & -5 & -11 & +1 & -1 & \\
 & & & +6 & +2 & -6 & (a) \\
 & & +6 & +2 & -6 & & (b) \\
\hline
 & 3 & +1 & -3 & -3 & -7 & (c)
\end{array}$$

The first term in line (c) is 3. We have $3(2+2) = 6 + 6$. We put first 6 in line (b) and another 6 in line (a) in a diagonal way as shown in Table 10.1. Since $-5 + 6 = 1$, this is placed in line (c) adjacent to 3. Again $1(2+2) = 2 + 2$ and these two 2's are placed in a diagonal way with 2 appearing in each of the lines (a) and (b). Again $-11 + 6 + 2 = -3$ and this is placed in line (c) adjacent to 1. Since $-3(2+2) = -6 + (-6)$, we place these numbers in a diagonal way as described earlier. We add the last two columns; $1 + 2 - 6 = -3$, $-1 - 6 = -7$. The quotient is $3x^2 + x - 3$ and the remainder is $-3x - 7$. The reckoning of Table 10.1 can be clearly understood by comparing it with actual division.

$$\begin{array}{l r l}
x^2-2x-2) & 3x^4+5x^3-11x^2+x-1 & (3x^2+x-3 \\
 & 3x^4-6x^3-6x^2 & \\
\hline
 & x^3-5x^2+\ x & \\
 & x^3-2x^2-2x & \\
\hline
 & -3x^2+3x-1 & \\
 & -3x^2+6x+6 & \\
\hline
 & -3x-7 &
\end{array}$$

EXAMPLE 6. *Divide* $3x^5 + 6x^4 - 2x^3 - x^2 - 2x + 4$ *by* $x^2 + 2x - 1$.

SOLUTION. With the usual reckoning of synthetic division, we have the following Table.

$$\begin{array}{r|rrrrrr}
1-(-2+1) & 3 & +6 & -2 & -1 & -2 & +4 \\
 & & & +3 & +0 & +1 & -3 \\
 & & -6 & -0 & -2 & +6 & \\
\hline
 & 3 & +0 & +1 & -3 & +5 & +1
\end{array}$$

Thus the quotient after division is $3x^3 + x - 3$ and the remainder is $5x + 1$.

We can also use synthetic division for expressing the given polynomial $a(x)$ as a polynomial in $x - \alpha$.

EXAMPLE 7. *Express the polynomial* $a(x) = x^3 + 2x^2 + x + 80$ *as a polynomial in* $x + 5$.

SOLUTION. Suppose $a(x) = \alpha_3 (x+5)^3 + \alpha_2 (x+5)^2 + \alpha_1 (x+5) + \alpha_0$, where $\alpha_0, \alpha_1, \alpha_2$ and α_3 are constants to be determined. Now we can write

$$a(x) = (x+5)\, q_1(x) + \alpha_0$$

for some polynomial $q_1(x)$. Hence α_0 is the remainder after the division of $a(x)$ by $x + 5$. The quotient $q_1(x)$ is given by

$$\begin{aligned} q_1(x) &= \alpha_3 (x+5)^2 + \alpha_2(x+5) + \alpha_1 \\ &= (x+5)\, q_2(x) + \alpha_1. \end{aligned}$$

This shows that α_1 is the remainder left by the division of $q_1(x)$ by $x + 5$. Once again, we have,

$$q_2(x) = \alpha_3(x+5) + \alpha_2$$

and hence α_2 in the remainder obtained after dividing $q_2(x)$ by $(x + 5)$; the constant α_3 can be got as the quotient of the division. Thus the constants α_0, α_1 α_2, and α_3 can be obtained systematically using the division process. We successively employ the synthetic division here.

$$\begin{array}{r|rrrr} 1-(-5) & 1\ + & 2\ + & 1\ + & 80 \\ & & -5\ + & 15\ - & 80 \\ \hline & 1\ - & 3\ + & 16\ + & 0 \end{array}$$

The preceeding Table shows that the quotient is

$$q_1(x) = x^2 - 3x + 16$$

and $\alpha_0 = 0$. Dividing $q_1(x)$ by $(x + 5)$, we get

$$\begin{array}{r|rrr} 1-(-5) & 1\ - & 3\ + & 16 \\ & & -5\ + & 40 \\ \hline & 1\ - & 8\ + & 56 \end{array}$$

the quotient

$$q_2(x) = x - 8$$

and the remainder $\alpha_1 = 56$. Once again we can write

$$q_2(x) = (x + 5) - 13$$

so that $\alpha_2 = -13$ and $\alpha_3 = 1$. Thus we can write

$$x^3 + 2x^2 + x + 80 = (x + 5)^3 - 13(x + 5)^2 + 56(x + 5).$$

EXAMPLE 8. *Express the polynomial $a(x) = x^3 + 4x^2 + 9x + 5$ in powers of $x + \frac{1}{3}$.*

SOLUTION. We use synthetic division and the idea established in example 7.

$$\begin{array}{r|lll|l} 1-(-1/3) & 1\ + & 4\ + & 9 & +\ 5 \\ & & -\ 1/3 & -\ 11/9 & -\ 70/27 \\ \hline & 1\ + & 11/3\ + & 70/9 & +\ 65/27 \\ & & -\ 1/3 & -\ 10/9 & \\ \cline{2-4} & 1\ + & \multicolumn{1}{l|}{10/3} & +\ 60/9 & \\ & & \multicolumn{1}{l|}{-\ 1/3} & & \\ \cline{2-3} & 1 + 3 & & & \end{array}$$

The preceeding table of division shows that

$$x^3 + 4x^2 + 9x + 5 = (x + 1/3)^3 + 3(x + 1/3)^2 + 60/9\ (x + 1/3) + 65/27.$$

EXERCISE 10.2

1. Divide $a(x)$ by $b(x)$ in the following
   - (*a*) $a(x) = 3x^6 + 7x^4 + 9x^2 + 2x + 1$, $b(x) = 2x + 2$.
   - (*b*) $a(x) = x^{10} + x^8 + x^6 + x^4 + x^2 + 1$, $b(x) = x^5 + x^3 + 1$.
   - (*c*) $a(x) = x^9 + 9x^5 + 4x$, $b(x) = x^3 + 3x^2 - 2$.
   - (*d*) $a(x) = x^6 = 2$, $b(x) = x - \sqrt[6]{2}$.
   - (*e*) $a(x) = 2x^3 - x^2 - 5x + 4$, $b(x) = x - 3$.
   - (*f*) $a(x) = 4x^4 - 2x^3 - 16x^2 + 5x + 9$, $b(x) = x^2 - 2x - 1$.

(*g*) $a(x) = 5x^3 - 2x^2 - 2x - 1,\ b(x) = x^2 + 4x + 3.$
(*h*) $a(x) = x^3 - 9x^2 + 27x - 27,\ b(x) = x^2 - 2x + 4.$
(*i*) $a(x) = -12x^4 + 4x^3 + 9x^2 - 1,\ b(x) = x^2 + 7.$
(*j*) $a(x) = -20x^4 - 12x^3 + 20x^2 + 7x + 6,\ b(x) = x^2 + x.$
(*k*) $a(x) = x^4 - x^2 + 3,\ b(x) = x^2 - 3.$
(*l*) $a(x) = x^5 + x^4 + x^3 + x^2 + x + 1,\ b(x) = x^2 - x - 2.$
(*m*) $a(x) = 3x^9 - 28x^8 + 65x^5 + 16x^2 - 80x,\ b(x) = 3x^5 - x^4 - 4x^2 + 1.$

2. Use synthetic division to divide $a(x)$ by $b(x)$ in the following set
(*a*) $a(x) = 4x^3 - 24x^2 + 21x - 5,\ b(x) = 2x - 1$
(*b*) $a(x) = 3x^3 + 19x^2 + 22x - 24,\ b(x) = x + 3$
(*c*) $a(x) = 5x^3 - 42x^2 + 81x + 18,\ b(x) = x - 3$
(*d*) $a(x) = 2x^3 - 18x^2 + 32x + 21,\ b(x) = x - 7$
(*e*) $a(x) = x^{10} - 9x^7 + 3x^4 + 4,\ b(x) = x - 7$
(*f*) $a(x) = x^4 - 11x^3 + 33x^2 - 37x - 14,\ b(x) = x^2 - 2x + 1$
(*g*) $a(x) = x^6 + x^4 + x^2 + 1,\ b(x) = x^2 - 1$
(*h*) $a(x) = 3x^4 + 9x^2 + 2,\ b(x) = x^2 + 2x + 1$
(*i*) $a(x) = x^4 - 3x^3 - 14x^2 + 12x + 40,\ b(x) = x^2 - 4$
(*j*) $a(x) = x^4 - x^3 - 49x^2 - 71x + 120,\ b(x) = x^2 + 8x + 15$
(*k*) $a(x) = 4x^5 + 3x^2 + 2,\ b(x) = x^3 + x - 1$
(*l*) $a(x) = x^{12} - 64,\ b(x) = x^4 - 2$
(*m*) $a(x) = x^4 - 10x^2 + 9,\ b(x) = x + 2$
(*n*) $a(x) = x^4 - 15x^2 + 10x + 24,\ b(x) = x^2 + 6x + 9$
(*o*) $a(x) = x^5 + 3x^4 - 20x^3 + 15x^2 + 4x - 20,\ b(x) = x^2 - x - 2$
(*p*) $a(x) = x^4 - 9x^3 + 9x^2 + 41x - 42,\ b(x) = x^2 + x - 2.$

3. Arrange the polynomial

$$a(x) = x^3 - 9x^2 + 27x - 27$$

in powers of $(x - 3)$.

4. Express $x^4 + 4x^3 + 4x + 1$ as a polynomial in $x + 1$
5. Express $x^5 - 32$ as a polynomial in $x - 2$
6. Express $x^{10} - 1$ as a polynomial in $x - 1$
7. Express $x^4 - 2$ in powers of $x - \sqrt[4]{2}$
8. Arrange $x^4 - 9x^3 + 36x^2 - 108x + 189$ in powers of $(x - 3)$
9. Express $x^8 + 12x^6 + 56x^4 + 119x^2 + 274$ in powers of $x^2 + 3$
10. Express $(x^6 + 9x^3 + 32x + 16)\,(x^5 - 7x + 6)$ in powers of $x - 1$.
11. Is the polynomial $18x^3 - 105x^2 + 77x - 10$ divisible by $x - 5$? Can your answer be based on an argument without performing actual division?

10.4 REMAINDER THEOREM AND FACTORIZATION

In section 10.3, we observed that given any polynomials $a(x)$ and $b(x)$ in $\mathbf{R}[x]$, there are unique polynomials $q(x)$ and $r(x)$ such that

$$a(x) = b(x)q(x) + r(x) \qquad (1)$$

where either $r(x)$ is zero or deg $r(x) <$ deg $b(x)$. This result is also true in $\mathbf{Z}[x]$, $\mathbf{Q}[x]$ and $\mathbf{C}[x]$. We shall use the division algorithm to get an elegant expression for the remainder when a polynomial $a(x)$ is divided by $(x - \alpha)$.

Theorem 1. (Remainder Theorem) If $a(x)$ is a polynomial in $\mathbf{R}[x]$ and α is a real number, then the remainder after dividing $a(x)$ by $x - \alpha$ is $a(\alpha)$.

Proof. Using division algorithm we can find unique polynomials $q(x)$ and $r(x)$ such that

$$a(x) = (x - \alpha)\, q(x) + r(x) \tag{2}$$

where either $r(x)$ is zero or $\deg r(x) < \deg (x - \alpha) = 1$. Hence if $r(x)$ is not zero, deg $r(x)$ 0; *i.e.*, $r(x)$ must be constant, say r. Putting $x = \alpha$ in (2),

$$a(\alpha) = r(\alpha) = r$$

and this in turn gives

$$a(x) = (x - \alpha)\, q(x) + a(\alpha) \tag{3}$$

Thus the remainder is $a(\alpha)$.

□

Definition 1. Let $a(x)$ be in $\mathbf{R}[x]$. A real or complex number α is called a *root* of the equation

$$a(x) = 0$$

if $a(\alpha) = 0$. We also say that α is a *zero* of $a(x)$.

REMARK. The remainder theorem is also valid in $\mathbf{Z}[x]$, $\mathbf{Q}[x]$ and $\mathbf{C}[x]$. If $a(x)$ is in $\mathbf{Z}[x]$ and α is an integer, then the remainder after dividing $a(x)$ by $(x - \alpha)$ is $a(\alpha)$. Similar results are true in $\mathbf{Q}[x]$ and $\mathbf{C}[x]$.

Corollary. If $a(x)$ is in $\mathbf{R}[x]$ and α is a real zero of $a(x)$, then

$$a(x) = (x - \alpha)\, q(x) \tag{4}$$

for some polynomial $q(x)$ in $\mathbf{R}[x]$.

If α is a complex zero of a polynomal $a(x)$ in $\mathbf{R}[x]$, then (4) is still valid with the understanding that $q(x)$ is now in general a polynomial with complex coefficients.

EXAMPLE 1. Let us consider the polynomial

$$a(x) = x^3 - x^2 + x - 1.$$

Then $a(1) = 0$. Hence the remainder theorem gives

$$a(x) = (x - 1)q(x)$$

for some polynomial $q(x)$. An easy computation gives $q(x) = x^2 + 1$. Thus we have,

$$x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1).$$

EXAMPLE 2. Consider the polynomial

$$a(x) = x^3 + 2x^2 + x + 80.$$

We have

$$\begin{aligned} a(-5) &= (-5)^3 + 2(-5)^2 + (-5) + 80 \\ &= -125 + 50 - 5 + 80 = 0. \end{aligned}$$

Hence using remainder theorem, we have

$$a(x) = (x + 5)q(x)$$

for some polynomial $q(x)$. We shall compare this with example 7 of section 10.3. We have expressed there $a(x)$ as a polynomial in $(x + 5)$;

$$\begin{aligned} a(x) &= (x + 5)^3 - 13(x + 5)^2 + 56(x + 5) \\ &= (x + 5)(x^2 - 3x + 16). \end{aligned}$$

Thus $\qquad q(x) = x^2 - 3x + 16.$

EXAMPLE 3. Let $a(x)$ be given by,

$$a(x) = x^4 + 3x^3 + 5x^2 + 9x + 6.$$

Now we see that

$$\begin{aligned} a(-2) &= (-2)^4 + 3(-2)^3 + 5(-2)^2 + 9(-2) + 6 \\ &= 16 - 24 + 20 - 18 + 6 = 0. \end{aligned}$$

Hence

$$a(x) = (x + 2)q_1(x) \tag{5}$$

for some polynomial $q_1(x)$. On the other hand

$$a(-1) = 1 - 3 + 5 - 9 + 6 = 0.$$

Putting $x = -1$ in (5), we get

$$0 = a(-1) = (-1 + 2)\, q_1(-1) = q_1(-1)$$

Hence we can find a polynomial $q_2(x)$ such that

$$q_1(x) = (x + 1)\, q_2(x) \tag{6}$$

Combining (5) and (6), we obtain

$$a(x) = (x + 2)(x + 1)\, q_2(x).$$

Definition 2. Let $a(x)$ and $b(x)$ be polynomials in $\mathbf{R}[x]$. We say $b(x)$ *divides* $a(x)$, or $b(x)$, is a *divisor* of $a(x)$, if there exists a polynomial $q(x)$ in $\mathbf{R}[x]$ such that

$$a(x) = b(x)q(x). \tag{7}$$

If $b(x)$ divides $a(x)$, we also say that $b(x)$ is a *factor* of $a(x)$ and we write $b(x) \mid a(x)$ (Read $b(x)$ divides $a(x)$).

Suppose $b(x) \mid a(x)$. Then we have

$$a(x) = b(x)q(x)$$

for some $q(x)$ in $\mathbf{R}[x]$. If $\alpha \neq 0$ is a real number we can write

$$a(x) = (\alpha b(x))\,((1/\alpha)q(x))$$

and $(1/\alpha)\, q(x)$ is again in $\mathbf{R}[x]$. Thus $\alpha\, b(x) | a(x)$ for every real number $\alpha \neq 0$.

Let us consider $\mathbf{Z}[x]$. We say a polynomial $m(x)$ divides a polynomial $n(x)$ in $\mathbf{Z}[x]$ if there is a polynomial $l(x)$ in $\mathbf{Z}[x]$ such that

$$n(x) = m(x)l(x).$$

If $km(x) \mid n(x)$ for some integer $k \neq 0$, then

$$n(x) = km(x)l(x) \text{ for some } l(x) \text{ in } \mathbf{Z}[x].$$

We can treat k also as an element of $\mathbf{Z}[x]$, it being a constant polynomial. Thus k also divides $n(x)$. Let us write

$$n(x) = kj(x)$$

where $j(x)$ has integer coefficients. Since

$$\deg(n(x)) = \deg(k) + \deg(j(x))$$

and $\deg(k) = 0$, $j(x)$ and $n(x)$ must have the same degree.

Suppose

$$n(x) = a_p x^p + a_{p-1} x^{p-1} + \ldots + a_0,\ a_p \neq 0,$$

and

$$j(x) = b_p x^p + b_{p-1} x^{p-1} + \ldots + b_0,\ b_p \neq 0.$$

Then we get a set of relations

$$a_p = kb_p,\ a_{p-1} = kb_{p-1},\ \ldots,\ a_0 = kb_0.$$

Thus k is a common factor of the coefficients $a_0, a_1, \ldots, a_p$ of $n(x)$. Hence, whenever $km(x) \mid n(x)$ for some integer k, then k can take only a finite set of values in **Z**, *viz.*, any of the common divisors of the coefficients of $n(x)$. In particular, if g.c.d $(a_0, a_1, \ldots, a_p) = 1$, then $k = +1$ or -1. In contrast, if $a(x)$ and $b(x)$ are in **R**$[x]$ and if $b(x) \mid a(x)$, then $\alpha b(x) \mid a(x)$ for every real number $\alpha \neq 0$. These things are inherent in the structure of **Z** and **R**. The only integers having multiplicative inverses in **Z** are $+1$ and -1. But every real number $\alpha \neq 0$ has a multiplicative inverse in **R**, namely $1/\alpha$.

Example 1 shows that $(x+1)$ and x^2+1 are divisors of $x^3 - x^2 + x - 1$. Example 2 shows that $x + 5$ is a divisor of $x^3 + 2x^2 + x + 80$. Similarly, example 3 shows that $(x+1)$, $(x+2)$, and $(x+1)(x+2)$ are all divisors of $x^4 + 3x^3 + 5x^2 + 9x + 6$. If the relation (7) holds, then $q(x)$ is also a divisor of $a(x)$.

Some important facts on divisibility in **R**$[x]$ are recorded in the following statements.

(1) If $p(x) \mid a(x)$ and $p(x) \mid b(x)$, then $p(x) \mid (a(x) + b(x))$.

(2) If $p(x) \mid a(x)$, then $p(x) \mid (a(x)b(x))$ for any polynomial $b(x)$ in **R**$[x]$.

(3) If $p(x) \mid a(x)$ and α is a real number, then $p(x) \mid \alpha a(x)$.

(4) If $p(x) \mid a(x)$ and $\alpha \neq 0$ is a real number, then $\alpha p(x) \mid a(x)$.

(5) If $q(x) \mid p(x)$ and $p(x) \mid a(x)$, then $q(x) \mid a(x)$.

If $a(x)$ can be written as a product of two polynomials, say

$$a(x) = b(x)c(x),$$

and if either $b(\alpha) = 0$ or $c(\alpha) = 0$, then $a(\alpha) = 0$. Thus any zero of $b(x)$ or $c(x)$ is also a zero $a(x)$. Conversely if α is a zero of $a(x)$, then $a(\alpha) = 0$ and hence

$$b(\alpha)c(\alpha) = 0.$$

This in turn implies that either $b(\alpha) = 0$ or $c(\alpha) = 0$ (or may be both). Thus every zero of $a(x)$ is either a zero of $b(x)$ or a zero of $c(x)$.

Suppose α_1, is a real zero of a polynomial $a(x)$ in **R**$[x]$. Then by division algorithm we can find a polynomial $q_1(x)$ in **R**$[x]$ such that

$$a(x) = (x - \alpha_1)q_1(x).$$

If α_2 is a real zero of $q_1(x)$, then it is also a zero of $a(x)$. On the other hand another application of division algorithm gives

$$q_1(x) = (x - \alpha_2)q_2(x)$$

for some polynomial $q_2(x)$ in **R**$[x]$. We can continue this process until we exhaust all the real zeros of $a(x)$, say $\alpha_1, \alpha_2, \ldots, \alpha_m$. In the end, we have

$$a(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_m)\, q_m(x) \tag{8}$$

for some polynomial $q_m(x)$ in **R**$[x]$, which has no real zeros.

A polynomial $a(x)$ in **R**$[x]$ may not have a real zero. This fact was observed in Chapter 5 on quadratic equations. For example, the polynomial

$$a(x) = x^2 + 1$$

has no real zero. Thus, in general, the polynomial $q_m(x)$ in (8) may have positive degree. We see from relation (8) that

$$\deg(a(x)) = m + \deg(q_m(x)). \tag{9}$$

Thus, we have

$$m \leq \deg(a(x)). \tag{10}$$

We record this observation in the following statement

If $a(x)$ is a polynomial with real coefficients, then the number of real zeros of $a(x)$ cannot exceed deg $(a(x))$.

EXAMPLE 4. Consider the polynomial

$$a(x) = x^3 - 1.$$

It has only one real zero, namely, 1. In fact

$$a(x) = (x - 1)(x^2 + x + 1)$$

and $x^2 + x + 1 > 0$ for all real numbers x, as we have observed in section 5.6. Thus the polynomial $x^2 + x + 1$ has no real zeros. Hence the number of real zeros of $x^3 - 1$ is only one and this is strictly less than the degree of $x^3 - 1$. It shows that the number of real zeros of a polynomial may be strictly less than its degree.

EXAMPLE 5. Let us consider the polynomial

$$a(x) = x^3 - x^2 - x + 1.$$

Since $a(1) = a(-1) = 0$, we can write

$$a(x) = (x - 1)(x + 1)q(x)$$

for some polynomial $q(x)$. We can easily compute $q(x)$, and it is equal to $x - 1$. Thus,

$$a(x) = (x - 1)^2(x + 1).$$

Hence $a(x)$ has zeros 1, 1 and -1. This example shows that a zero of a polynomial may repeat itself. Given a polynomial $a(x)$, we say that α is a *zero of $a(x)$ of multiplicity m* if there exists a polynomial $q(x)$ such that

$$a(x) = (x - \alpha)^m q(x) \text{ where } q(\alpha) \neq 0.$$

Thus $a(x)$ has m zeros $\alpha, \alpha, \ldots, \alpha$, and the remaining zeros of $a(x)$ are precisely the zeros of $q(x)$. If α is a zero of $a(x)$ of multiplicity m, then we count α totally m times when we consider the number of zeros of $a(x)$. Such a process of counting the zeros of $a(x)$ is called '*counting according to multiplicity*'.

In example 5, 1 is a zero of multiplicity 2. Thus counting the zeros of $a(x)$ given in example 5 according to multiplicity, we see that the total number of zeros of $a(x)$ is equal to its degree.

Consider the polynomial

$$a(x) = x^2 + 1.$$

Since $x^2 + 1 > 0$ for all real numbers x, $a(x)$ has no real zeros. This in turn implies that there are no real numbers α_1 and α_2 such that

$$x^2 + 1 = (x - \alpha_1)(x - \alpha_2).$$

In other words, $x^2 + 1$ cannot be written as a product of linear factors with coefficients in **R**. This is an inadequacy of the real number system itself. However, if we consider $x^2 + 1$ as a polynomial over **C**, then $x^2 + 1$ has zeros in **C** Infact, $+i$ and $-i$ are the zeros of $x^2 + 1$, and

$$x^2 + 1 = (x + i)(x - i).$$

This is indeed true of any polynomial $a(x)$. It is a consequence of a deep result known as fundamental theorem of algebra that any polynomial $a(x)$ in **R**$[x]$ can be factored as

$$a(x) = \beta(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n), \tag{11}$$

for some complex numbers $\beta, \alpha_1, \alpha_2, \ldots, \alpha_n$; $n = \deg a(x)$.

Fundamental Theorem of Algebra If $a(x)$ is a nonconstant polynomial with complex coefficients, then $a(x)$ has at least one zero in **C**.

If $a(x)$ is in **R**$[x]$, then the fundamental theorem of algebra implies that $a(x)$ has a zero α_1 in **C**. Hence.

$$a(x) = (x - \alpha_1)q_1(x)$$

for some polynomial $q_1(x)$ in **C**$[x]$. Applying the fundamental theorem of algebra to $q_1(x)$, we conclude that $q_1(x)$ has a zero α_2 in **C**. Hence, we get

$$a(x) = (x - \alpha_1)(x - \alpha_2)q_2(x)$$

for some polynomial $q_2(x)$. Continuing this, we conclude that a factorization of the form (11) holds for $a(x)$. We can always get a factorization of a polynomial $a(x)$ in **R**$[x]$ involving only quadratic factors and linear factors. Suppose

$$w = \alpha + i\beta,\ \beta \neq 0$$

is a zero of $a(x)$. If

$$a(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0,$$

where $a_0, a_1, \dots a_n$, are real numbers, then

$$a_n w^n + a_{n-1} w^{n-1} + \dots + a_0 = 0.$$

Taking the complex conjugation on both sides, we get

$$a_n \overline{w}^n + a_{n-1}\overline{w}^{n-1} + \dots + a_0 = 0.$$

Thus $\overline{w}$ is also a zero of $a(x)$. This reasoning shows that the nonreal zeros of a polynomial $a(x)$ in **R**$[x]$ appear in pairs. But then $(x - w)$, $(x - \overline{w})$ and hence $(x - w)(x - \overline{w})$ are all factors of $a(x)$. However, we can write

$$\begin{aligned}(x - w)(x - \overline{w}) &= (x - \alpha - i\beta)(x - \alpha + i\beta)\\ &= (x - \alpha)^2 + \beta^2.\end{aligned}$$

Hence $(x - \alpha)^2 + \beta^2$ is a factor of $a(x)$. Since $(x - \alpha)^2 + \beta^2$ is in **R**$[x]$ and $a(x)$ is in **R**$[x]$, the quotient after the division of $a(x)$ by $(x - \alpha)^2 + \beta^2$ gives again a polynomial in **R**$[x]$. Applying the same reasoning to this quotient, we can further factor $a(x)$. Finally we get a factorization in the form

$$a(x) = c(x - y_1)(x - y_2) \dots (x - y_k)q_1(x)q_2(x)\dots q_m(x) \tag{12}$$

where $y_1, y_2, \dots, y_k$ are real zeros of $a(x)$ and $q_j(x)$ are quadratic factors of the form $(x - \alpha)^2 + \beta^2$.

In the factorization (12), k could be zero or m could be zero. If we take the polynomial

$$a(x) = x^2 - 3x + 2$$

then it can be factored as

$$a(x) = (x - 2)(x - 1)$$

so that $a(x)$ has no quadratic factors and $m = 0$ in (12). On the other hand, let us consider

$$a(x) = x^4 + 3x^2 + 2.$$

Then $a(x) > 0$ for all real numbers x and hence $a(x)$ has no real zero. Again, we can have a factorization

$$a(x) = (x^2 + 1)(x^2 + 2)$$

involving only quadratic factors and $k = 0$ in (12).

If $a(x)$ is a polynomial in $\mathbf{R}[x]$ of odd degree, then the factorization (12) shows that $a(x)$ has at least one linear factor. Hence $a(x)$ has at least one real zero. We record this in the following.

A polynomial of odd degree with real coefficients has at least one real zero.

EXERCISE 10.3

1. Verify whether $a(x)$ is divisible by $b(x)$ in $\mathbf{R}[x]$ in the following problems
   (*a*) $a(x) = x^4 + 2x^2 - 8$, $b(x) = x - 2$.
   (*b*) $a(x) = x^5 + 5x^3 + 3x^2 + 9$, $b(x) = 2x - 3$.
   (*c*) $a(x) = x^4 - 5x^2 + 6$, $b(x) = x - 2$.
   (*d*) $a(x) = x^3 - 3x^2 - 3x + 1$, $b(x) = x - 1$.
   (*e*) $a(x) = x^9 - 6x^6 + 12x^3 - 8$, $b(x) = x - 2$.
   (*f*) $a(x) = 35x^3 - 124x^2 - 67x + 12$, $b(x) = 5x + 3$.
   (*g*) $a(x) = 18x^3 - 5x^2 + 77x - 10$, $b(x) = x - 5$
   (*h*) $a(x) = 63x^3 - 149x^2 + 48x - 4$, $b(x) = x - 2$.
   (*i*) $a(x) = 6x^3 + 17x^2 - 23x - 70$, $b(x) = -2x + 5$.
   (*j*) $a(x) = 2x^4 - x^3 - 29x^2 + 26x + 48$, $b(x)\ x - 3$.
   (*k*) $a(x) = 2x^4 + 5x^3 - 50x^2 + 25x + 28$, $b(x) = x^2 + 3x - 28$.
   (*l*) $a(x) = x^6 + 4x^5 + 8x^3 - 9x + 2$, $b(x) = x^2 + 3x + 2$.
   (*m*) $a(x) = x^5 - x^3 - x^2 + 4x + 2$, $b(x) = x^2 + 2x + 2$.
2. Prove that $x^m + \alpha^m$ is not divisible by $x - \alpha$.
3. Prove that $x^{2k+1} - \alpha^{2k+1}$ is not divisible by $x + \alpha$.
4. Prove that $x^{2k} + \alpha^{2k}$ is not divisible by $x + \alpha$.
5. Prove that $2n^3 - 3n^2 + n$ is divisible by 6 for any natural number n.
6. Prove that $x^n - a^n$ is divisible by $x - a$.
7. Show that $x^n + a^n$ is divisible by $x + a$ iff n is odd.

10.5 GCD AND LCM OF POLYNOMIALS

In Chapter 2, we studied the concepts of greatest common divisor and least common multiple of two integers. Recall that if m and n are two integers, then an integer r is a greatest common divisor (gcd for short) of m and n if

(*i*) $r \mid m$ and $r \mid n$, and
(*ii*) if $l \mid m$ and $l \mid n$, then $l \mid r$.

Thus r is a common divisor of m and n, and it is largest, only in the sense of (*ii*). For example, a gcd of 12 and 18 is 6. We observe that -6 also is a gcd of 12 and 18. This brings out a fundamental aspect of gcd of two integers that it is not uniquely defined. Whenever r is a gcd of m and n, $-r$ is also a gcd of m and n. However, *if we require that gcd be positive*, then it is uniquely determined.

All these ideas can be carried to polynomials. We shall begin with an example.

EXAMPLE 1. Consider the polynomials

$$a(x) = x^4 - 2x^3 - x^2 + 4x - 2,$$
$$b(x) = x^4 - 3x^2 + 2.$$

We observe that $a(1) = 0 = a(\sqrt{2}) = a(-\sqrt{2})$. Similarly $b(1) = b(-1) = (b\sqrt{2}) = b(-\sqrt{2}) = 0$. Thus we can get factorizations,

$$a(x) = (x^2 - 2)(x - 1)^2,$$
$$b(x) = (x^2 - 2)(x^2 - 1).$$

This factorization shows that $(x^2 - 2) \mid a(x)$ and $(x^2 - 2) \mid b(x)$. Similarly $(x - 1)$ divides both $a(x)$ and $b(x)$. Infact, $(x - 1)$, $(x - \sqrt{2})$, $(x + \sqrt{2})$, $(x^2 - 2)$, $(x - 1)(x - \sqrt{2})$, $(x - 1)(x + \sqrt{2})$ and $(x^2 - 2)(x - 1)$ are common divisors of $a(x)$ and $b(x)$. Of course, we have seen that whenever $q(x) \mid a(x)$ then $\alpha\, q(x) \mid a(x)$ for any real number $\alpha \neq 0$. But apart from these nonzero constant factors, the only common divisors of $a(x)$ and $b(x)$ are the ones just listed earlier. We note that $(x - 1)(x^2 - 2)$ is divisible by every common divisor of $a(x)$ and $b(x)$. Thus we can expect $(x - 1)(x^2 - 2)$ to be a candidate for greatest common divisor of $a(x)$ and $b(x)$.

EXAMPLE 2. Consider the polynomials

$$a(x) = (3x + 2)(x + 1)(x^2 - 4),$$
$$b(x) = (3x + 2)(x - 2)(x^2 + 3).$$

A set of common divisors of $a(x)$ and $b(x)$ is $\{(3x + 2), (x - 2), (3x + 2)(x - 2)\}$. Again a nonzero real multiple of each of these common divisors is again common divisor. Thus, upto constant factors, $(3x + 2)$, $(x - 2)$ and $(3x + 2)(x - 2)$ are the only common divisors of $a(x)$ and $b(x)$. Again, $(3x + 2)(x - 2)$ is divisible by every common divisor of $a(x)$ and $b(x)$.

The polynomials behave very much like integers and the above two examples tell us that it is possible to imitate the notions of gcd and l cm in **Z** to define gcd and l cm of polynomials in **R**[x].

Definition 3. Let $a(x)$ and $b(x)$ be polynomials in **R**[x]. A polynomial $q(x)$ in **R**[x] is called a *greatest common divisor* of $a(x)$ and $b(x)$ if

(*i*) $q(x) \mid a(x)$ and $q(x) \mid b(x)$, and

(*ii*) whenever $r(x) \mid a(x)$ and $r(x) \mid b(x)$ then $r(x) \mid q(x)$.

Thus $q(x)$ is a common divisor of $a(x)$ and $b(x)$, and every common divisor of $a(x)$ and $b(x)$ is also a divisor of $q(x)$.

We have already observed earlier that whenever $q(x) \mid a(x)$ then $\alpha q(x) \mid a(x)$ for every $\alpha \neq 0$. Hence if $q(x)$ is a common divisor of $a(x)$ and $b(x)$, $\alpha q(x)$ is also a common divisor of $a(x)$ and $b(x)$ for every $\alpha \neq 0$. Therefore we observe that if $q(x)$ is a gcd of $a(x)$ and $b(x)$ then $\alpha q(x)$ is also a gcd of $a(x)$ and $b(x)$ for every real number $\alpha \neq 0$. This shows that gcd of two polynomials is not uniquely determined. It is determined only upto a constant real factor. However, *if we require that the leading coefficient be 1*, then gcd of $a(x)$ and $b(x)$ is uniquely determined. A gcd of two polynomials that has leading coefficient 1 is called *the gcd* of the given polynomials.

A polynomial $a(x)$ whose leading coefficient is 1 is called a *monic* polynomial. Thus the gcd of two polynomials is a gcd which is also monic.

EXAMPLE 3. *Find the gcd of*

$$a(x) = (3x + 1)(3x + 2)(x^2 - 3)$$
$$b(x) = (3x + 1)(x - \sqrt{3})^2(x - 4).$$

SOLUTION. A set of common divisors of $a(x)$ and $b(x)$ is

$$\{(3x + 1), (x - \sqrt{3}), (3x + 1)(x - \sqrt{3})\}.$$

These are the only common divisors of $a(x)$ and $b(x)$ upto constant real factors. Hence a gcd of $a(x)$ and $b(x)$ is $(3x+1)(x-\sqrt{3})$. We also observe that $(x+(1/3))(x-\sqrt{3})$ is also a gcd and it is monic. Hence the gcd of $a(x)$ and $b(x)$ is

$$(x+1/3)(x-\sqrt{3}).$$

The least common multiple of two polynomials may be defined imitating the definition of lcm of two integers.

Definition 4. Let $a(x)$ and $b(x)$ be two polynomials in $\mathbf{R}[x]$. Then an element $q(x)$ of $\mathbf{R}[x]$ is called a *least common multiple* of $a(x)$ and $b(x)$ if

(*i*) $a(x) \mid q(x)$ and $b(x) \mid q(x)$, and

(*ii*) whenever $a(x) \mid r(x)$ and $b(x) \mid r(x)$, then $q(x) \mid r(x)$.

We also observe that whenever $q(x)$ is an lcm of $a(x)$ and $b(x)$ then $\alpha q(x)$ is also an l cm of $a(x)$ and $b(x)$ for every $\alpha \neq 0$. Thus the l cm of two polynomials in $\mathbf{R}[x]$ is not uniquely determined but only upto a constant real multiple. If we require that l cm of two polynomils be monic, then it is uniquely determined. Such an lcm of two polynomials is called *the lcm* of the given polynomials.

EXAMPLE 4. Let us consider

$$a(x) = (x^2-2)(x-1)^2$$
$$b(x) = (x^2-2)(x^2-1).$$

If we take

$$q(x) = (x^2-2)(x-1)^2(x+1),$$

then $a(x) \mid q(x)$ and $b(x) \mid q(x)$. If we drop any factor of $q(x)$, then the resulting polynomial is not divisible by at least one of $a(x)$ and $b(x)$. Thus $q(x)$ is the lcm of $a(x)$ and $b(x)$.

EXAMPLE 5. *Find the lcm of*

$$a(x) = (3x+2)(x+1)(x^2-4)$$
$$b(x) = (3x+2)(x-2)(x^2-1).$$

SOLUTION. If we take

$$q(x) = (3x+2)(x^2-1)(x^2-4),$$

then $q(x)$ is a lcm of $a(x)$ and $b(x)$. Hence the lcm is given by

$$(x+2/3)(x^2-1)(x^2-4).$$

EXERCISE 10.4

1. Find the gcd and lcm of $a(x)$ and $b(x)$ in the following set

   (*a*) $a(x) = (x^2+2)(x+9)(x^5-1)$, $b(x) = (x^2+2)(x-1)(x^4+2)$

   (*b*) $a(x) = (x^2+2x-8)(2x^2-x-1)$, $b(x) = (x+4)\,(6x^2-5x-4)$

   (*c*) $a(x) = (\sqrt{2}x^2-5x+3\sqrt{2})\,(x^2-9)$, $b(x) = (x^2-2)(x-3)$

   (*d*) $a(x) = (12x^2-5x-2)(9x^2+5x-4)$,
   $b(x) = (4x^2+5x+1)(x^3-12x^2+47x-60)$

   (*e*) $a(x) = x^4-1$, $b(x) = x^3-x^2+x-1$

   (*f*) $a(x) = (\sqrt{2}x+2)(x^2+1)(x-2)2$, $b(x) = (x+\sqrt{2})(x^3+x)(x-2)$

   (*g*) $a(x) = (3x^2+2\sqrt{3}x+1)(x+4x+3)$, $b(x) = (x+1\sqrt{3})(x+3)$

   (*h*) $a(x) = (\sqrt{3}x^2+4x+\sqrt{3})(x^5+6x^4+8x^3)$, $b(x) = (x^2+4x)\,(\sqrt{3}x+1)\,(x+\sqrt{5})$

2. If $p(x)$ is a gcd of $a(x)$ and $b(x)$, show that $p(x)$ is also a gcd of $\alpha a(x)$ and $\beta b(x)$ where α and β are nonzero constants.

10.6 EUCLID'S ALGORITHM

In Chapter 2, we have developed a systematic method called Euclid's algorithm for finding a gcd of two integers. Now let us consider the problem of finding a gcd of two polynomials $a(x)$ and $b(x)$ in $\mathbf{R}[x]$. If we have a complete factorization of both $a(x)$ and $b(x)$, then we can find their gcd by looking at the common factors of $a(x)$ and $b(x)$. However, given a polynomial, it may be extremely difficult to find its factors. In this section we develop, imitating what we did in $\mathbf{Z}$, an elegant method called Euclid's algorithm which can be used to find the gcd of two polynomials without worrying about the factorizations of the given polynomials. A basis for the algorithm is the division algorithm for polynomials described in Section 10.3. Given two polynomials $a(x)$ and $b(x)$ in $\mathbf{R}[x]$, we can find unique polynomials $q(x)$ and $r(x)$ in $\mathbf{R}[x]$ such that

$$a(x) = b(x)q(x) + r(x) \tag{1}$$

where

$$r(x) \equiv 0 \text{ or } \deg r(x) < \deg b(x). \tag{2}$$

Suppose $p(x)$ is a gcd of $a(x)$ and $b(x)$. The definition of the gcd of two polynomials shows that $p(x) \mid a(x)$ and $p(x) \mid b(x)$, and every polynomial $l(x)$ dividing both $a(x)$ and $b(x)$ also divides $p(x)$. The relation (1) gives

$$r(x) = a(x) - b(x)q(x). \tag{3}$$

Thus $p(x) \mid r(x)$. If $l(x) \mid b(x)$ and $l(x) \mid r(x)$, then the relation (1) shows that $l(x) \mid a(x)$ Hence $p(x)$, being the gcd of $a(x)$ and $b(x)$, is divisible by $l(x)$. Thus we have shown that $p(x) \mid b(x)$, $p(x) \mid r(x)$ and every polynomial $l(x)$ dividing both $b(x)$ and $r(x)$ also divides $p(x)$. Now it follows that $p(x)$ is a gcd of $b(x)$ and $r(x)$.

Conversely, if $p(x)$ is a gcd of $b(x)$ and $r(x)$, then $p(x)$ $b(x)$ and $p(x) \mid r(x)$. This implies that $p(x) \mid a(x)$. If $l(x) \mid b(x)$ and $l(x) \mid a(x)$, then $l(x) \mid r(x)$ by (3). Since $p(x)$ is a gcd of $b(x)$ and $r(x)$, $l(x) \mid p(x)$. Thus $p(x)$ is also a gcd of $a(x)$ and $b(x)$.

Thus the relation (1) implies that $p(x)$ is a gcd of $a(x)$ and $b(x)$ iff it is a gcd of $b(x)$ and $r(x)$. Moreover, relation (2) shows that $\deg r(x) < \deg b(x)$. Thus we have been able to reduce the problem of finding the gcd of $a(x)$ and $b(x)$ to the problem of finding the gcd of polynomials of lower degree. Further application of division algorithm (1) brings in further reduction in degrees of polynomials. These ideas are clarified in a few examples.

EXAMPLE 1. *Find the gcd of the polynomials*

$$a(x) = x^4 - 2x^3 - x^2 + 4x - 2,$$
$$b(x) = x^4 - 3x^2 + 2.$$

SOLUTION. We shall begin with the division of $a(x)$ by $b(x)$. We use synthetic division for finding the quotient $q_1(x)$ and the remainder $r_1(x)$.

$$\begin{array}{r|rrrrr}
1 - (0 + 3 + 0 - 2) & 1 & -2 & -1 & +4 & -2 \\
 & & & & & -2 \\
 & & & & +0 & \\
 & & & +3 & & \\
 & & +0 & & & \\
\hline
 & 1 & -2 & +2 & +4 & -4
\end{array}$$

The division table shows that

$$q_1(x) = 1$$
$$r_1(x) = -2(x^3 - x^2 - 2x + 2)$$

Thus we can write

$$a(x) = b(x)q_1(x) + r_1(x) \tag{4}$$

where

$$\deg r_1(x) = 3 < 4 = \deg b(x).$$

But when a relation of the form (4) holds, we have seen that $p(x)$ is a gcd of $a(x)$ and $b(x)$ iff it is a gcd of $b(x)$ and $r_1(x)$. Thus it is sufficient to find a gcd of $b(x)$ and $r_1(x)$. Again we use division algorithm to find the quotient $q_2(x)$ and the remainder $r_2(x)$ such that

$$b(x) = r_1(x)q_2(x) + r_2(x). \tag{5}$$

Since gcd of two polynomials is determined only upto a constant multiple, it is sufficient to determine a gcd of $b(x)$ and the polynomial $x^3 - x^2 - 2x + 2$ (see exercise 2 in section 10.5). We use synthetic division.

$$\begin{array}{r|rrrrr}
1-(1+2-2) & 1 & +\ 0 & -\ 3 & +\ 0 & +\ 2 \\
 & & & & -\ 2 & -\ 2 \\
 & & & +\ 2 & +\ 2 & \\
 & & +\ 1 & +\ 1 & & \\
\hline
 & 1 & +\ 1 & +\ 0 & +\ 0 & +\ 0
\end{array}$$

The division table gives us

$$q_2(x) = x + 1,$$
$$r_2(x) = 0.$$

Thus the relation (5) reduces to

$$b(x) = (x^3 - x^2 - 2x + 2)(x + 1). \tag{6}$$

The relation (6) shows that $x^3 - x^2 - 2x + 2$ is a gcd of $b(x)$ and $r_1(x)$. Hence it is also a gcd of $a(x)$ and $b(x)$. Since the leading coefficient of $x^3 - x^2 - 2x + 2$ is 1, it is the gcd of $a(x)$ and $b(x)$.

EXAMPLE 2. *Find the gcd of*

$$a(x) = 3x^3 + x + 4,$$
$$b(x) = 2x^3 - x^2 + 3.$$

SOLUTION. We begin with the observation that

$$a(x) = \left(\frac{3}{2}\right) b(x) + r_1(x) \tag{7}$$

where $$r_1(x) = \frac{3}{2}\, x^2 + x - \frac{1}{2}. \tag{8}$$

Hence it is sufficient to find a gcd of $b(x)$ and $r_1(x)$. But

$$b(x) = \left(\frac{4}{3}x - \frac{14}{9}\right) r_1(x) + r_2(x) \tag{9}$$

where $$r_2(x) = \left(\frac{20}{9}\right)(x + 1). \tag{10}$$

It is sufficient to find a gcd of $r_1(x)$ and $r_2(x)$. We can now write

$$r_1(x) = \left(\left(\frac{3}{2}\right)x - \frac{1}{2}\right)(x+1) = \left(\frac{9}{20}\right)\left(\left(\frac{3}{2}\right)x - \frac{1}{2}\right)\left(\frac{20}{9}\right)(x+1)$$

$$= \left(\frac{9}{20}\right)\left(\left(\frac{3}{2}\right)x - \frac{1}{2}\right)r_2(x).$$

This relation shows that $r_2(x) \mid r_1(x)$. Hence $r_2(x)$ is a gcd of $r_1(x)$ and $r_2(x)$. Retracing the steps back, we conclude that $r_2(x)$ is a gcd of $a(x)$ and $b(x)$. Hence the gcd of $a(x)$ and $b(x)$ is $x + 1$.

Previous examples bring out the main feature of Euclid's method. The problem of finding a gcd of $a(x)$ and $b(x)$ is successively reduced to a problem of finding a gcd of lower degree polynomials, until we come to a stage where one of the polynomials divides the other.

Theorem 2. (Euclid's Algorithm) Let $a(x)$ and $b(x)$ be any two polynomials with real coefficients. Define polynomials $q_j(x)$ and $r_j(x)$ recursively by

$$\left.\begin{aligned} a(x) &= b(x)\,q_1(x) + r_1(x) \\ b(x) &= r_1(x)\,q_2(x) + r_2(x) \\ r_1(x) &= r_2(x)\,q_3(x) + r_3(x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ r_{k-2}(x) &= r_{k-1}(x)\,q_k(x) \end{aligned}\right\} \qquad (11)$$

where $\deg r_{j+1}(x) < \deg r_j(x)$ for $j = 1, 2, \ldots, (k-2)$. then $r_{k-1}(x)$ is a gcd of $a(x)$ and $b(x)$.

Proof. We begin with the observation that there is always an integer k such that $r_k(x) \equiv 0$. Since $\deg r_{j+1}(x) < \deg r_j(x)$, we can find l such that $\deg r_{l-1}(x) \equiv 0$. Hence $r_{l-1}(x) = c$ for some constant c. If $c = 0$, we can take $k = l - 1$ since $r_{l-1}(x) \equiv 0$. If $c \neq 0$, then $r_{l-2}(x) = c_{q_l}(x)$ and hence $r_l(x) \equiv 0$. We can now take $k = l$. In any case, we can find k such that $r_k(x) \equiv 0$.

Suppose $p(x) \mid a(x)$ and $p(x) \mid b(x)$. Then

$$p(x) \mid [a(x) - b(x)q_1(x)].$$

Thus $p(x) \mid r_1(x)$. Now, since $p(x) \mid b(x)$ and $p(x) \mid r_1(x)$, we get

$$p(x) \mid [b(x) - r_1(x)q_2(x)].$$

Hence $p(x) \mid r_2(x)$. Continuing the process, we conclude that $p(x) \mid r_j(x)$ for $j = 1, 2, 3, \ldots, k-1$.

Conversely suppose $p(x) \mid r_{k-1}(x)$ and $p(x) \mid r_{k-2}(x)$. Since

$$r_{k-3}(x) = r_{k-2}(x)q_{k-1}(x) + r_{k-1}(x),$$

we conclude that $p(x) \mid r_{k-3}(x)$. Continuing the argument, we conclude that $p(x) \mid b(x)$ and finally $p(x) \mid a(x)$.

Thus $p(x)$ is a common divisor of $a(x)$ and $b(x)$ iff $p(x)$ is a common divisor of $r_{k-2}(x)$ and $r_{k-1}(x)$. But the choice of k is such that

$$r_{k-2}(x) = r_{k-1}(x)q_k(x). \qquad (12)$$

Thus $r_{k-1}(x) \mid r_{k-2}(x)$. This implies that $r_{k-1}(x)$ is a common divisor of $r_{k-2}(x)$ and $r_{k-1}(x)$. Hence $r_{k-1}(x)$ is a common divisor of $a(x)$ and $b(x)$.

Moreover, (12) shows that $r_{k-1}(x)$ is a gcd of $r_{k-2}(x)$ and $r_{k-1}(x)$. If $l(x)$ is a common divisor of $a(x)$ and $b(x)$, then $l(x)$ is also a common divisor of $r_{k-2}(x)$ and $r_{k-1}(x)$. Since $r_{k-1}(x)$ is a gcd of $r_{k-2}(x)$ and $r_{k-1}(x)$, it now follows that $l(x) \mid r_{k-1}(x)$. Thus $r_{k-1}(x)$ is a common divisor of $a(x)$ and $b(x)$, and any common divisor $l(x)$ of $a(x)$ and $b(x)$ also divides $r_{k-1}(x)$. We conclude that $r_{k-1}(x)$ is a gcd of $a(x)$ and $b(x)$.

❑

EXAMPLE 3. *Find the gcd of a(x) and b(x) where*

$$a(x) = x^5 + 9x^4 + x^3 + 9x^2 - 2x - 18,$$
$$b(x) = x^4 - 4.$$

SOLUTION. Our strategy is to use Euclid's algorithm. The division process is shown in the following table

$$\begin{array}{r|rrrrrr}
1-(0+0+0+4) & 1 & +9 & +1 & +9 & -2 & -18 \\
 & & & & & -4 & +36 \\
 & & & & +0 & +0 & \\
 & & & +0 & +0 & & \\
 & & +0 & +0 & & & \\
\hline
 & 1 & +9 & +1 & +9 & +2 & +18
\end{array}$$

$$\therefore \qquad q_1(x) = x + 9,\ r_1(x) = x^3 + 9x^2 + 2x + 18.$$

$$\begin{array}{r|rrrrr}
1-(-9-2-18) & 1 & +0 & +0 & +0 & -4 \\
 & & & & -18 & +162 \\
 & & & -2 & +18 & \\
 & & & -9 & +81 & \\
\hline
 & 1 & -9 & +79 & +0 & +158
\end{array}$$

$$\therefore \qquad q_2(x) = x - 9,\ r_2(x) = 79x^2 + 158 = 79(x^2 + 2).$$

But
$$\begin{aligned} r_1(x) &= x^3 + 9x^2 + 2x + 18 \\ &= x(x^2+2) + 9(x^2+2) \\ &= (x+9)\,(x^2+2). \end{aligned}$$

Hence
$$\begin{aligned} r_1(x) &= (1/79)(x+9)\,(79)(x^2+2) \\ &= (1/79)(x+9)r_2(x). \end{aligned}$$

By Euclid's algorithm, $r_2(x)$ is a gcd of $a(x)$ and $b(x)$. This implies that $x^2 + 2$ is the gcd of $a(x)$ and $b(x)$.

If m and n are integers, and if d is a gcd of m and n, then $l = \dfrac{mn}{d}$ is an lcm of m and n.

This relation between lcm and gcd is also true for polynomials. If $p(x)$ is a gcd of $a(x)$ and $b(x)$, then,

$$q(x) = \frac{a(x)\,b(x)}{p(x)}$$

is a lcm of $a(x)$ and $b(x)$. Thus a gcd $p(x)$ and a lcm $q(x)$ of two polynomials $a(x)$ and $b(x)$ are related by

$$\alpha p(x)q(x) = a(x)b(x) \tag{13}$$

where $\alpha \neq 0$ is a real constant (A proof of this is relegated to problems at the end of this chapter). This relation (13) is often used to find a lcm of two polynomials.

EXAMPLE 4. *Find the lcm of*

$$a(x) = 3x^4 - 4x^3 + 1,$$
$$b(x) = 4x^4 - 5x^3 - x^2 + x + 1.$$

SOLUTION. We first find the gcd of $a(x)$ and $b(x)$ using Euclid's algorithm. We can write

$$a(x) = \left(\frac{3}{4}\right)b(x) + \left(\frac{1}{4}\right)(x^3 - 3x^2 + 3x - 1).$$

Hence it is sufficient to find a gcd of $b(x)$ and $r_1(x) = x^3 - 3x^2 + 3x - 1$. We use synthetic division.

$$\begin{array}{l|lllll} 1-(3-3+1) & 4 & -5 & -1 & +1 & +1 \\ & & & & +4 & +7 \\ & & & -12 & -21 & \\ & & +12 & +21 & & \\ \hline & 4 & +7 & +8 & -16 & +8 \end{array}$$

This division process shows that

$$\begin{aligned} b(x) &= (4x+7)(x^3 - 3x^2 + 3x - 1) + 8x^2 - 16x + 8 \\ &= (4x+7)(x-1)^3 + 8(x-1)^2. \end{aligned}$$

Hence the problem is now reduced to find a gcd of $(x-1)^3$ and $8(x-1)^2$. We observe that $(x-1)^2$ is the required gcd of $a(x)$ and $b(x)$.

Now we use the relation (13) for finding the lcm of $a(x)$ and $b(x)$. Since $(x-1)^2$ is the gcd of $a(x)$ and $b(x)$, it is a factor of both $a(x)$ and $b(x)$. We use synthetic division for finding the remaining factor. Division of $a(x)$ by $(x-1)^2$ is as follows.

$$\begin{array}{l|lllll} 1-(2-1) & 3 & -4 & +0 & +0 & +1 \\ & & & -3 & -2 & -1 \\ & & +6 & +4 & +2 & \\ \hline & 3 & +2 & +1 & +0 & +0 \end{array}$$

Thus we get

$$3x^4 - 4x^3 + 1 = (3x^2 + 2x + 1)(x-1)^2.$$

Division of $b(x)$ by $(x-1)^2$ is shown in the following table.

$$\begin{array}{l|lllll} 1-(2-1) & 4 & -5 & -1 & +1 & +1 \\ & & & -4 & -3 & +1 \\ & & +8 & +6 & +2 & \\ \hline & 4 & +3 & +1 & +0 & +0 \end{array}$$

This gives

$$4x^4 - 5x^3 - x^2 + x + 1 = (4x^2 + 3x + 1)(x-1)^2.$$

An lcm of $a(x)$ and $b(x)$ is given by

$$q(x) = \frac{a(x)\,b(x)}{(x-1)^2} = (3x^2 + 2x + 1)\,(4x^2 + 3x + 1)\,(x - 1)^2$$

The lcm of $a(x)$ and $b(x)$ is

$$\left(x^2 + \left(\frac{2}{3}\right)x + \left(\frac{1}{3}\right)\right)\left(x^2 + \left(\frac{3}{4}\right)x + \left(\frac{1}{4}\right)\right)(x-1)^2.$$

EXAMPLE 5. *Find the gcd and the lcm of*

$$a(x) = 2x^7 + 18x^6 + 6x^5 + 19x^4 + 13x^3 + 3x^2 + 9x + 2,$$
$$b(x) = 2x^3 + 19x^2 + 13x + 2.$$

SOLUTION. It is sufficient to find a gcd of

$$a_1(x) = x^7 + 9x^6 + 3x^5 + \left(\frac{19}{2}\right)x^4 + \left(\frac{13}{2}\right)x^3 + \left(\frac{3}{2}\right)x^2 + \left(\frac{9}{2}\right)x + 1$$

$$b_1(x) = x^3 + \left(\frac{19}{2}\right)x^2 + \left(\frac{13}{2}\right)x + 1.$$

We use synthetic division

$$\begin{array}{r|rrrrrrrr}
1-(-19/2\ -13/2-1) & 1 & +\ 9 & +\ 3 & +\ 19/2 & +\ 13/2 & +\ 3/2 & +\ 9/2 & +\ 1 \\
 & & & & -\ 1 & +\ 1/2 & -\ 5/4 & +\ 1/8 & -\ 1/16 \\
 & & & -\ 13/2 & +\ 13/4 & -\ 65/8 & +\ 13/16 & -\ 13/32 & \\
 & & -\ 19/2 & +\ 19/4 & -\ 95/8 & +19/16 & -\ 19/32 & & \\
\hline
 & 1 & -\ 1/2 & +\ 5/4 & -\ 1/8 & +\ 1/16 & +\ 15/32 & +\ 135/32 & +\ 15/16
\end{array}$$

Thus the remainder $r_1(x)$ is given by

$$r_1(x) = \left(\frac{15}{32}\right)(x^2 + 9x + 2).$$

Now it is sufficient to find a gcd of $b_1(x)$ and $x^2 + 9x + 2$.

$$\begin{array}{r|rrrr}
1-(-9\ -2) & 1 & +\ 19/2 & +\ 13/2 & +\ 1 \\
 & & & -\ 2 & -\ 1 \\
 & & -\ 9 & -\ 9/2 & \\
\hline
 & 1 & +\ 1/2 & +\ 0 & +\ 0
\end{array}$$

The division table shows that $x^2 + 9x + 2$ divides $b_1(x)$. Thus the gcd of $a_1(x)$ and $b_1(x)$ and hence that of $a(x)$ and $b(x)$ is $x^2 + 9x + 2$.

Now we get other factors of $a(x)$ and $b(x)$ dividing them by $x^2 + 9x + 2$.

$$\begin{array}{r|rrrrrrrr}
1-(-9\ -2) & 2 & +\ 18 & +\ 6 & +\ 19 & +\ 13 & +\ 3 & +\ 9 & +\ 2 \\
 & & & -\ 4 & +\ 0 & -\ 4 & -\ 2 & +\ 0 & -\ 2 \\
 & & & -18 & +\ 0 & -\ 18 & -\ 9 & +\ 0 & -\ 92 \\
\hline
 & 2 & +\ 0 & +\ 2 & +\ 1 & +\ 0 & +\ 1 & +\ 0 & +\ 0
\end{array}$$

This shows that

$$\begin{aligned} a(x) &= (x^2 + 9x + 2)(2x^5 + 2x^3 + x^2 + 1) \\ &= (x^2 + 9x + 2)(2x^3 + 1)(x^2 + 1). \end{aligned}$$

Similarly dividing $b(x)$ by $x^2 + 9x + 2$, we have.

$$\begin{array}{r|rrrr} 1 - (-9\ -2) & 2\ + & 19\ + & 13\ + & 2 \\ & & & -\ 4 & -\ 2 \\ & & -\ 18 & -\ 9 & \\ \hline & 2\ + & 1\ + & 0\ + & 0 \end{array}$$

We see that

$$b(x) = (2x + 1)(x^2 + 9x + 2)$$

An lcm of $a(x)$ and $b(x)$ is given by

$$q(x) = \frac{a(x)\,b(x)}{x^2 + 9x + 2} = (2x + 1)(x^2 + 9x + 2)(2x^3 + 1)(x^2 + 1)$$

Hence the lcm is

$$(x + \frac{1}{2})(x^2 + 9x + 2)(x^3 + \frac{1}{2})(x^2 + 1).$$

EXERCISE 10.5

1. Find the gcd and the lcm of the following polynomials:
   (*a*) $a(x) = 4x^4 + 5x^2 + 7x + 2$, $b(x) = 16x^3 + 10x + 7$
   (*b*) $a(x) = 2x^4 - 13x^2 + x + 15$, $b(x) = 3x^4 - 2x^3 - 17x^2 + 12x + 9$
   (*c*) $a(x) = x^5 + 5x^2 - 2$, $b(x) = 2x^5 - 5x^3 + 1$
   (*d*) $a(x) = 2x^5 - 5x^2 + 3$, $b(x) = 3x^5 - 5x^3 + 2$
   (*e*) $a(x) = 2x^3 + 9x^2 + 8x - 5$, $b(x) = x^2 + 5x + 6$
   (*f*) $a(x) = 2x^3 + 9x^2 + 8x - 4$, $b(x) = x^2 + 5x + 6$
   (*g*) $a(x) = x^6 + x^3 + y + 1$, $b(x) = x^2 + 1$
   (*h*) $a(x) = x^{10} - 3x^9 + 3x^8 - 11x^7 + 11x^6 - 11x^5 + 19x^4 - 13x^3 + 8x^2 - 9x + 3$
   $b(x) = x^6 - 3x^5 + 3x^4 - 9x^3 + 5x^2 - 5x + 2$
2. Find a pair of polynomials $a(x)$ and $b(x)$ when the gcd and the lcm are given by
   (*a*) the gcd $(a(x), b(x)) = x + 1$,
   the lcm $(a(x), b(x)) = x^4 + 4x^3 + 5x^2 + 8x + 6$
   (*b*) the gcd $(a(x), b(x)) = x^2 + 1$
   the lcm $(a(x), b(x)) = x^8 - 1$
   (*c*) the gcd $(a(x), b(x)) = x + 1$
   the lcm $(a(x), b(x)) = (x^6 + 3x^3 + 2)(x^3 + 1)$
   (*d*) the gcd $(a(x), b(x)) = (x + 1)^2$
   the lcm $(a(x), b(x)) = (x^2 - 1)(x^2 + 3x + 2)$
3. Find a pair of polynomials $a(x)$ and $b(x)$ in the following cases
   (*a*) $a(x) + b(x) = x^6 - 1$, the gcd $(a(x), b(x)) = x + 1$
   (*b*) $a(x) + b(x) = (x^2 + 1)(x + 1)^2$, the gcd $(a(x), b(x)) = x + 1$

(*c*) $a(x) + b(x) = (x^2 + 5x + 6)(x^2 - 1)$, the gcd $(a(x), b(x)) = (a^2 + x - 2)$

(*d*) $a(x) + b(x) = x^6 - 27$, the gcd $(a(x), b(x)) = x - \sqrt{3}$

PROBLEMS

1. Given $p(x)$ and $q(x)$ show that there exist a pair of polynomials $a(x)$ and $b(x)$ such that
$$a(x) + b(x) = p(x)$$
$$\text{gcd } (a(x), b(x) = q(x)$$
$$\text{iff} \quad q(x) | p(x)$$

2. Given $p(x)$ and $q(x)$, show that there exist a pair of polynomials $a(x)$ and $b(x)$ such that
$$\text{gcd } (a(x), b(x)) = p(x),$$
$$\text{lcm } (a(x), b(x)) = q(x)$$
$$\text{iff} \quad p(x) | q(x).$$

3. Let $p(x)$ be a gcd of $a(x)$ and $b(x)$. Show that there are polynomials $l(x)$ and $m(x)$ such that
$$a(x)l(x) + b(x)m(x) = p(x)$$

4. Find $l(x)$ and $m(x)$ such that
$$(x - 1)^2 \, l(x) - (x + 1)^2 m(x) = 1$$
holds as an identity.

5. We say $a(x)$ and $b(x)$ are relatively prime if gcd of $a(x)$ and $b(x)$ is a constant. Suppose $a(x)$ and $b(x)$ are relatively prime and
$$a(x) | b(x)c(x),$$
for some polynomial $c(x)$. Prove that $a(x) | c(x)$.

6. If $p(x)$ is a gcd of $a(x)$ and $b(x)$, prove that $\frac{a(x)}{p(x)}$ and $\frac{b(x)}{p(x)}$ are relatively prime.

7. If $q(x)$ is a lcm of $a(x)$ and $b(x)$, show that $q(x)r(x)$ is a lcm of
$a(x)r(x)$ and $b(x)r(x)$ for any polynomial $r(x)$.

8. Prove that for any two polynomials $a(x)$ and $b(x)$
$$a(x)b(x) = \alpha p(x)q(x)$$
where α is a real, $p(x)$ is a gcd of $a(x)$ and $b(x)$, and $q(x)$ is a lcm of $a(x)$ and $b(x)$.

9. Let α and β be distinct zeros of a polynomial $p(x)$, and suppose
$$p(x) = (x - \alpha)q(x),$$
$$p(x) = (x - \beta)r(x)$$
for some polynomials $q(x)$ and $r(x)$. Prove that the remaining zeros of $p(x)$ are the roots of the equation
$$q(x) - r(x) = 0.$$

10. Prove the division algorithm in $\mathbf{R}[x]$; given polynomials $a(x)$ and $b(x)$ in $\mathbf{R}[x]$, there are unique polynomials $q(x)$ and $r(x)$ such that
$$a(x) = b(x)q(x) + r(x)$$
where either $r(x)$ is the zero polynomial or $\deg r(x) < \deg b(x)$.

11. Find a polynomial $p(x)$ of degree 5 such that $(x - 1)^3$ divides $p(x) - 1$ and x^3 divides $p(x)$.

12. Show that for every integer n,
$$\frac{x}{1-x} - \frac{1-x^n}{(1-x)^2}$$
is a polynomial of degree $n - 2$.

13. Consider the cubic equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

where $ac - b^2 \neq 0$. Show that this equation has two equal roots iff

$$(bc - ad)^2 = 4(ac - b^2)\,(bd - c^2),$$

and in this case the equal root is given by

$$\alpha = \frac{bc - ad}{2(ac - b^2)}.$$

14. Suppose $p(x)$ is a polynomial over **Z** such that there exists a positive integer k for which none of the integers $p(1), p(2), \ldots, p(k)$ is divisible by k. Prove that $p(x)$ has no integer zeros.

15. Find all polynomials $p(x)$ such that

$$p(q(x)) = q(p(x))$$

for every polynomial $q(x)$.

16. Find a necessory and sufficient condition that the polynomial

$$ax^4 + bx^3 + cx^2 + dx + e(a \neq 0),$$

is of the form $p(q(x))$ for some quadratic polynomials p and q.

17. Let $p(x)$, $q(x)$ and $r(x)$ be quadratic polynomials with positive leading coefficients and having real zeros. Suppose each pair of them has a common zero. Show that $p(x) + q(x) + r(x)$ has only real zeros.

18. Let $p(x)$ be a polynomial in **R**$[x]$ of degree m, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n distinct real number. Prove that

$$p(x) = a_0 + a_1(x - \alpha_1) + a_2(x - \alpha_1)(x - \alpha_2) + \ldots$$
$$+ a_n(x - \alpha_1)\,(x - \alpha_2)\,\ldots\,(x - \alpha_n),$$

for some real numbers $a_0, a_1, \ldots, a_n$.

19. Given $n + 1$ distinct real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ and real numbers $\beta_1, \beta_2, \ldots, \beta_{n+1}$ (not necessarily distinct), show that there is a unique polynomial $p(x)$ of degree less than or equal to n such that

$$p(\alpha_i) = \beta_i,\ 1 \leq i \leq n + 1.$$

20. If α is a zero of

$$p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0,$$

where a_i may be complex, show that

$$|\alpha| \leq \max\,\{1, |a_0| + |a_1| + \ldots + |a_{n-1}|\}.$$

Polynomials in two variables An expression of the form

$$p(n, j) = \sum_{k=0}^{m} \sum_{l=0}^{n} c_{kl}x^k y^l$$

is called a polynomial in two variables. Here the coefficients C_{kl} may be integers, rationals, reals or complex numbers. As in the case of polynomials in one variable, we have **Z**$[x, y]$, **Q**$[x, y]$, **R**$[x, y]$ and **C**$[x, y]$. We can also view, for example, **R**$[x, y]$ as the set of all expressions of the form

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \ldots + a_0(x)$$

where $a_i(x)$ are in **R**$[x]$. Similarly **R**$[x, y]$ can also be thought as the collection of all expressions of the form

$$b_m(y)x^m + b_{m-1}(y)x^{m-1} + \ldots + b_0(y)$$

where $b_i(y)$ are in **R**$[y]$. The degree of a term of the form $c_{kl}x^ky^l$ is $k + l(c_{kl} \neq 0)$. The *degree* of $p(x, y)$ is defined as the maximum of the degrees of its terms.

21. Let $p(x, y)$ be a symmetric polynomial in two variables x and y, *i.e.* $p(x, y) = p(y, x)$. Suppose $(x - y)$ is a factor of $p(x, y)$. Show that $(x - y)^2$ is a factor of $p(x, y)$.

22. Consider the polynomial

$$p(x, y) = x^4 + y^4 + x^2 + y^2.$$

Express $p(x, y)$ as the sum of squares of three polynomials over $\mathbf{R}$ in x and y.

23. Define

$$p(x, y) = x^2y + xy^2,\ Q(x, y) = x^2 + xy + y^2.$$

For each n, consider F_n and G_n;

$$F_n(x, y) = (x + y)^n - x^n - y^n,$$
$$G_n(x, y) = (x + y)^n + x^n + y^n.$$

Prove that, for each positive integer n, either F^n or G^n is a polynomial in P and Q over $\mathbf{Z}$.

24. Express $x^3 + y^3$, $x^2y + xy^2$, $1/x + 1/y$, $x^3 + y^3 + xy$, $x^4 + y^4 + x^3y + xy^3$ as polynomials in two variables P and Q where

$$P(x, y) = x + y$$
$$Q(x, y) = xy.$$

Can you conjecture something?

25. Let $p(x)$ be a polynomial of degree n over $\mathbf{Z}$. Suppose $p(k)$ is a prime number for $2n + 1$ distinct integers k. Show that $p(x)$ is irreducible over $\mathbf{Z}$; *i.e.* we cannot find nonconstant polynomial $q(x)$ and $r(x)$ in $\mathbf{Z}(x)$, each of degree less than n, such that

$$p(x) = q(x)r(x).$$

26. Find all polynomials $p(x)$ such that

$$p(x^2) = p(x)p(x + 1).$$

27. Suppose $p(x)$ is a polynomial which leaves remainder 2 and 1 when divided by $x - 1$ and $x - 2$ respectively. What is the remainder when $p(x)$ is divided by $(x - 1)(x - 2)$?

28. Let $a_1, a_2, \ldots, a_n$ be distinct integers. Show that the polynomial

$$(x - a_1)^2 (x - a_2)^2 \ldots (x - a_n)^2 + 1$$

is irreducible over $\mathbf{Z}$.

29. Suppose $p(x)$ is a polynomial in $\mathbf{Z}[x]$ such that $p(0)$ and $p(1)$ are odd numbers. Prove that $p(x)$ has no integral zeros.

30. Determine all polynomials $p(x)$ such that

$$p(x^2 + 1) = (p(x))^2 + 1 \text{ and } p(0) = 0.$$

31. Let $p(x)$ be a polynomial in $\mathbf{R}[x]$ such that $p(x) \geq 0$ for all real values of x. Prove that

$$p(x) = q_1^2(x) + q_2^2(x) + \ldots + q_k^2(x)$$

for some polynomials $q_1, q_2, \ldots, q_n$.

32. Let $p(x)$ be the polynomial

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

of degree n and a_i are real or complex numbers. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the zeros of $p(x)$. Prove that

$$\sum_{i=1}^{n} \alpha_i = \frac{a_{n-1}}{a_n}$$

$$\sum_{1 \leq i < j \leq n} \alpha_i\alpha_j = \frac{a_{n-2}}{a_n} \ \ldots$$

$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

33. Let a, b and c be real numbers such that $a + b + c = 0$. Prove that

$$\frac{a^5 + b^5 + c^5}{5} = \left(\frac{a^3 + b^3 + c^3}{3}\right)\left(\frac{a^2 + b^2 + c^2}{2}\right)$$

34. Show that the zeros of the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_3 x^3 + x^2 + x + 1$$

with real coefficients $a_n, a_{n-1}, \dots, a_3$, cannot all be real.

35. Let $p(x)$ be *a* monic polynomial in $\mathbf{Z}[x]$. Prove that any rational zero of $p(x)$ must be an integer.

CHA

11

INEQUALITIES

11.1 INTRODUCTION

We referred in Chapter 1 to certain basic properties of the real number system. One of the most important properties of real numbers is that these numbers have an ordering. We can speak of a number a to be larger or smaller than another number b. This concept of ordering in **R** is, as we saw in Chapter 1, an extension of the natural ordering in **N**. Recall that as we extended **N** to larger and larger systems upto **C** through **Z**, **Q** and **R** we were able to keep the natural ordering at all stages except the last one. We now denote by $\mathbf{R}^+$ the set of all positive real numbers, and by $\mathbf{R}^-$ the set of all negative real numbers. Thus **R** is the union of $\mathbf{R}^+$, $\mathbf{R}^-$ and $\{0\}$. Given a real number a, either a is in $\mathbf{R}^+$ or a is in $\mathbf{R}^-$ or $a = 0$. One and only one of these possibilities is true. We note that, given any two real numbers a and b, a is said to be less than b (or b is greater than a) if $b - a$ is a positive real number. We write this $a < b$ (or $b > a$). We record the following important properties of this ordering in **R**.

(*i*) Given any two real numbers a and b, one and only one of the following three conditions is true

$$a < b \text{ or } a = b \text{ or } a > b.$$

(*ii*) If $a < b$ and c is any real number, then $a + c < b + c$.

(*iii*) If $a < b$ and $c > 0$, then $ac < bc$.

(*iv*) If $a > 0$, $b > 0$ and $a < b$, then $(1/a) > (1/b)$.

(*v*) For any real number a, $a^2 \geq 0$.

11.2 SOME BASIC INEQUALITIES

We know that $a^2 \geq 0$ for any real number a. This is an important inequality in itself. As a consequence of this property, we can derive many inequalities.

Let c and d be any two real numbers. Then we have $(c - d)^2 \geq 0$. Expanding this, we get

$$c^2 - 2cd + d^2 \geq 0.$$

So,
$$\frac{c^2 + d^2}{2} \geq cd. \tag{1}$$

If a and b are nonnegative reals, by taking $c = \sqrt{a}$, $d = \sqrt{b}$ in the relation (1), we get,

$$\sqrt{ab} \leq \frac{a+b}{2} \tag{2}$$

Here the number $\frac{a+b}{2}$ is called the **Arithmetic Mean (A.M.)** of a and b.

Similarly $\sqrt{ab}$ is called the **Geometric Mean (G.M.)** of a and b. Thus the relation (2) asserts that the geometric mean of two nonnegative real numbers is always smaller than (and utmost equal to) their arithmetic mean. Since

$$\frac{c^2+d^2}{2} = cd \text{ iff } c = d,$$

it follows that

$$\sqrt{ab} = \frac{a+b}{2} \text{ iff } a = b$$

for any two nonnegative real numbers. Thus equality holds in (2) iff $a = b$.

Inequality (2) has an interesting geometric interpretation. Let us consider a semicircle with diameter AB. If C is any point on the semicircle, then ABC forms a right angled triangle with the right angle at C. Let CD be the perpendicular to AB dropped from C. Let $AD = a$ and $DB = b$ (See Fig. 11.1).

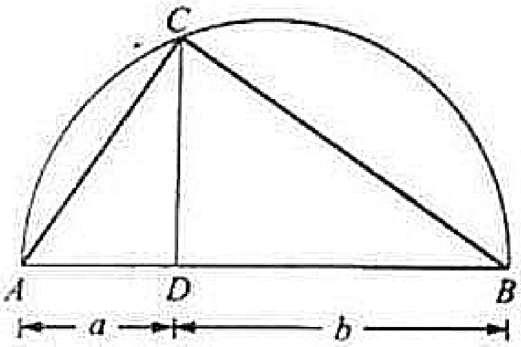

Fig. 11.1

The triangles DAC and DCB are similar, since $\angle ADC = \angle BDC$, $\angle DCA = \angle DBC$ and $\angle DAC = \angle DCB$. Hence

$$\frac{DA}{DC} = \frac{DC}{DB}$$

Therefore $\qquad DC^2 = DA \cdot DB = ab$

and hence $\qquad DC = \sqrt{ab}$

If r is the radius of the semicircle, then $DC \leq r$. But $r = \frac{a+b}{2}$.

Thus we get $\qquad \sqrt{ab} \leq \frac{a+b}{2}$

Thus we can interpret the inequality (2) as the statement that the perpendicular is the shortest distance from a point to a straight line. We also observe that $DC = r$ iff D is the centre of the semicircle. And this is equivalent to $a = b$.

Note. Compare this with the construction 10, Section 4.5 for the mean proportional between two segments.

EXAMPLE 1. *For any three positive reals a, b, and c show that*

$$a^2 + b^2 + c^2 \geq ab + bc + ca \tag{3}$$

SOLUTION. Using the inequality (2), we have,

$$ab \le \frac{a^2+b^2}{2},\ bc \le \frac{b^2+c^2}{2},\ ca \le \frac{c^2+a^2}{2}.$$

Adding these three relations, we get

$$ab+bc+ca \le \frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}$$
$$= a^2+b^2+c^2.$$

EXAMPLE 2. *For any positive integer n, prove that*

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} \qquad (4)$$

SOLUTION. Let a and b be positive reals such that $a < b$. We have

$$b^{n+1}-a^{n+1} = (b-a)\,(b^n + a\,b^{n-1} + \ldots + a^n).$$

By a repeated use of $a < b$ we get

$$(n+1)a^n < b^n + a\,b^{n-1} + \ldots + a^n < (n+1)b^n.$$

Multiplying by $b-a$, we get

$$(n+1)\,a^n\,(b-a) < b^{n+1} - a^{n+1} < (n+1)\,(b-a)b^n. \qquad (5)$$

Taking $a = \left(1+\frac{1}{n+1}\right)$ and $b = \left(1+\frac{1}{n}\right)$ in (5), we get

$$(n+1)\left(1+\frac{1}{n+1}\right)^n\left(\frac{1}{n}-\frac{1}{n+1}\right) < \left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1}$$
$$< (n+1)\left(1+\frac{1}{n}\right)^n\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

This gives

$$\frac{1}{n}\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1} < \frac{1}{n}\left(1+\frac{1}{n}\right)^n.$$

The second part of this inequality reduces to

$$\left(1+\frac{1}{n}\right)^{n+1} - \frac{1}{n}\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

which gives

$$\left(1+\frac{1}{n}\right)^n\left\{1+\frac{1}{n}-\frac{1}{n}\right\} < \left(1+\frac{1}{n+1}\right)^{n+1}$$

and this is precisely (4).

EXAMPLE 3. (*Weierstrass inequality*). *If $a_1, a_2, \ldots, a_n$ are positive real numbers, each being less than 1, and if $s_n = a_1 + a_2 + \ldots + a_n$ prove that,*

$$\text{(i) } 1 - s_n < (1-a_1)\,(1-a_2)\cdots(1-a_n) < \frac{1}{1+s_n} \qquad (6$$

$$\text{(ii) } 1 + s_n < (1+a_1)\,(1+a_2)\cdots(1+a_n) < \frac{1}{1-s_n} \qquad (7$$

where $s_n < 1$ in (ii).

SOLUTION. We have, $(1-a_1)(1-a_2) = 1-a_1-a_2+a_1a_2$
$> 1-(a_1+a_2)$.

Similarly,

$$\begin{aligned}(1-a_1)(1-a_2)(1-a_3) &> \{1-(a_1+a_2)\}(1-a_3)\\ &= 1-(a_1+a_2)-a_3+a_3(a_1+a_2)\\ &> 1-(a_1+a_2+a_3).\end{aligned}$$

Continuing, we get

$$(1-a_1)(1-a_2)\cdots(1-a_n) > 1-(a_1+a_2+\cdots+a_n) = 1-s_n.$$

Similarly, we can prove that,

$$(1+a_1)(1+a_2)\ldots(1+a_n) > 1+s_n.$$

Since $0 < a_j < 1$ for $j = 1, 2, \ldots, n$, we also have

$$(1-a_j)(1+a_j) = 1-a_j^2 < 1$$

Therefore, $1-a_j < \dfrac{1}{1+a_j}$, $1+a_j < \dfrac{1}{1-a_j}$ for $j = 1, 2, \ldots, n$.

Thus $(1-a_1)(1-a_2)\cdots(1-a_n) < \dfrac{1}{(1+a_1)(1+a_2)\cdots(1+a_n)}$

$$< \frac{1}{1+s_n}$$

The other result may be proved similarly. Note however that at the last step one uses the hypothesis $s_n < 1$.

EXAMPLE 4. *If* $a_j \geq 1$ *for* $j = 1, 2, \ldots, n$, *prove that*

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq (1+a_1+a_2+a_3+\cdots+a_n)\frac{2^n}{1+n}$$

[Note that this is a stronger inequality than the left part of (7)]

SOLUTION. $(1+a_1)(1+a_2)\cdots(1+a_n)$

$$\begin{aligned}&= 2^n\left(\frac{1}{2}+\frac{a_1}{2}\right)\left(\frac{1}{2}+\frac{a_2}{2}\right)\cdots\left(\frac{1}{2}+\frac{a_n}{2}\right)\\ &= 2^n\left(1+\frac{a_1-1}{2}\right)\left(1+\frac{a_2-1}{2}\right)\cdots\left(1+\frac{a_n-1}{2}\right)\\ &\geq 2^n\left(1+\frac{a_1-1}{2}+\frac{a_2-1}{2}+\cdots+\frac{a_n-1}{2}\right)\\ &\geq 2^n\left(1+\frac{a_1-1}{n+1}+\frac{a_2-1}{n+1}+\cdots+\frac{a_n-1}{n+1}\right)\\ &= \frac{2^n}{n+1}(1+a_1+a_2+\ldots+a_n).\end{aligned}$$

EXAMPLE 5. *If* a, b, *and* c *are positive numbers such that* $a+b>c$, $b+c>a$ *and* $a+c>b$, *prove that*

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c} > \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \tag{8}$$

The required inequality is equivalent to

$$\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}\right)+\left(\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)+\left(\frac{1}{a+b-c}+\frac{1}{b+c-a}\right)>\frac{2}{a}+\frac{2}{b}+\frac{2}{c} \tag{9}$$

Consider the first term on the left hand side;

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}=\frac{2c}{(c+a-b)(c-(a-b))}$$

$$=2\frac{c}{c^2-(a-b)^2}.$$

Since $c^2-(a-b)^2<c^2$ and $c^2-b)^2$ (because of the hypothesis $b+c>a$), we get

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}>\frac{2}{c}.$$

Similarly, we get

$$\frac{1}{c+a-b}+\frac{1}{a+b-c}>\frac{2}{a} \text{ and } \frac{1}{a+b-c}+\frac{1}{b+c-a}>\frac{2}{b}.$$

Adding these we get the inequality (9).

EXAMPLE 6. *If a, b, and c are positive real numbers then prove that*

(*i*) $\dfrac{a+c}{b+c}<\dfrac{a}{b}$ *if* $a>b$

(*ii*) $\dfrac{a+c}{b+c}>\dfrac{a}{b}$ *if* $a<b$

SOLUTION. To prove (*i*) we start with $a>b$.

$\therefore$ $ac>bc$. So $ab+ac>ab+bc$.

$\therefore$ $a(b+c)>b(a+c)$.

which means $\dfrac{a+c}{b+c}<\dfrac{a}{b}$.

(*ii*) is similar.

EXAMPLE 7. *For any n, prove that*

$$\frac{1}{2\sqrt{n+1}}<\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}<\frac{1}{\sqrt{2n+1}}$$

SOLUTION. Start with

$$\frac{2k-1}{2k}<\frac{2k}{2k+1}$$

which can be verified to be true by cross-multiplication. Then

$$s_n=\frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{2n-1}{2n}<\frac{2}{3}\frac{4}{5}\frac{6}{7}\cdots\frac{2n}{2n+1}.$$

So $$s_n^2<\frac{1}{2}\frac{2}{3}\frac{3}{4}\frac{4}{5}\frac{5}{6}\frac{6}{7}\cdots\frac{2n-1}{2n}\cdot\frac{2n}{2n+1}=\frac{1}{2n+1}.$$

Hence $$s_n < \frac{1}{\sqrt{2n+1}}$$

Again, since
$$(2n+1)\, s_n = \frac{3}{2}\frac{5}{4}\frac{7}{6}\cdots\frac{2n-1}{2n-2}\frac{2n+1}{2n}$$
and because of the verifiable inequality
$$\frac{2k+1}{2k} > \frac{2k+2}{2k+1}$$
we get, as before
$$((2n+1)s_n)^2 > \frac{3}{2}\frac{4}{3}\frac{5}{4}\frac{6}{5}\cdots\frac{2n-1}{2n-2}\frac{2n}{2n-1}\frac{2n+1}{2n}\cdot\frac{2n+2}{2n+1}$$
$$= \frac{2n+2}{2} = n+1$$

This gives $$s_n > \frac{\sqrt{n+1}}{2n+1} > \frac{1}{2\sqrt{n+1}}$$

the last inequality being verifiable by cross-multiplication.

EXERCISE 11.1

Prove the following inequalities.

1. $a^2 + 2ab + 4b^2 \geq 0$ for all reals a and b.
2. For any real number a,
$$4a^4 - 4a^3 + 5a^2 - 4a + 1 \geq 0.$$
3. $\dfrac{a^2}{1+a^4} \leq \dfrac{1}{2}$ for any real a.
4. $a^4 + b^4 \geq a^3b + ab^3$ for all real numbers a and b.
5. $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, a, b, c in R.
6. $a^2 + \dfrac{1}{1+a^2} \geq$ for all real a.
7. $\dfrac{a^2+3}{\sqrt{a^2+2}} > 2.$
8. $2a^2 + b^2 + c^2 \geq 2a\,(b+c)$ for all reals a, b, c.
9. $a^2 + b^2 + c^2 \geq 2\,(a+b+c) - 3$ for all reals a, b, c.
10. $a + b + c \geq \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$ for positive a, b, c.
11. $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \geq \dfrac{1}{\sqrt{ab}} + \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ca}}$ for positive a, b, c.
12. $(1+a)(1+b) \geq 4$ if $a > 0$, $b > 0$ and $ab = 1$.
13. $a^2 + b^2 + c^2 \geq 3$ if a, b, c are nonnegative and $a + b + c \geq 3$.
14. $a^2 + b^2 \geq \dfrac{c^2}{2}$ if $a + b \geq c \geq 0$.

15. $a^4 + b^4 \geq \frac{c^4}{8}$ if $a + b \geq c \geq 0$.

16. $a^8 + b^8 > \frac{c^2}{128}$ if $a + b \geq c \geq 0$.

17. $(b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2 \geq ab + bc + ca$ for any reals a, b and c.

18. If a, b, c and d are real numbers greater than 1, then $8(abcd + 1) > (a + 1)(b + 1)(c + 1)(d + 1)$.

19. For any positive real a, b, c

$$\frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} + \frac{a^2 + b^2}{a + b} \geq a + b + c.$$

20. For any positive integer n, $n^{n/2} < n!$ for $n > 2$.

21. For any positive a, b and c, $(a + b)(b + c)(c + a) \geq 8abc$.

22. For any positive a, b and c,

$$a^2b^2 + b^2c^2 + c^2a^2 \geq abc\,(a + b + c).$$

23. If a, b and c are positive such that $a + b + c = 1$ then $ab + bc + ca \leq \frac{1}{3}$.

24. Let $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$, be two sets of reals such that $b_j > 0$ for $1 \leq j \leq n$. Let m and M be respectively the minimum and maximum of n fractions

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots, \frac{a_n}{b_n}.$$

Prove that
$$m \leq \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots b_n} \leq M.$$

25. A quadrilateral is called *convex* if both the diagonals lie inside the quadrilateral. In any convex quadrilateral, prove that the sum of two diagonals is less than its perimeter but larger than half the perimeter. Recall also the definition of convex polygon under Theorem 17, Section 3.3.

26. Prove that

$$a^2(1 + b^4) + b^2(1 + a^4) \leq (1 + a^4)(1 + b^4) \text{ for any reals } a \text{ and } b.$$

27. Prove that

$$|a| + |b| + |c| \geq \sqrt{a^2 + b^2 + c^2} \quad \text{for all reals } a, b, c.$$

28. Prove that

$$(|a| + |b|)(|b| + |c|)(|c| + |a|) \geq 8\,|abc|.$$

29. Prove that

$$\frac{bc}{b + c} + \frac{ca}{c + a} + \frac{ab}{a + b} \leq \frac{1}{2}(a + b + c)$$

for any real numbers a, b and c.

30. For positive reals a, b and c, prove that
$ab(a + b) + bc(b + c) + ca(c + a) \geq 6abc$.

31. If $a^2 + b^2 + c^2 = 1$ prove that $-\frac{1}{2} \leq ab + bc + ca \leq 1$.

11.3 AM-GM INEQUALITY

We have seen in section 11.2 that for any two positive reals a and b

$$\frac{a+b}{2} \geq \sqrt{ab}. \tag{1}$$

This was derived as a consequence of the non-negativity of the square of a real number. This is indeed true for any finite set of positive real numbers, not just for two. If $a_1, a_2, \ldots, a_n$ are n real numbers, the real number

$$\frac{a_1 + a_2 + \ldots + a_n}{n};$$

is called the *arithmetic mean* of $a_1, a_2, \ldots, a_n$. If $a_i \geq 0$ for $i = 1, 2, \ldots, n$, we define their *geometric mean* as the real number

$$(a_1\, a_2 \ldots a_n)^{1/n}.$$

A generalization of (1) to a set of n positive numbers $a_1, a_2, \ldots, a_n$ is the inequality.

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n}. \tag{2}$$

This inequality is known as '*Arithmetic Mean – Geometric Mean Inequality*' (*AM – GM inequality for short*). We write (2) also in the form

$$a_1 a_2 \ldots a_n \leq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^n. \tag{3}$$

Equality holds in (3) iff $a_1 = a_2 = a_3 = \ldots a_n$.

The proof of this inequality is relegated to the problem at the end of this chapter. We study some applications of $AM - GM$ inequality in the following examples.

EXAMPLE 1. *Show that an equilateral triangle is a triangle of maximum area for a given perimeter and a triangle of minimum perimeter for a given area.*

SOLUTION. If A is the area of the triangle with sides a, b and c then A and s are related by

$$A^2 = s(s-a)\,(s-b)\,(s-c) \tag{4}$$

Using $AM - GM$ inequality for the positive numbers $s - a$, $s - b$ and $s - c$, we have

$$A^2 \leq s\left\{\frac{(s-a)+(s-b)+(s-c)}{3}\right\}^3 = s\left\{\frac{3s-2s}{3}\right\}^3 = \frac{s^4}{3^3}.$$

This gives

$$A \leq \frac{s^2}{3\sqrt{3}}. \tag{5}$$

If p denotes the perimeter of the triangle with sides a, b, and c then $p = 2s$. Hence (5) takes the form

$$A \leq \frac{p^2}{12\sqrt{3}}. \tag{6}$$

Thus given a perimeter p, the maximum possible area that a triangle of perimeter p can have is $\frac{p^2}{12\sqrt{3}}$. Using the condition for equality in $AM - GM$ inequality, we see that

$$A = \frac{p^2}{12\sqrt{3}} \text{ iff } s - a = s - b = s - c$$

which happens iff $a = b = c$.

Thus for a given perimeter p, the maximum area $\frac{p^2}{12\sqrt{3}}$ is attained by an equilateral triangle of side $\frac{p}{3}$.

Conversely, given the area A, (6) shows that

$$p \geq \sqrt{(12\sqrt{3}A)}. \tag{7}$$

Thus (7) gives a lower bound for the perimeter of a triangle in terms of its area. Again this minimum perimeter $p = \sqrt{12\sqrt{3}A}$ is assumed for a given area iff equality holds in $AM - GM$ inequality; that is iff $a = b = c$.

EXAMPLE 2. *Let $a_1, a_2, \ldots, a_n$ be n positive real numbers, and $b_1, b_2, \ldots, b_n$ be a permutation of these numbers. Prove that*

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} \geq n.$$

SOLUTION. Using $AM - GM$ inequality for n fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$, we get

$$\frac{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n}}{n} \geq \left(\frac{a_1}{b_1}\frac{a_2}{b_2}\ldots\frac{a_n}{b_n}\right)^{1/n}.$$

But $a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_n$ since $b_1, b_2, \ldots b_n$ is a permutiition of $a_1, a_2, \ldots, a_n$. Hence the RHS in the above inequality is 1. We get

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} \geq n.$$

EXAMPLE 3. *For any positive integer n, prove that*

$$(2n)! < [n(n+1)]^n$$

SOLUTION. We write

$$\begin{aligned}(2n)! &= 1.2.3.4.5.6\ldots(2n-1)2n \\ &= \{1.3.5\ldots.(2n-1)\}\{2.4.6\ldots.2n\} \\ &= 2^n\{1.3.5\ldots(2n-1)\}\{1.2.3\ldots n\}\end{aligned}$$

But $$1.3.5\ldots(2n-1) < \left[\frac{1+3+5+\ldots+(2n-1)}{n}\right]^n,$$

$$1.2.3.\ldots n < \left[\frac{1+2+\ldots+n}{n}\right]^n.$$

We can compute these sums. We have

$$\begin{aligned}&2[1+3+\ldots+(2n-3)+(2n-1)] \\ &= [1+(2n-1)]+[3+(2n-3)]+\ldots+[(2n-3)+3]+[(2n-1)+1] \\ &= \underbrace{2n+2n+\ldots+2n+2n}_{n\ terms} = 2n^2.\end{aligned}$$

$$\therefore \quad 1+3+\ldots+(2n-1) = n^2.$$

Similarly, we can compute the other sum to get

$$1+2+\ldots+n=\frac{n(n+1)}{2}.$$

Hence $$(2n)! < \left(\frac{n^2}{n}\right)^n\left(\frac{n(n+1)}{2n}\right)^n = [n(n+1)]^n.$$

EXAMPLE 4. *Let the bisector of the angle C of a triangle ABC meet the side AB at D. Show that*

$$CD^2 < AC\,.\,BC$$

SOLUTION. We have $\Delta ABC = \Delta DC + \Delta BDC,$

where Δ denotes area. Therefore,

$$\frac{1}{2}AC\,BC\sin\angle ACB=\frac{1}{2}AC\,AD\sin\angle ACD+\frac{1}{2}BC\,CD\sin\angle DCB \tag{8}$$

Since CD is the angle bisector of $\angle ACB$,

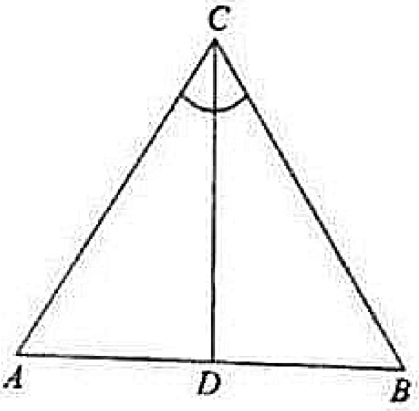

Fig. 11.2

$$\angle ACD=\frac{\angle ACB}{2}\text{ and }\angle DCB=\frac{\angle ACB}{2}.$$

Hence (8) reduces to

$$AC\,.\,BC\,.\,\sin\angle ACB=(AC+BC)\,CD\sin\frac{\angle ACB}{2}. \tag{9}$$

Using the trigonometric identity

$$\sin\theta = 2\sin\frac{\theta}{2}\quad\cos\frac{\theta}{2}$$

we can write (9) in the form

$$2AC\,BC\sin\frac{\angle ACB}{2}\cos\frac{\angle ACB}{2}=(AC+BC)\,CD\sin\frac{\angle ACB}{2}.$$

$$\therefore\quad \frac{1}{CD}\cos\frac{\angle ACB}{2}=\frac{1}{2}\left(\frac{1}{AC}+\frac{1}{BC}\right) \tag{10}$$

Since $\frac{\angle ACB}{2}$ is an acute angle, $0<\cos\frac{\angle ACB}{2}<1$. This gives us

$$\frac{1}{CD}>\frac{1}{2}\left(\frac{1}{AC}+\frac{1}{BC}\right)\geq\left(\frac{1}{AC}\times\frac{1}{BC}\right)^{1/2},$$

the last part by the use of $AM - GM$ inequality.

Hence $$CD^2 < AC\,BC.$$

EXERCISE 11.2

1. For positive real numbers a, b, and c prove that

$$a^4 + b^4 + c^4 \geq abc\,(a + b + c).$$

2. Prove that $(a^3 + b^3)^2 \leq (a^2 + b^2)\,(a^4 + b^4)$ for all real numbers a and b.

3. If $a_1, a_2, \ldots, a_n$ are positive real numbers prove that

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2.$$

4. If a, b, c are positive, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

5. For positive a, b and c, prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$

6. If $a > 0$, $b > 0$ and $c > 0$, prove that

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 6.$$

7. If a, b and c are positive, prove that

$$(a + b + c)\,(ab + bc + ca) \geq 9abc.$$

8. If a, b, c are positive, prove that

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{9}{a+b+c}.$$

9. If a, b, c, are positive and $a + b + c = 1$ prove that

$$8\,abc \leq (1 - a)\,(1 - b)\,(1 - c) \leq \frac{8}{27}.$$

10. If a, b and c are the sides of the triangle, prove that

$$(a + b + c)^3 \geq 27\,(a + b - c)\,(b + c - a)\,(c + a - b).$$

11. If a, b, c and d are positive, show that

$$\frac{1}{b+c+d} + \frac{1}{c+d+a} + \frac{1}{d+a+b} + \frac{1}{a+b+c} \geq \frac{16}{a+b+c+d}.$$

12. Show that the square is a rectangle of maximum area for a given perimeter and a rectangle of minimum perimeter for a given area.

13. If s is the semi-perimeter of a triangle with in radius r, prove that

$$s^2 \geq 27\,r^2.$$

14. If a, b and c are the sides of a triangle ABC with area Δ, prove that

$$ab + bc + ca \leq 4\sqrt{3}\,\Delta.$$

with equality iff $\Delta\,ABC$ is equilateral.

15. If a, b, c are the sides of a triancle, prove that

$$(abc)^2 \geq \left(\frac{4\Delta}{\sqrt{3}}\right), \text{ where } \Delta \text{ is the area of the triangle.}$$

16. If a, b, c are positive real numbers, such that $(1 + a)\,(1 + b)\,(1 + c) = 8$, prove that $abc \leq 1$.

17. If a, b, c are positive real numbers, prove that

$$(a^2b + b^2c + c^2a)\,(a^2c + b^2a + c^2b) \geq 9\,a^2\,b^2\,c^2.$$

18. Prove for any two positive numbers $a \neq b$ and a positive integer n

$$ab^n < \left(\frac{a+nb}{n+1}\right)^{n+1}.$$

19. If s_n is the sum

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} (n > 2), \text{ prove that}$$

$$n(n+1)^{1/n} - n < S_n < n - (n-1)n^{-\frac{1}{(n-1)}}$$

20. Show that

$$\overset{n}{\underset{k=0}{\pi}} \binom{n}{k} < \left(\frac{2^n - 2}{n-1}\right)^{n-1}$$

21. Prove that $n! < \left(\frac{n+1}{2}\right)^n$.

22. Prove that $1.3.5 \dots (2n-1) < n^n$.

23. If a, b, c are positive, prove that $a^2b + b^2c + c^2a \geq 3\,abc$.

24. If a, b and c are positive real numbers such that $a + b + c = 1$, prove that

$$\frac{b(1-b)}{ac} + \frac{c(1-c)}{ab} + \frac{a(1-a)}{bc} \geq 6.$$

25. If a, b and c are positive real numbers, not all equal, prove that

$$6abc < a^2(b+c) + b^2(c+a) + c^2(a+b) < 2(a^3 + b^3 + c^3).$$

26. If $a_1, a_2, \dots, a_n$ are positive real numbers, prove that

$$\sum_{i<j} \sqrt{a_i a_j} \leq \frac{n-1}{2} (a_1 + a_2 + \dots + a_n).$$

11.4 CAUCHY–SCHWARZ INEQUALITY

Another inequality which is useful in applications is the Cauchy-Schwarz inequality. We express this as a theorem.

Theorem 1. Let $a_1\ a_2, \dots a_n$ and $b_1, b_2, \dots b_n$ be two sets of real numbers.

Then
$$\left[\sum_{i=1}^{n} a_i b_i\right]^2 \leq \left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} b_i^2\right] \tag{1}$$

and equality holds in (1) iff

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}. \tag{2}$$

Proof. Let us put

$$A = \sum_{i=1}^{n} a_i^2,\ B = \sum_{i=1}^{n} b_i^2,\ C = \sum_{i=1}^{n} a_i b_i.$$

Then (1) is equivalent to

$$C^2 \leq AB. \tag{3}$$

If $B = 0$, then $b_i = 0$ for $i = 1, 2, \ldots, n$. Hence $C = 0$ and (3) is true. Therefore it is sufficient to consider the case $B \neq 0$. This implies that $B > 0$. We now have

$$\begin{aligned} 0 \leq \sum_{i=1}^{n}(Ba_i - Cb_i)^2 &= \sum_{i=1}^{n}(B^2a_i^2 - 2BC\,a_ib_i + C^2b_i^2) \\ &= B^2\sum_{i=1}^{n}a_i^2 - 2BC\sum_{i=1}^{n}a_ib_i + C^2\sum_{i=1}^{n}b_i^2 \\ &= B(AB - C^2). \end{aligned}$$

Since $B > 0$, we get $AB - C^2 \geq 0$

which is the required inqualty (3). Moreover, equality holds iff

$$\sum_{i=1}^{n}(Ba_i - Cb_i)^2 = 0.$$

This is equivalent to

$$\frac{a_i}{b_i} = \frac{C}{B} \text{ for } i = 1, 2, \ldots, n. \qquad \square$$

REMARK. The Cauchy-Schwarz inequality is also true for complex numbers with a little modification. If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two sets of complex numbers, then

$$\left|\sum_{i=1}^{n}a_ib_i\right|^2 \leq \left(\sum_{i=1}^{n}|a_i|^2\right)\left(\sum_{i=1}^{n}|b_i|^2\right) \tag{4}$$

with equality in (4) iff $a_i = \lambda\, b_i$ for some constant λ, $i = 1, 2, \ldots n$. The proof of this inequality is left to the problems at the end of the chapter.

EXAMPLE 1. *If $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \ldots + a_n = 1$, prove that*

$$a_1^2 + a_2^2 + \ldots + a_n^2 \geq \frac{1}{n}.$$

SOLUTION. We have

$$\begin{aligned} 1 = (a_1 + a_2 + \ldots + a_n)^2 &= (a_1.1 + a_2.1 + \ldots + a_n.1)^2 \\ &\leq (a_1^2 + a_2^2 + \ldots + a_n^2)(1 + \ldots + 1) \\ &= n(a_1^2 + a_2^2 + \ldots + a_n^2) \end{aligned}$$

$$\therefore \quad a_1^2 + a_2^2 + \ldots + a_n^2 \geq \frac{1}{n}.$$

EXAMPLE 2. *Let P be a point inside a triangle ABC, let r_1, r_2, r_3 denote the distances from P to the sides BC, CA and AB respectively. If R is the circumradius of ΔABC, show that*

$$\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} < \frac{1}{\sqrt{2R}}(a^2 + b^2 + c^2)^{1/2} \tag{5}$$

where $BC = a$, $CA = b$ and $AB = c$. Show also that the equality holds in (5) iff ABC is an equilateral triangle; and P is the incentre of ΔABC.

SOLUTION. We have

$$\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3} = \sqrt{ar_1}\,\frac{1}{\sqrt{a}}+\sqrt{br_2}\,\frac{1}{\sqrt{b}}+\sqrt{cr_3}\,\frac{1}{\sqrt{c}}.$$

Applying Cauchy-Schwarz inequality for the sets

$\{\sqrt{ar_1}, \sqrt{br_2}, \sqrt{cr_3}\}$ and $\left\{\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right\}$ we get

$$\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3} \le (ar_1+br_2+cr_3)^{1/2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{1/2} \tag{6}$$

Equality holds in (6) iff

$$\sqrt{a}\,\sqrt{ar_1}=\sqrt{b}\,\sqrt{br_2}=\sqrt{c}\,\sqrt{cr_3}$$

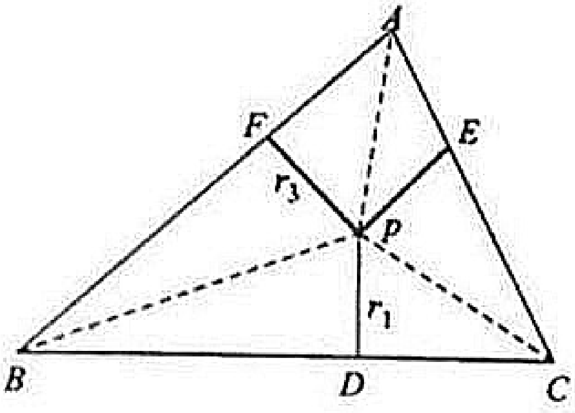

Fig. 11.3

which happens iff

$$a^2r_1=b^2r_2=c^2r_3. \tag{7}$$

But $\qquad ar_1=2\Delta PBC;\ br_2=2\Delta PCA;$ and $cr_3=2\Delta PAB.$

Hence,

$$ar_1+br_2+cr_3=2(\Delta PBC)+\Delta PCA+\Delta PAB)=2\Delta ABC. \tag{8}$$

Now we use the identity (See Chapter 6. Properties of triangles Theorem 3. Corollary 2.)

$$\Delta ABC=\frac{abc}{4R}. \tag{9}$$

This reduces (8) to

$$ar_1+br_2+cr_3=\frac{abc}{2R}. \tag{10}$$

Hence (6) takes the form

$$\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3} \le \left(\frac{abc}{2R}\right)^{1/2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{1/2}$$

$$=\frac{1}{\sqrt{2R}}(ab+bc+ca)^{1/2}.$$

But the Cauchy-Schwarz inequality gives again

$$(ab+bc+ca) \le (a^2+b^2+c^2)^{1/2}(b^2+c^2+a^2)^{1/2}$$
$$=a^2+b^2+c^2$$

with the equality being true iff $\frac{a}{b}=\frac{b}{c}=\frac{c}{a}$.

Thus $\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3} \le \frac{1}{\sqrt{2R}}(a^2+b^2+c^2)^{1/2}$

with equality being true iff

$$a^2r_1 = b^2r_2 = c^2r_3 \text{ and } \frac{a}{b}=\frac{b}{c}=\frac{c}{a}.$$

This reduces to $a=b=c$ and $r_1=r_2=r_3$.

Hence the equality in (5) holds iff ABC is an equilateral triangle and P is the incentre.

EXAMPLE 3. *If a, b, c and d are positive real numbers such that*

$$c^2+d^2=(a^2+b^2)^3,$$

prove that $\frac{a^3}{c}+\frac{b^3}{d} \ge 1$, *with equality iff* $ad = bc$.

SOLUTION. Using Cauchy-Schwarz inequality, we get

$$(a^2+b^2)^2 = \left(\sqrt{\frac{a^3}{c}}\sqrt{ac}+\sqrt{\frac{b^3}{d}}\sqrt{bd}\right)^2$$

$$\le \left(\frac{a^3}{c}+\frac{b^3}{d}\right)(ac+bd)$$

where equality holds iff $a^2d^2 = b^2c^2$.

Thus
$$\left(\frac{a^3}{c}+\frac{b^3}{d}\right)(ac+bd) \le (a^2+b^2)^2$$
$$= (a^2+b^2)^{1/2}(a^2+b^2)^{3/2}$$
$$= (a^2+b^2)^{1/2}(c^2+d^2)^{1/2}$$
$$\ge ac+bd$$

again by another application of Cauchy-Schwarz inequalilty. Equality holds in the last step iff $\frac{a}{c}=\frac{b}{d}$.

Combining both, we get $\frac{a^3}{c}=\frac{b^3}{d}\ge 1$

and equality holds iff $ad = bc$.

EXCRCISE 11.3

1. Prove that, if $n > 2$.

$$\sum_{k=1}^{n}\sqrt{\binom{n}{k}} \le \sqrt{n(2^n-1)}.$$

2. If $a > 0, b > 0$ and $a + b = 1$, prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \ge \frac{25}{2}.$$

3. If $a > 0, b > 0$ and $a + b = 1$, prove that

$$\left(a+\frac{1}{a}\right)^3+\left(b+\frac{1}{b}\right)^3 \ge \frac{125}{4}.$$

4. If $a > 0, b > 0, c > 0$ and $a + b + c = 6$, prove that

$$a^2 + b^2 + c^2 \ge 12.$$

5. If a, b, c are positive, prove that $a \cos^2 \theta + b \sin^2 \theta < c$ implies that

$$\sqrt{a} \cos^2 \theta + \sqrt{b} \sin^2 \theta < c.$$

6. If a, b, c are positive, prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}.$$

7. If a, b, c are positive, prove that

$$a + b + c \ge \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

8. If a and b are positive, and $a + b = 1$, prove that

$$\sqrt{4a+1} + \sqrt{4b+1} \le 2\sqrt{3}.$$

Prove the following inequalities:

9. $(a + b + c + d)^2 \le 4(a^2 + b^2 + c^2 + d^2)$.
10. $(a^3 + b^3)^2 \le (a^2 + b^2)(a^4 + b^4)$.
11. $ab + bc + ca \le a^2 + b^2 + c^2$.
12. $(|a| + |b| + |c|)^2 \le 3(a^2 + b^2 + c^2)$.
13. For any positive numbers a, b and c, prove that

$$(a^2b + b^2c + c^2a)(a^2c + b^2a + c^2b) \ge 9a^2b^2c^2.$$

PROBLEMS

1. Complete the formal proof of the *A.M. – G.M.* inequality. For this you have to prove two results: (*i*) If the inequality is true for 2^k numbers then it is also true for 2^{k+1} numbers; and (*ii*) If the inequality is true for every 2^k numbers, then it is also true for n where $2^k < n < 2^{k+1}$.

2. (*i*) Let a_i, b_i, $1 \le i \le n$ be nonnegative real numbers. Show that the discriminant of the quadratic polynomial

$$p(x) = \sum_{i=1}^{n} (a_i x + b_i)^2$$

is non-positive and use this to prove Cauchy–Schwarz inequality.

(*ii*) Let a and b be two nonnegative real numbers. Use the fact that the polynomial $p(x) = (x - a)(x - b)$ has only real zeros to establish the *A.M. - G.M.* inequality.

3. Let ABC be an equilateral triangle, K, L and M be arbitrary points on the sides BC, CA and AB respectively. Show that the area of at least one of the triangles AML, BKM and CLM is smaller than one-fourth of the area of the triangle ABC.

4. For any triangle with angles α, β and γ, prove

$$\cos\alpha\cos\beta\cos\gamma \le \frac{1}{8}.$$

Show that the equality holds iff the triangle is equilateral.

5. Suppose $a_1, a_2, \ldots, a_n$ are real numbers such that

$$A + \sum_{k=1}^{n} a_k^2 < \frac{1}{(n-1)}\left(\sum_{k=1}^{n} a_k\right)^2$$

for some constant A. Prove that

$$A < 2a_i a_j,\ i \ne j,\ i, j = 1, 2, \ldots, n.$$

6. Let $a_i > 0$ for $i = 1, 2, \ldots, n$. For any integer $k \ge 1$, prove that

$$\frac{a_1^k + a_2^k + \ldots + a_n^k}{n} \le \frac{a_1^{k+1} + a_2^{k+1} + \ldots + a_n^{k+1}}{a_1 + a_2 + \ldots + a_n}.$$

7. Let P be a point inside a triangle ABC. Let D, E and F be the feet of the perpendiculars from P to BC, CA and AB respectively. Find all P for which the sum

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF} \text{ is least.}$$

8. For any triangle with angles α, β and γ, prove that

$$0 \le \sin\alpha + \sin\beta + \sin\gamma \le \frac{3}{2}\sqrt{3}.$$

Show also that equality holds iff the triangle is equilateral.

9. A real valued function f in an interval $[a, b]$ is said to *be convex* in $[a, b]$ if for any x and y in $[a, b]$ and λ in $[0, 1]$, f satisfies

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y).$$

Suppose f is convex in $[a, b]$. Let $x_1, x_2, \ldots, x_n$ be any n numbers in $[a, b]$ and $\lambda_1, \ldots, \lambda_n$ be n numbers in $[0, 1]$ such that $\lambda_1 + \ldots + \lambda_n = 1$.

Prove that

$$f\left(\sum_{j=1}^{n} \lambda_j x_j\right) \le \sum_{j=1}^{n} \lambda_j f(x_j).$$

10. If a is a positive real numbers such that $a \ne 1$ and if p and q are positive rationals such that $p > q$, prove that

$$\frac{a^p - 1}{p} > \frac{a^q - 1}{q}.$$

11. If $n > 0$ is an integer and $a > 1$ is a real number, prove that

$$a^{n+1} - \frac{1}{a^{n+1}} > \frac{n+1}{na}\left(a^n - \frac{1}{a^n}\right).$$

12. Suppose all the zeros of the polynomial $p(x)$ in $\mathbf{R}(x)$,

$$p(x) = a_n x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 - n^2bx + b$$

are real and positive. Prove that all the zeros are equal (**Hint.** Find relations between the zeros and the coefficients of the polynomial).

13. Show that the polynomial

$$x^n + ax^{n-1} + bx^{n-1} + \dots + c$$

has at least one nonreal zero if

$$a^2 - 2b < n\,(c^2)^{1/n}.$$

14. Let α, β, and γ be angles of a triangle. Prove that

$$\tan^2\frac{\alpha}{2} + \tan^2\frac{\beta}{2}\tan^2\frac{\gamma}{2} \geq 1.$$

15. Find all real numbers x, y and z such that

$$(1-x)^2 + (x-y)^2 + (y-z)^2 + z^2 = \frac{1}{4}.$$

16. If x and y are positive real numbers, and m and n are positive integers, prove that

$$\left(\frac{x+y}{m+n}\right)^{m+n} \geq \left(\frac{x}{m}\right)^m \left(\frac{y}{n}\right)^n.$$

17. Find the largest y such that

$$\frac{1}{1+x^2} \geq \frac{y-x}{y+x} \quad \text{for all } x > 0.$$

18. Find the maximum and minimum values of

$$\frac{x+1}{xy+x+1} + \frac{y+1}{yz+y+1} + \frac{z+1}{zx+z+1}.$$

19. Is there a set of real numbers u, v, w, x, y and z such that

$$u^2 + v^2 + w^2 + 3(x^2 + y^2 + z^2) = 6,\ ux + vy + wz = 2?$$

20. If $x_1 x_2 \dots x_n = a^n$ for some constant a and $x_i > 0$ for $i = 1, 2, \dots, n$, find the least possible value of

$$(x_1 + k)(x_2 + k)\dots(x_n + k)$$

where k is a positive real number.

21. For any two positive integers n and k with $k \leq n$

Prove that

$$2 < \left(1 + \frac{1}{n}\right)^n < 3 \quad \text{for all } n.$$

CHAP

12

ELEMENTARY COMBINATORICS

Combinatorics is that part of Mathematics which deals with counting and enumeration of specified objects, patterns or designs. Already we have seen in Chapter 9, the fundamental concepts, viz., permutations and combinations, which constitute the warp and woof of any counting problem. In this chapter we shall study two basic principles which have been used by man many times in his scientific thinking and have now been streamlined, and written down as part of the foundational principles in the subject of combinatorics. These are:

(*i*) (*IEP*): The Inclusion and Exclusion Principle; and (*ii*) (*PHP*): The Pigeon Hole Principle. We shall take these one by one.

12.1 THE INCLUSION AND EXCLUSION PRINCIPLE (IEP)

We start with an interesting problem which will dramatically illustrate the principle. On a rainy day, five gentlemen A, B, C, D, E attend a party after leaving their umbrellas in a checkroom. After the party is over, the umbrellas get mixed up and are returned to the gentlemen in such a manner that none receives his own umbrella. In how many ways can this happen?

Denote this number by D_5. The total number of distributions of the five umbrellas to the five gentlemen is $5! = 120$. To get D_5 we may have to subtract, from 120, the number of distributions which allow an umbrella in the hands of its owner. This latter situation may be called a 'hit'. Precisely, a 'hit' is a distribution of umbrellas by which one or more gentlemen get back their umbrellas. Thus we have:

$$D_5 = 120 - (\text{number of hits}).$$

How many hits are there? To answer this let us start counting the number of distributions in which a gentleman, say A, gets back his umbrella. This counting is done by first assuming A's umbrella to be in A's hands and distributing the remaining four randomly. This latter can be done in $4! = 24$ ways. Thus the number of distributions which give A his umbrella is 24. Call this (*). Similarly, the number of distributions which give B his umbrella is 24. Call this (**). And so on, for C, D and E. So the number of ways in which A or B or C or D or E gets back his umbrella is $5 \times 24 = 120$. Is this the number of hits? If it were so, D_5 would become

$$120 - 120 = 0 \qquad (i).$$

But this answer for D_5 is obviously wrong, because we know there are ways of distributing the umbrellas all wrongly. So there must be some error in the above argument. What is the error?

The error is this. In subtracting the number of hits, we have oversubtracted. When A gets back his umbrella and the remaining umbrellas are distributed randomly, there are situations when A and B both get back their umbrella. These ways are included in (*). Similarly when B gets back his umbrella and the remaining are distributed randomly, there are situations when B and A both get back their umbrellas. These are counted in (**). Thus the number of distributions that give A and B their own umbrellas is subtracted twice in the entry − 120 that appears in (*i*). In fact, every time two gentleman get back their umbrellas, it will be counted twice in the entry − 120 in (*i*). In order to allow for this excess subtraction we have to add to the count in (*i*) once the number of times two gentlemen get back their own umbrellas (irrespective of what the others get). This number is

$\binom{5}{2} \times 3! = 60$. So the count (*i*) should be corrected as

$$120 - 120 + 60 \qquad (ii).$$

Pursuing the same logic we see that we have again over-corrected the count of hits. For, those hits in which three gentlemen get back their own umbrellas, have been

$$\text{subtracted thrice} = \binom{3}{2} \text{ in } -120 \text{ and}$$

$$\text{added thrice} = \binom{3}{2} \text{ in } +60$$

In other words, these hits have been counted $-3 + 3 = 0$ times in (*ii*). So the correction has to be done by subtracting them once. Their number, *i.e.*, the number of times three gentlemen get back their own umbrellas, is

$$\binom{5}{3} \times 2! = 20$$

Hence the updated count would be

$$120 - 120 + 60 - 20 \qquad (iii).$$

Again there has been an over-correction. Those hits in which four gentleman get back their own umbrellas have been counted

$$\binom{4}{1} = 4 \text{ times in the entry } -120,$$

$$\binom{4}{2} = 6 \text{ times in the entry } 60,$$

and

$$\binom{4}{3} = 4 \text{ times in the entry } -20.$$

This means the counting of such hits has been done

$$-4 + 6 - 4 = -2 \quad \text{times.}$$

So the correction necessary to (*iii*) is to add once the number of distributions which give four gentlemen their own umbrellas. This number is

$$\binom{5}{4} \; 1! = 5$$

So (*iii*) has to be updated as

$$120 - 120 + 60 - 20 + 5 \qquad (iv).$$

Lastly, it is clear that the one single way in which all five get back their own umbrellas has been counted, in (*iv*),

$\binom{5}{1} = 5$ times in the entry – 120,

$\binom{5}{2} = 10$ times in the entry 60,

$\binom{5}{3} = 10$ times in the entry – 20 .

$\binom{5}{4} = 5$ times in the entry 5

The correction to (*iv*) therefore is to subtract this number once. This gives us the final count and also the answer to D_5 as follows.

$$D_5 = 120 - 120 + 60 - 20 + 5 - 1 = 44.$$

This essentially is the principle of inclusion and exclusion. The name comes from a set theoretic visualisation, in the form of *Venn diagrams*. Look at the following situation:

In a village there are 1000 families, of which 480 have male children, 540 have female children and 275 have both male and female children. What is the number of families which have neither male nor female children?

Let the portion covered by the rectangle in Fig. 12.1 stand for the set of all 1000 families. Let A and B stand for the set of all families with male children and female children, respectively. So we have

$$|A| = 480, |B| = 540 \text{ and } |A \cap B| = 275.$$

Denote the complement of A in the set of all families by A' and similarly let B' stand for the complement of B. Then what we require to calculate is $|A' \cap B'|$. This set $A' \cap B'$ is clearly what lies outside both the sets A and B and is shown by the shaded portion in the figure. It is equal to

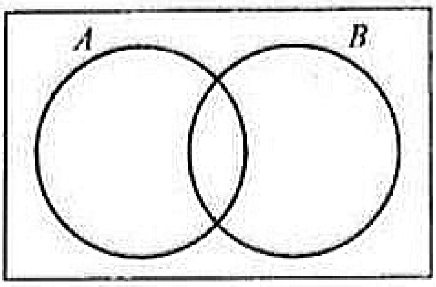

Fig. 12.1

$$\begin{aligned} &1000 - (|A| + |B| - |A \cap B|) \\ &= 1000 - (480 + 545) + 275 \\ &= 250. \end{aligned}$$

Thus 250 families have neither male nor female childern.

We can write the above as a formula without reference to the problem of the families and their children. If N is the population of the universe and A and B are two sets — it does not matter whether they intersect or are disjoint or one is a subset of the other — then,

$$|A' \cap B'| = N - (|A| + |B|) + |A \cap B|.$$

Using the set product notation AB for the intersection $A \cap B$ we can rewrite the above as

$$|A'B'| = N - \Sigma|A| + |AB|.$$

One can similarly proceed to the case of three sets A, B, C. We have

$$|A'B'C'| = N - \Sigma|A| + \Sigma|AB| - |ABC|.$$

(The student reader is advised to verify this by drawing the Venn diagram). The generalization of the above to n sets

$$A_1, A_2, \ldots, A_n$$

is the statement of the *SIEVE FORMULA*, which is only another name for the *IEP*.

SIEVE FORMULA (IEP) If $A_1, A_2, \ldots A_n$ are n subsets of a universe with population N,

$$|A_1'A_2' \ldots, A_n'| = N - \Sigma|A_i| + \Sigma|A_iA_j| - \Sigma|A_iA_jA_k| + \ldots + (-1)^n A_1A_2 \ldots A_n \quad (1)$$

We shall see several applications of this formula now.

EXAMPLE 1. *Apply the formula to the problem of the 5 gentlemen not getting their umbrellas at the end of the party.*

SOLUTION. Denote by A_i the set of distributions in which the ith gentleman gets his umbrella. Then, $N = 120$ and

$$\text{each } |A_i| = 24;$$
$$\text{each } |A_iA_j| = 6;$$
$$\text{each } |A_iA_jA_k| = 2;$$
$$\text{each } |A_iA_jA_kA_l| = 1;$$

and $|A_1A_2A_3A_4A_5| = 1.$

Hence the number of ways in which none gets his umbrella is equal to

$$|A_1'A_2'A_3'A_4'A_5'| = 120 - \binom{5}{1} \times 24 + \binom{5}{2} 6 - \binom{5}{3} \times 2 + \binom{5}{4} \times 1 - 1$$
$$= 120 - 120 + 60 - 20 + 5 - 1$$
$$= 44.$$

EXAMPLE 2. *Let N be the population of a universe. Let the elements in the universe be associated with certain properties (= qualities, conditions) called $P_1, P_2, \ldots P_t$. Then the number of elements which have none of the t properties is given by*

$$n(1) - n(2) + n(3) - \ldots, + (-1)^t n(t)$$

where $n(i)$ = is the number of elements with property i, $i = 1, 2, \ldots, t$.

SOLUTION. This is nothing but the Sieve Formula restated. Transfer the situation to the setting of a Venn diagram. Let A_i, $i = 1, 2, \ldots, t$ be the set of elements with property P_i. Then

$$n(1) = |A_1| + |A_2| + \ldots |A_t| = \Sigma|A_i|$$
$$n(2) = \Sigma|A_iA_j|;\ n(3) = \Sigma|A_iA_jA_k| \text{ and so on.}$$

Calculation of $n(1)$: To find the number of permutations in which (12) appears, keep (12) together as one entity. The remaining $(k-2)$ objects together with the single object (12) make up $(k-1)$ distinct objects. These can be permuted in $(k-1)!$ ways. The same argument applies for the cases of the number of permutations in which (23) appears, (34) appears and so on.

So $$n(1) = \binom{k-1}{1} \times (k-1)! \text{ where the factor } \binom{k-1}{1}$$

is because there are $k-1$ patterns and we have to choose one of them.

Calculation of $n(2)$: We first choose 2 patterns. This can be done in $\binom{k-1}{2}$ ways. If the 2 patterns chosen are overlapping like 12 and 23, then we have one single entity 123 which along with the remaining $k-3$ objects make $k-2$ objects in all and these can be permuted in $(k-2)!$ ways. If, on the other hand, the 2 patterns chosen are 'disjoint' like 12 and 34 then we have two distinct objects 12, 34. These two together with the remaining $k-4$ objects make up $k-2$ objects in all, again giving rise to the same $(k-2)!$ permutations. Thus whether the patterns chosen are overlapping, or not, the resulting number of permutations is the same $(k-2)!$ Hence

$$n(1) = \binom{k-1}{2}(k-2)!$$

We shall observe that in the case of choosing 3 patterns also a similar situation happens. The three patterns chosen may belong to one of the following three 'types' as far as 'overlapping' of patterns is concerned:

12, 23, 34 : (Type 1)
12, 23, 45 : (Type 2)
and 12, 34, 56 : (Type 3).

In the case of type 1, there is one object of the form 1234 and there are $(k-4)$ remaining objects. These together can be permuted in $(k-3)!$ ways. In the case of type 2, there are two objects of the form 123 and 45 and there are $(k-5)$ remaining objects. These together can be permuted in $(k-3)!$ ways. Again in the case of type 3, there are three objects of the form 12, 34, 56 and there are $k-6$ other objects. These together can be permuted in $(k-3)!$ ways.

Thus

$$n(3) = \binom{k-1}{3}(k-3)!$$

And so on. The final answer can be seen to be

$$k! - \binom{k-1}{1}(k-1)! + \binom{k-1}{2}(k-2)! - \binom{k-1}{3}(k-3)! + \ldots + (-1)^{k-1}\binom{k-1}{k-1}1!$$

EXAMPLE 8. *If $A_1, A_2, \ldots A_n$ are n subsets of a universe with population N,*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \Sigma|A_i| - \Sigma|A_iA_j| + \Sigma|A_iA_jA_k| \ldots + (-1)^{n-1}|A_1A_2 \ldots A_n|$$

SOLUTION. This follows from the Sieve Formula, since

$$|A_1'A_2' \ldots A_n'| = N - |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

EXAMPLE 9. *How many integer solutions are there of* $x_1 + x_2 + x_3 + x_4 = 30$ *with* $0 \le x_i < 10$?

SOLUTION. Let P_1 stand for the property of "$x_1 \ge 10$". Let P_2 stand for the property of "$x_2 \ge 10$" and so on. Then we have four properties P_1, P_2, P_3, P_4. We want the number of integer solutions of the equation, with *none* of the four properties. The total number of non-negative integer solutions, without any restriction on the x_i's is,

$$\binom{30+4-1}{4-1}$$

from Example 5, Section 9.2. Thus **N**, the total number of all solutions is $\binom{33}{3}$. We know $n(0) = N - n(1) + n(2) - n(3) + \ldots$. To get $n(1)$ we first write $x_1 = 10 + y_1$. The equation becomes

$$y_1 + x_2 + x_3 + x_4 = 20.$$

There is no restriction on the unknowns now. The number of non-negative integer solutions is, by the same Example, quoted above,

$$\binom{20+4-1}{4-1}$$

i.e. $\binom{23}{3}$. This is the number of solutions of the original equation with $x_1 \ge 10$. The same is true for the number of solutions with $x_2 \ge 10$ or $x_3 \ge 10$ or $x_4 \ge 10$. Thus

$$n(1) = 4\binom{23}{3}.$$

To get $n(2)$ we write, as a typical case, $x_1 = 10 + y_1$ and $x_2 = 10 + y_2$. The equation becomes

$$y_1 + y_2 + x_3 + x_4 = 10$$

for which, the number of non-negative integer solutions is

$$\binom{10+4-1}{4-1} = \binom{13}{3}.$$

Hence $$n(2) = 6\binom{13}{3}.$$

At the next stage where we have to make three of the x_i's greater than or equal to 10, there exists only one non-negative integer solution of the original equation. So $n(3) = 1$. At the next stage where we have to make all the x_i's greater than or equal to 10, there exists no non-negative integer solution of the original equation. Hence $n(4)$ is zero. So we obtain

$$n(0) = \binom{33}{3} - 6\binom{23}{3} + 4\binom{13}{3} - 1$$

which is therefore the number of required types of solutions.

$$\{99, 98, 97, 96, 95, 94, 93, 92, 91\}.$$

We take only nine numbers counting from the topmost two-digit number 99, because, if we take all ten numbers there would be nothing left for the other non-empty subset. The sum of the above nine numbers is 855.

It is then clear that, whatever non-empty subsets we take from our set of numbers, the sum of the numbers in the subsets will vary from 10 to 855 only. In an actual case the variation could be even a smaller spread than this, but this is the utmost possible spread. In other words there are a possible variety of $(855 - 9) = 846$ values for the sums of subsets.

Now the pigeon-hole principle applies. On the one hand we have 1023 possible subsets and on the other hand there are only 846 possible sum-values. Since each subset must have a sum-value, it follows there certainly exist more than one subset having the same sum-value! If these two subsets are disjoint, we are done. If they are not, *i.e.*, they overlap, by throwing away the common elements, we can arrive at disjoint subsets which have the same sum. An illustration of this by an example (Exercise 12.2 No. 10) is left as an exercise for the student.

One would wonder at the power of the *PHP* used in the above example. Without much mathematical equipment, formula, or manipulation, we were able to solve Example 1. The *PHP* has its applications in several totally unexpected situations. We shall see a few such examples before we actually come to the mathematics implied by the *PHP*.

EXAMPLE 2. *A lattice point (x, y, z) in three-space is one all of whose coordinates are integers. Nine such points are taken at random. Show that of the 36 line segments joining pairs of these points, a least one passes through a lattice point of the space.*

SOLUTION. We are just given two pieces of information, namely, there are 9 points and 36 line-segments joining them, pair by pair. In fact the "36-line" information is redundant, we could oursleves have calculated that when there are 9 points there should be (a maximum of) $\binom{9}{2} = 36$ line segments joining them. Thus the only real piece of information is that there are 9 lattice points. But the fact they are lattice points is the real clue. The coordinates, x, y, z of a lattice point are all integers. These integers have only two possibilities in terms of parity (*i.e.*, of being even or odd). Each x_i, each y_i, and each z_i of the nine points (x_i, y_i, z_i) has only to be either odd or even. In other words there are only $2^3 = 8$ possibilities for the (x_i, y_i, z_i) in terms of the parity of the coordinates. But there are 9 points in all. These nine points have each to fall into one of these 8 possibilities. Here comes the use of the *PHP*. So there should exist 2 points out of the nine, whose coordinates (a, b, c) and (a', b', c') have the same parity, coordinate by coordinate. More precisely a and a' are either both even or both odd; b and b' are either both even or both odd; c and c' are either both even or both odd.

Now comes the mathematical consequence. Therefore $a + a'$, $b + b'$, $c + c'$ are all even. This means that the mid-point of the line segment (one of the 36 line segments stated in the problem!) joining (a, b, c), (a', b', c'), which is nothing but

$$\frac{a+a'}{2}, \frac{b+b'}{2}, \frac{c+c'}{2}$$

has all its three coordinates integers and so is a lattice point in 3-space. Hence the result!

Here is another interesting application of the *PHP*, which is the foundational starting point for a whole branch of combinatorics, called "RAMSEY THEORY".

EXAMPLE 3. *Six people meet in a party. Show that either there are at least three who have mutually shaken hands before or there are at least three, no two of whom have shaken hands before. Show also that the number 'six' in the statement cannot be replaced by the number 'five' or less.*

SOLUTION. For convenience we shall refer to any two people who have shaken hands before by the term "friends" or "acquaintances" and any two who have never done so, by the term "strangers". So three persons who, two by two, are friends would be called "mutual friends"; and in the same manner, three people who, two by two, are strangers would be called "mututal strangers". So the problem requires us to show that in any party of six people either three of them are mutual strangers or three of them are mutual acquaintances. The proof of this statement requires nothing but a three-step cold logic.

In order to help the understanding, it is conveninent to convert the problem to a graph-theoretic setting. Right here one should experience the thrill of the ascent to mathematical abstraction from a concrete situation. Suppose we had a graph with 6 vertices and every pair of points was joined by an edge. Such a graph is called a *complete graph* — the completeness indicating that there are no more pairs of points to be joined by edges. A complete graph on n vertices is denoted by the symbol K_n. Figure 12.2 shows K_2, K_3, K_4, K_5 and K_6.

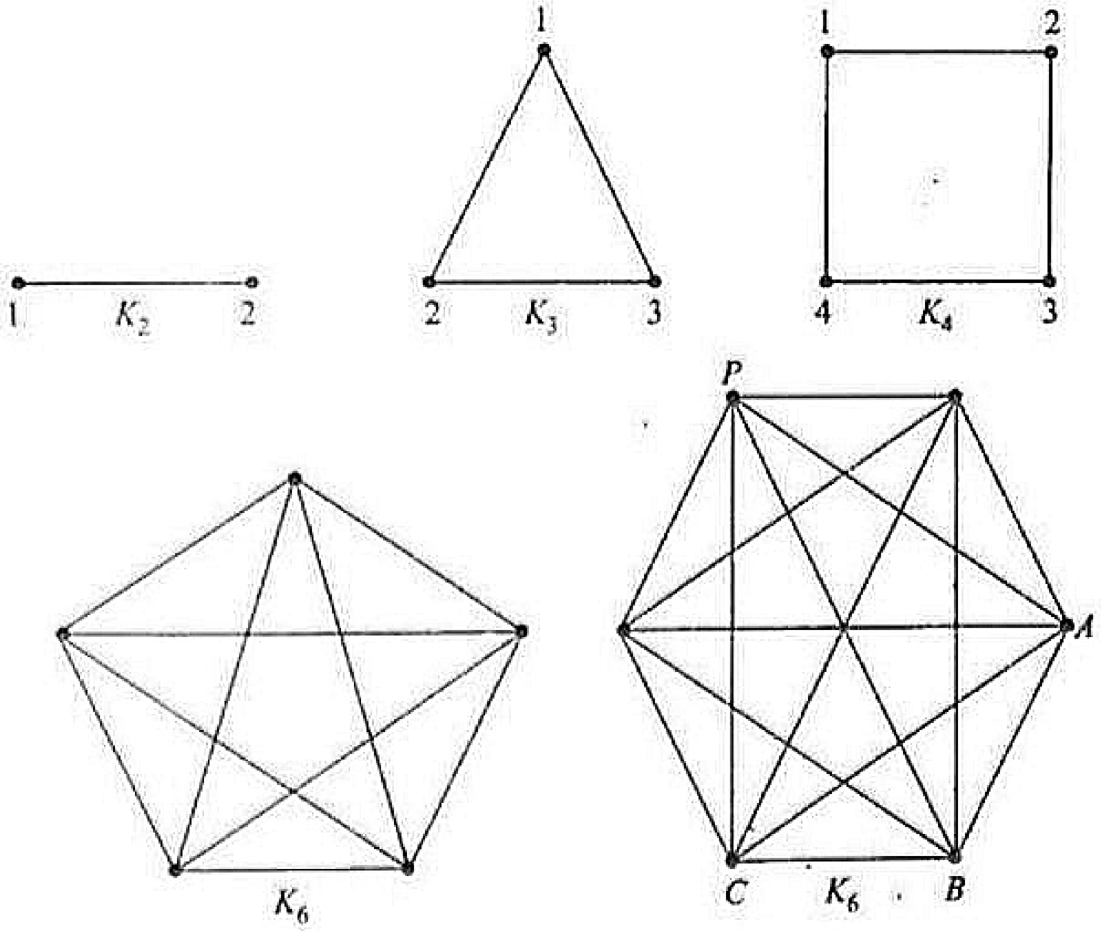

Fig. 12.2

Look at K_6. It has $\binom{6}{2} = 15$ edges. Let the 6 vertices of K_6 stand for the $\binom{6}{2}$ people in our party. Let the edges be coloured red or green according as the two people represented by the vertices connected by the edge are mutual strangers or acquaintances. The problem now asserts: In whatever way you may colour the 15 edges of a K_6 red and green, you can never avoid either a red triangle — that is, a triangle all of whose three sides are red — or, in the alternative, a green triangle. The interesting proof goes as follows.

Consult K_6 in Fig. 12.2. Focus your attention on any one vertex, say P. There are five lines going forth from P. They are coloured red or green — some red, some green. We do not know how many of them are red and how many of them are green. It could be all five red; four red and one green; three red and two green; two red and three green; one red and four green; or all five green. The beauty here is the relevance of the *PHP*. Since there are only two colours and we have five lines which fall into either one of them, the *PHP* says there are at least three of the same colour. (The above listing of the different possibilities certainly confirms this conclusion — to arrive at which, however, the listing was not necessary).

Suppose A, B, C are the other ends of these three lines, all of the same colour, say, red. If either one of AB, BC, CA is red then that line with the two edges from P meeting it at its ends would give a red triangle. If none of AB, BC, CA is red, then all three are green and we have a green triangle ABC.!

This concludes the proof that either there exist three mutual strangers or there exist three mutual acquaintances.

Finally, we note that the number 'six' cannot be replaced by 'five'. For, if we had a K_5, and we coloured its 10 edges red and green, it is not always true that there exists either a red triangle or green triangle; for, look at the edge-colouring of K_5 exhibited in Fig. 12.3 where the letters r and g indicated on the edges stand for red and green respectively. We note that, for this particular colouring, there is neither a red triangle nor a green triangle. This single case tells us that the existence of a monochromatic triangle (triangle whose sides are all of the same colour) is not universally true in the case of K_5. Certainly it cannot be so in the case of K_4 or K_3.

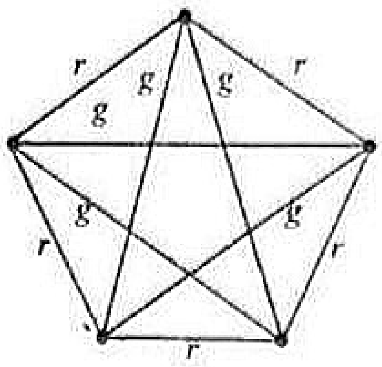

Fig. 12.3

Thus 6 is the least possible value of n for which K_n has this property. Note that when K_6 has the property, all K_n, $n > 6$ has also the property.

This elaborate discussion of Example 3, enables us to assert the following statement, which is sometimes called the Friendship Theorem.

Friendship Theorem. In a party of six people there always exists either three mutual acquaintances or three mutual strangers. The number '6' is the smallest positive integer for which this result is true.

EXAMPLE 4. *There are 1958 computers which can communicate among themselves in 6 languages — with the proviso that any two computers communicate only in one of these 6 languages. Prove that there exist at least 3 computers whose mutual language of communication, two by two, is the same.*

SOLUTION. The graph-theoretic analogy of Example 3 will help as also the edge-colouring analogy. Imagine a polygon with 1958 vertices, with its edges coloured with 6 colours. Fix a vertex P. 1957 lines go forth from P. They fall into 6 colours. Since $6 \times 326 = 1956$, the *PHP* tells us that there exist at least 327 edges of one colour say C_1. The other ends of these 327 edges shooting forth from P constitute a complete graph with 327 vertices. If there exists a single pair of vertices say (a, b) out of these, that have the same colour as C_1, then we have a C_1 coloured triangle of which one vertex is P and the other two are a, b. If not, this means, all the edges of the K_{327} have only the remaining five colours.

Repeat the process now for this K_{327} and the five colours. Since $5 \times 65 = 325$, it follows that, out of the 326 lines shooting forth from one fixed vertex, say Q there exists, by an application of *PHP*, 66 lines of the same, colour. As before this leads us to a K_{66} with all its edges being coloured by four colours. Again since $4 \times 16 = 64$, the 65 edges shooting forth from a fixed vertex R of this K_{66} fall into four categories of colours and so, by, *PHP*, enable us to assert the existence of either a monochromatic triangle with one vertex R or in the alternative that of a K_{17} whose edges are all coloured by 3 colours.

One more reduction gives a K_5 whose edges are all coloured by 2 colours. And we know by Example 3, this certainly leads to a monochromatic triangle.

It is now time for us to state the pigeon hole principle formally as a self evident proposition:

PHP If $kn + 1$ pigeons $(k \geq 1)$ are distributed among n pigeon-holes, one of the pigeon-holes will contain at least $k + 1$ pigcons. A stronger version of this would be the following:

PHP If m pigeons are placed into n pigeon-holes, then at least one pigeon-hole will contain more than $\left[\frac{m-1}{n}\right]$ pigeons, where $\left[\frac{m-1}{n}\right]$ is the largest integer in $\frac{m-1}{n}$.

EXAMPLE 5. *Given a sequence of 10 distinct numbers, show that there exists either an increasing subsequence of length 4 or else a decreasing subsequence of length 4.*

SOLUTION. It is convenient to keep a concrete case in front of us. Let us have the following sequence for this purpose.

$$24 \quad 3 \quad 5 \quad 4 \quad 17 \quad 14 \quad 21 \quad 8 \quad 22 \quad 10$$

The increasing subsequences starting with 3 are: 3 5 17 21 22; 3 4 17 21 22; 3 17 21 22; 3 14 21 22; 3 8 10 and so on. Therefore, we observe that the length of the longest increasing subsequence starting from 3 is 5. Thus for each a_i, $1 \leq i \leq 10$ in the sequence $a_1, a_2 \dots a_{10}$ we may obtain t_i, the length of the longest increasing sequence starting with a_i.

In our illustration, $t_1 = 1$, $t_2 = 5$, $t_3 = 4$ and so on. Let us write these t_i's below the terms of the sequence thus:

a_i:	24	3	5	4	17	14	21	8	22	10
t_i:	1	5	4	4	3	3	2	2	1	1

If we do this for the general case, we have

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & \dots a_{10} \\ t_1 & t_2 & t_3 & t_4 & \dots t_{10} \end{matrix}$$

Now we claim that there exists a_i such that the corresponding t_i is 4. In other words there exists a term starting from which the length of the longest increasing subsequence is 4. If this does not happen, then we shall prove there exists a decreasing subsequence of length 4 starting from some term.

Now what does it mean to say that there exists no term, starting from which the length of the longest increasing subsequence is 4? It means $t_i = 3$ or less for all i. Now there are ten terms $a_1, a_2 \dots a_{10}$. These are the 10 pigeons. The pigeon holes are the three t-values, namely 1, 2, 3. So by the *PHP* there is at least one t-value

(= pigeon hole) which contains more than $\left[\dfrac{10-1}{3}\right] = 3$ pigeons. Thus there exists a t-value which is common to at least 4 terms of the sequence. Suppose this t-value is, say, 2 (The argument is the same for any other t-value). And further suppose that the terms of the sequence having this t-value are

$$a_2\, a_3\, a_5\, a_8$$

Each has a value 2. We therefore have the situation as follows:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
t-value :		2	2		2			2		

Our claim is : $a_2 > a_3 > a_5 > a_8$.

To see this, first note that if $a_2 < a_3$ then since a_3 is the starting point of an increasing subsequence of length 2, by appending a_2 to the beginning of this subsequence we will have an increasing subsequence of length 3, starting from 2. This is a contradiction to $t_2 = 2$. Thus $a_2 > a_3$.

Again $a_3 > a_5$ for a similar reason and $a_5 > a_8$ for the same reason. We now have

$$a_2 > a_3 > a_5 > a_8.$$

which is a decreasing subsequence of length 4.

This result is known in a general form and was proved by Erdos and Szekeres in the following form. We shall not prove it here.

Erdos-Szekeres Theorem: Given a sequence of $n^2 + 1$ distinct integers either there is an increasing subsequence of $n + 1$ terms or a decreasing subsequence of $n + 1$ terms.

EXAMPLE 5. is a special case of this.

EXAMPLE 6. *A set of numbers is called a sum-free set if no two of them add up to a member of the same set and if no member of the set is double another member. How big could be a sum free subset of $\{1, 2, 3, \dots, 2n + 1\}$.?*

SOLUTION. First note that the set

$$\{n + 1, n + 2, n + 3, \dots, 2n + 1\}$$

is a sum-free subset and its size is $n + 1$. We shall now prove that no sum-free subset could be bigger.

Suppose a subset is of size $n + 2$. Let the largest number in the subset S be l. Thus $S = \{a, b, \ldots, l\}$ where l is the largest. Certainly $l \leq 2n + 1$ since S is a subset of

$$\{1, 2, 3, \ldots, 2n + 1\}.$$

Now consider the set of numbers

$$\{1, 2, 3, \ldots l\}$$

If l is odd, the $l - 1$ numbers $1, 2, 3, \ldots, l - 1$ pair off into $\frac{l-1}{2}$ pairs

$$(1, l - 1), (2, l - 2) \ldots \qquad (*)$$

such that the sum of each pair of numbers is l. If l is even, the $l - 1$ numbers $1, 2, 3, \ldots l - 1$ pair off into $\frac{l-2}{2}$ pairs

$$(1, l - 1), (2, l - 2), \ldots \left(\frac{l}{2} - 1, \frac{l}{2} + 1\right) \qquad (**)$$

and a singleton $\frac{l}{2}$, such that the sum of each pair is l.

Since $\frac{l}{2} \leq n + \frac{1}{2} < n + 1$ we can apply the *PHP* to the pigeons which are the $n + 1$ members of the set other than l and the smaller number of pigeonholes which are the pairs listed in (*) and (**). The *PHP* will then imply that two members of the set S other than l would fall into one of these pairs. In other words there exist two members in S whose sum is l. So is not sum-free!.

Thus any subset of size $> n + 1$ cannot be sum-free. So the maximum size of a sum-free subset of $\{1, 2, 3, \ldots, 2n + 1\}$ is $n + 1$.

EXAMPLE 7. *Given a set of $n + 1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.*

SOLUTION. Let the given set be $\{x_1, x_2, \ldots, x_{n+1}\}$

and let $$x_i = 2^{n_i} y_i$$

where n_i is a non-negative integer and y_i is odd. What we have done is to break each x_i into its even component 2^{n_i} and its odd component y_i. As an illustration, $48 = 2^4 \times 3$; $35 = 2^0 \times 35$; $8 = 2^3 \times 1$, etc. Let

$$T = \{y_i;\ i = 1, 2, \ldots, n + 1\}$$

i.e. T is the set of all y_i's. T therefore is a collection of $n + 1$ odd integers, each less than $2n$. But there are only n odd numbers less than $2n$. So by the *PHP*, this means two numbers (= pigeons) in T must be equal — say, $y_i = y_j$ with $i < j$.

Then

$$x_i = 2^{n_i} y_i \text{ and } x_j = 2^{n_j} y_j$$

Here, if $n_i \leq n_j$ then x_i divides x_j, and if $n_i > n_j$, then x_j divides x_i. Thus in all cases there are two numbers in the given set such that one divides the other!

EXAMPLE 8. *We are given a set M of 100 distinct positive integers none of which has a prime factor greater than 12. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.*

SOLUTION. There are only 5 primes less than 12. They are: 2, 3, 5, 7, 11. So each $m \in M$ has a prime factorisation of the form

$$2^{k_2}3^{k_3}5^{k_5}7^{k_7}11^{k_{11}} \tag{1}$$

where $2^{k_2}, 3^{k_3}, 5^{k_5}, 7^{k_7}, 11^{k_{11}}$, are non-negative integers. With each $m \in M$ associate an ordered 5-tuple (= vector with 5 coordinates) as follows.

If m has the factorisation (1) the vector corresponding to it is

$$x_1x_2x_3x_4x_5.$$

where $\quad x_i = 0$ if k_i is even;

and $\quad x_i = 1$ if k_i is odd.

Just to illustrate, the vector corresponding to $48 = 2^4 \times 3$ is 01000 and the vector corresponding to $2310 = 2 \times 3 \times 5 \times 7 \times 11$ is 11111. The number of all such possible vectors is $2^5 = 32$. Since $100 > 32$, by (*PHP*) there exist two numbers (= pigeons) say a_1 and b_1 in M which have the same associated vector (= pigeon hole). This means a_1 and b_1 have, for each i, their k_i's both even or both odd. Hence the product $a_1\, b_1$ has all its k's even numbers. In other words a_1b_1 is a perfect square and so may be written as c_1^2 for some integer c_1.

Now remove this pair of numbers a_1, b_1 from M. We then have 98 elements and since $98 > 32 = 2^5$, again by another application of *PHP*, there exists two numbers a_2, b_2 such that, as in the previous case, $a_2b_2 = c_2^2$ for some integer c_2.

Remove a_2 and b_2 from M. Proceed like this until we have removed 33 pairs of elements from M. At every stage we have a set which has more than 32 elements and so the above arguments are valid. Finally we have $100 - 66 = 34$ elements.

Now look at the removed set of 66 elements (*i.e.*, 33 pairs : $a_1, b_1, \ldots, a_{33}, b_{33}$. Each product $a_i \times b_i = c_i^2$ for some integer c_i. Therefore

$$c_i = \sqrt{a_ib_i}$$

We thus have 33 positive integers and each of these integers c_i has no other prime factor other than 2, 3, 5, 7, 11. Since $33 > 32 = 2^5$, again by an application of *PHP*, there exist at least two integers c_i, c_j whose exponent vectors are the same. This means $c_ic_j = d^2$ Now

$$d^4 = (c_ic_j)^2 = c_i^2c_j^2 = a_ib_ia_jb_j \text{ for some } a_i, b_i, a_j, b_j \text{ in } M \text{ and we are done!}$$

We shall conclude this section with an indication of a famous theorem of Ramsey — which is the blossoming out of the *PHP* into a full fledged mathematical research activity in modern times. Ramsey proved in 1931 the following theorem by a terse logic that was a supreme extension in a tight-rope fashion of the logic of the Friendship Theorem. The statement of Ramsey's theorem may be described as follows.

Let t be a positive integer. Let $r, q_1, q_2, \ldots, q_t$ be positive integers such that $1 \le r \le q_i$ for every i.

Then there exists, says Ramsey's theorem, a smallest positive integer n (which of course depends on the r and the q's) such that the following holds: Let the

r-subsets (*i.e* subsets containing r elements) of S be partitioned into t distinct classes listed as:

$$A_1, A_2, \ldots, A_r.$$

In other words each r-subset of S is precisely in one of these t classes. Then for some $i = 1, 2, \ldots, t$ there exists a subset X of S wth q_i elements such that all r-subsets of S belong to the same A_i.

Flabbergasting! Is'nt it? But this terseness will vanish if you care to see how this theorem is an abstraction of the Friendship Theorem. There are three levels of abstraction, all happening simultaneously.

Take $r = 2$. This means we are interested only in 2-subsets. So if the set S is the set of vertices of a graph, we are interested in the edges of the graph. This was the case in the Friendship Theorem. The use of a general number r instead of 2 is one level of abstraction.

Take $t = 2$. This means that the r-subsets (*i.e* the edges of the graph, in the speical case $r = 2$) are partitioned only into 2 calsses, namely those which are coloured red and those which are coloured green. The use of the general t instead of 2 is another level of abstraction.

Then Ramsey's theorem says that there exists a smallest positive integer n (in the case of the Friendship Theorem it is 6) which has the following property. If the complete graph K_n has its edges partitioned into a red class and a green class, there exists a triangle (q_i-subset, i.e, a 3-subset; the use of q_i instead of 3 is another level of abstraction) all of whose sides belong to one colour.

Thus the Friendship Theorem is the special case of Ramsey's theorem with $r = 2$, $t = 2$ and q_1, q_2 both equal to 3. But there is one difference between the Friendship Theorem and Ramsey's Theorem. We pay a big price for abstraction. While the Friendship Theorem asserts that 6 is the magic number with the stated property, Ramsey's theorem only says **there exists** such a number. It does not tell you what it is or how to calculate it. Calculation of these numbers, called Ramsey numbers, is not easy. The Ramsey number is denoted by

$$\mathrm{N}\,(q_1, q_2, \ldots, q_t; r)$$

showing its dependence on the q's and the r's. Thus the Friendship Theorem says

$$\mathrm{N}\,(3, 3; 2) = 6$$

The general Ramsey numbers for various r's and all possible q's are therefore only known to exist. Their actual determination has given rise to several continuing research problems. Most of the numbers which are known, are for the case $t = 2$. For $t = 3$, the only known number is

$$\mathrm{N}\,(3, 3, 3; 2) = 17.$$

This means 17 is the smallest positive integer n such that in whatever way K_n is edge-coloured with three colours, there will always exist a monochromatic triangle.

Recall Example 4 where we proved

$$\mathrm{N}\,(3,3,3; 2) \leq 17$$

$$\mathrm{N}\,(3,3,3\ 3; 2) \leq 66$$

$$\mathrm{N}\,(3,3,3,3,3; 2) \leq 327$$

and

$$\mathrm{N}\,(3,3,3,3,3,3; 2) \leq 1958.$$

That N (3,3,3;2) is actually 17 and not any number less than 17, needs an example of a complete 16-gon whose edges are coloured with three colours in such a way that no monochromatic triangle exists. This needs deep Mathematics. As far as the other numbers are concerned, particularly for higher values of $t > 3$, very little is known.

EXERCISE 12.2

1. If there are 40 people in a room show that there exists a subset of more than 3 people who will have a common month of birth.
2. Concoct a problem similar to Problem 1 in respect of date of birth.
3. What is the smallest possible number(s) which have the stated property in Problems 1 and 2?
4. If a factory has 100 electrical outlets with a total of 25000-volt capacity show that there exists at least one outlet with a capacity of 250 or more volts.
5. An international society has its members from six different countries. The list of members contains 1994 names numbered 1, 2, ..., 1994. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country or twice as large as the number of one member from his own country.
6. Complete the proof of the last part of Example 1 by means of the illustration required in the proof.

PROBLEMS

1. How many n-digit decimal sequences are there, using digits 0, 1, 2, ... 9, but in which the digits 2, 4, 6, 8 all appear?
2. How many positive integers ≤ 462 are relatively prime to 462? Relate this problem to a function defined in Chapter 2.
3. Show that if the 21 edges of a complete 7-gon is coloured red and blue, there exist at least 3 monochromatic triangles.
4. A chess player plays 132 games in 77 days. Prove that for a certain number of consecutive days he has played exactly an aggregate of 21 games.
5. How many derangements of $1, 2, \ldots, n$ are there in which only the even integers occupy new positions?
6. Suppose that 1985 points are given inside a unit cube. Show that we can always choose 32 of them in such a way that every (possibly degenerate) closed polygon with these points as vertices has perimeter less than $8\sqrt{3}$. **Hint:** $1985 = 31 \times 64 + 1$.
7. Let x be any real number. Prove that among the numbers
$$x, 2x, \ldots, (n-1)x$$
there is one that differs from an integer by at most $1/n$.
8. a, b, c, d, e, f, g, are non-negative real numbers adding up to 1. If M is the maximum of the five numbers.
$$a+b+c,\ b+c+d,\ c+d,\quad d+e+f,\ e+f+g.$$
find the minimum possible value that M can take as a, b, c, d, e, f, g vary.
Hint: Append the four numbers a, $a+b$, $f+g$, g to the five given.
9. Given a set M of 1992 positive integers none of which has prime factors > 28, prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer. What is the smallest number that can replace 1992 in this problem?
Hint: Refer to Example 8 and imitate it.

10. A particular case of Ramsey's theorem says that if the edges a complete n-gon be coloured red and blue then there exists either a triangle all of whose edges are of the same colour or a complete 4-gon all of whose six edges are of the other colour. Prove by constructing an appropriate example that $n > 8$.

11. Let $S = \{1, 2, \ldots, n\}$, $i \in S$ is said to be a *fixed point* of a permutation p or S if $p(i) = i$. Let $p_n(k)$ be the number of permutations of S which have k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!$$

[redacted]

13

BEGINNINGS OF PROBABILITY THEORY

We use the words 'probable and probability' in our everyday language without realising that there is mathematics involved in the usage of the language. When we say, 'probably it may rain today' it is usually an innocuous statement reflecting our understanding of the local weather. When the newsreader announces that there is a high probability that it may rain tomorrow, she actually has, behind her, the support of all the mathematical analysis made by the weather forecasting section of the Meteorological Department. Though she herself may not understand the mathematics of it, somebody somewhere in the higher echelons must have precise statements before them which look very much like the following: There is an eighty per cent, chance that it will rain tomorrow. Here it is not the number 'eighty' that matters. What matters is that there is such a number floating around. Where did this number 'eighty' come from? It comes from the fact that scientists have analyses several similar situations in the past, presumably a large number of them. Eighty per cent, of those times it had really rained and that is the reason it is being said now there is an 80% chance of its raining tomorrow. Thus, in order to make a statement; 'It is probable that such and such an event might happen', one must have a knowledge of a record of such events in the past or one must be capable of counting all possible occurrences and non-occurences of the event.

For instance one says that if we toss an unbiased coin ('unbiased' means: no side of the coin is unduly loaded; or, in other words, nature has nothing to distinguish between the two sides of the coin), the probability that we get heads is the same as the probability that we get tails. Or, what is the same thing, 'head' and 'tail' are *equally likely* in the free toss of a single unbiased coin. So we say, the probability of each is 1/2. In this language, we are tacitly assuming that there is a total of 'one' for the sum of all probabilities. What does one mean by this 'all'? The meaning of this 'all is the starting point of the theory of probability. The 'all' means the 'universe of all events'. We have a technical name for it. It is called the 'SAMPLE SPACE'.

We shall elaborate this concept now. In the experiment of tossing a single coin, though the actual outcome of a single toss is not predictable, the set of all possible outcomes can be visualised in advance. This set of all possible outcomes of an experiment is called the sample space of the experiment. We usually denote it by S. Here are some illustrations, to which we shall come back repeatedly.

(*a*) Suppose our experiment consists in the tossing or flipping of a single coin. The sample space is

$$S = \{H, T\}$$

where H means the outcome of the toss is a head and T means the outcome is a tail.

(*b*) If on the other hand the experiment consists of a single throw of a six-faced die, whose faces are marked with 1, 2, 3, 4, 5, 6, the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

where the outcome 'i' means that i appeared on the top face of the die.

(*c*) Suppose the experiment is the tossing of two coins. The sample space then, is

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

where (H, H) means both coins turn up heads; (H, T) means the first coin turns up heads, while the second coin turns up tail; (T, H) means the first coin turns up tails and the second coin turns up heads; and finally, (T, T) means both the coins turn up tails.

(*d*) Suppose the experiment consists of throwing two dice. The sample space would be

$$S = \{(i, j) \mid i = 1, 2, \ldots, 6; j = 1, 2, \ldots, 6\}.$$

Here (i, j) means i on the first die and j on the second die.

(*e*) Suppose the experiment consists of measuring in hours the life time of a torch light cell. The sample space consists of all nonnegative reals, so that

$$S = \{x : 0 \leq x < \infty\}.$$

Now we define an **event** as a **subset of the sample space.** In (*a*) above, the event that a head appears is the subset $\{H\}$ of the sample space. In (*c*), the event that a Head appears in the second coin is the subset

$$\{(H, H), (T, H)\}.$$

In (*b*), the event that the die throws up an odd number is the subset $\{1,3,5\}$ of the sample space.

In (*d*), the event that the two dice throw up a sum of 8 is the subset

$$\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}.$$

In (*e*), the event that the torch light cell has a life not greater than 12 hours is the subset $\{x : 0 \leq x \leq 12\}$.

As soon as we define 'events' as subsets of the sample space we can transfer the language of set theory to the world of 'events'. Thus the union of two events is the union of the two subsets defining the two events. As illustration, in (*b*) above, if E is the event $\{1, 2, 3\}$ and F is the event $\{2, 3, 4\}$, then

$$E \cup F \text{ is the event } \{1, 2, 3, 4\} \tag{*}$$

and

$$E \cap F \text{ is the event } \{2, 3\} \tag{**}$$

The intersection $E \cap F$ of two events is usually written EF. Translating the meaning of unions and intersections in the case of events we have the following interpretations: $E \cup F$ means the occurrence of either E or F which means, in the illustration (*), the throwing of any one of the numbers 1, 2, 3, 4. The event EF, on the other hand, means both E and F and in the illustration (**) it means the occurrence of 2 or 3.

Another concept that we use from set theory in the algebra of events is E and not-E. If E is the event, $\{2, 3, 4\}$ in (*b*) not-E means the non-occurrence of 2, 3, 4, that is, the occurrence of $\{1, 5, 6\}$. So E and not-E (which is also written as E^c) are complements of each other. E^c occurs iff E does not occur.

The union and intersection of two events can be extended to any number of events, and in fact, to an infinite number of events without difficulty. Thus if $E_1, E_2, E_3, \ldots$ are an infinite number of events, the event

$$\bigcup_{i=1}^{\infty} E_i$$

is the event which denotes the occurrence of either one of the E_i's. In the same manner,

$\bigcap_{i=1}^{\infty} E_i$, which is written as $E_1 E_2 E_3 \ldots$ is the event which denotes the occurrence of all of them simultaneously. In (e) for example, let

$$E_i = \{x \mid 0 \le x \le i\}, \text{ for each } i.$$

Then $$\bigcap_{i=1}^{\infty} E_i = E_1 E_2 E_3 \ldots = \{x \mid 0 \le x \le 1\}.$$

In other words the life of the torch cell being one hour or less is an event which means, E_1, has occurred, E_2 has occurred, E_3 has occurred and so on, an infinitum.

To summarise what we have done so far, we have called the set of all outcomes (of an experiment) the **sample-space,** and every subset of it an **event**. The third fundamental concept in the subject is the association of a number, called PROBABILITY, to every event of the sample space. But we are not supposed to do this arbitrarily. This association of probability $P(E)$ to every event E has to satisfy the following three Axioms, in order to be both meaningful and useful.

> AXIOM 1 $\qquad 0 \le P(E) \le 1$
>
> In other words, the probability of every event, that is, an outcome or a set of outcomes, is a number between 0 and 1, both inclusive.
>
> AXIOM 2 $\qquad P(S) \le = 1$
>
> In other words, the probability of the whole sample space considered as an event (= subset of itself), has to be 1. It is therefore called the sure event. One of the outcomes listed under S is bound to happen.

Before we take up the third Axiom, we need to explain what are known as 'mutually exclusive' events. Recall that, already in chapter 9, we referred to two occurrences as 'mutually exclusive' if they cannot occur simultaneously. That is, when one event is occurring, the other event is (by that very fact) not happening. Now that we have defined events as subsets of the sample space, it is easier to define 'mutually exclusive' events.

Definition. Two events are said to be **'mutually exclusive'** if their intersection (as subsets of the sample space) is empty. We shall use the contraction '*m.e.*' for 'mutually exclusive'.

We shall take up specific examples of events from the five sample spaces we have already introduced. This will give a better understanding of the concepts.

In (a), where the experiment is the tossing of a single coin, $S = \{H, T\}$. So the events are φ (= the empty set); $\{H\}$; $\{T\}$ and the whole space $S = \{H, T\}$.

We define $P(H) = 1/2 = P(T)$. Already $P(S) = 1$. We shall come to $P(\varphi)$ later. Why did we define $P(H)$ and $P(T)$ as 1/2? Here lies the intuitive assumption that 'heads' and 'tails' are equally likely whenever an unbiased coin is tossed. In other words, the two events constituting the sample space are assumed to have the same probability. Incidentally, the events that constitute the sample space are called **'Elementary events'**; — 'elementary', in the sense that they are not made up of other events. For instance, the event {1, 2, 3} in the throw of a single die (sample space (*b*)) is actually the union of three events {1}, {2} and {3}. Whereas, the events {1}, {2}, {3}, {4}, {5} and {6} stand by themselves and make up the sample space. They are the 'elementary events' of (*b*). In the case of (*c*), in which we toss two coins, the elemetary events are

HH; *HT*; *TH*; and *TT*.

Now let us make an important assumption in the case of (*a*), (*b*), (*c*) and (*d*): *viz.*, that **the elementary events of these sample spaces are equally likely**. This enables us to assign an equal probability to the elementary events of each of these sample spaces. Thus we assign

$$\frac{1}{\text{Total number of elementary events (in the sample space)}} \qquad (*)$$

to each elementary event. This gives,

in (*a*) : $P(H) = 1/2 = P(T)$

in (*b*) : $P(1) = 1/6 = P(2) = P(3) = P(4) = P(5) = P(6)$

in (*c*) : $P(H, H) = 1/4 = P(H, T) = P(TH) = P(TT)$ and

in (*d*) : $P(i, j) = 1/36$ where $i = 1, 2, \ldots, 6$ and $j = 1, 2, \ldots, 6$.

Now let us discuss the rationale for (*). For instance, why did we choose to give the number 1/2 to the probabilities of the two elementary events {*H*} and {*T*}? Here lies a deeper perception of what is happening. Note that the two probabilities $P(H)$ and $P(T)$ add up to 1. But already $P(S) = 1$. Is there a connection between these two 1's? This connection is what is going to be postulated in the Third Axiom of probability. Note that {*H*} and {*T*} are 'mutually exclusive' events.

AXIOM 3 For any sequence (finite or infinite) of *m.e.* (= mutually exclusive) events $E_1, E_2, \ldots$ that is, events for which the intersection of any two of them is empty,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Thus $P(H) = 1/2$; $P(T) = 1/2$; further since {*H*} and {*T*} are *m.e.*, $P(H \cup T) = P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$ and this confirms with $P(S) = 1$.

As another illustration, take (*b*). Let $E = \{1, 2\}$ and $F = (3, 4, 5, 6\}$. Here $EF = \varphi$. So *E* and *F* are *m.e.* The Axiom 3 then says : $P(E \cup F) = P(E) + P(F)$.

Here $P(E) = P(\{1, 2\}) = P(1) + P(2)$, since {1} and {2} are *m.e.*

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Similarly, $P(F) = P(\{3, 4, 5, 6\}) = P(3) + P(4) + P(5) + P(6)$,

since {3}, {4}, {5} {6} are *m.e.*

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$$

Then $P(E \cup F) = P(E) + P(F) = 1/3 + 2/3 = 1$ and this corroborates with $P(S) = 1$.

Again, probability of throwing an odd number with a single die is

$$= P(\{1, 3, 5\}) = P(1) + P(3) + P(5) = 1/2.$$

Probability of throwing a sum of 5 with 2 dice (*i.e.*, in sample space (*d*))

$$= P(1, 4) + P(2, 3) + P(3, 2) + P(4, 1)$$
$$= 4 \times 1/36 = 1/9.$$

EXAMPLE 1. *Suppose there are 4 red balls and 7 green balls — all identical in size, except for the colour, as stated — in a bag. You are asked to close your eyes and, from the bag,*

(*a*) *if you pick one ball, what is the probability that the ball is red?*

(*b*) *if you pick two balls, in one shot, what is the probability that both are red?*

(*c*) *if you pick two balls one after another, without replacing the ball that has been drawn, what is the probability that* (*i*) *one is red and the other is green?* (*ii*) *the first one is red and the second one is green?*

(*d*) *if you pick two balls one after another, replaing the ball each time after the colour is noted, what is the probability that* (*i*) *one is red and the other is green?* (*ii*) *the first one is red and the second one is green?*

SOLUTION. We shall do each of these by two styles of approach. One is by looking at the total number of events and then taking the proportion of so-called 'favourable' events. The other is by looking at the elementary events in the sample space and then making up the necessary event by a union of elementary events.

First Method (*a*) Picking one ball from 11 balls can happen in 11 ways. Picking a red ball from the 4 balls available can happen in 4 ways. These latter are the favourable events. So the probability of drawing a red ball is 4/11.

(*b*) Picking 2 balls from 11 balls can be done in $\binom{11}{2} = 55$ ways. We assume that each of these ways is equally likely. The number of favourable ways is the number of choices of 2 balls from the 4 red balls. This number is $\binom{4}{2} = 6$.

So the required probability = 6/55.

(*c*) (*i*) Two balls can be drawn one after another (without replacement) from a bag of 11 balls in $11_2 = 11 \times 10 = 110$ ways. Of these, the number favourable is the number of ways of choosing one red and one green from a total of 4 red and 7 green balls - which is 28 + 28 = 56 so the required probability is 56/110 = 28/55.

(*ii*) As above, the total number is 110 ways. Of these, the number that is favourable is 4 × 7 (Red first, Green next) = 28. So the required probability is 28/110 = 14/55.

(*d*) (*i*) Total number of ways in this case = 11 × 11 = 121. Number of ways of drawing one red and the other green is = 4 × 7 + 7 × 4 = 56. So the required probability is = 56/121.

(*ii*) Total number of ways = 121. Number of ways of drawing the first one red and the second one green is = $4 \times 7 = 28$. So required probability = 28/121.

Second Method We use the elementarty events constituting the sample space. We take our elementary events in such a way that their equal likelihood is reasonable.

(*a*) Picking of each particular ball is the elementary event. There are 11 such. They are equally likely. Each of them has probability 1/11. There are four red balls. So prob (red ball) = $4 \times 1/11 = 4/11$.

(*b*) Picking of two balls in one shot is the elementary event. There are $\binom{11}{2} = 55$ such. Each of them has probability 1/55. Of these, six combinations of red balls are there. So the required probability is $6 \times 1/55 = 6/55$.

(*c*) (*i*) Picking two balls, one after another, without replacement, gives the elementary events. So there are $11_2 = 11 \times 10 = 110$ such. Of these, the combination of one red and one green is satisfied by $28 + 28 = 56$. So the probability is $56 \times 1/110 = 28/55$.

(*ii*) The sample space as above has 110 points. Those which represent 'red-first-and-green-next', are $4 \times 7 = 28$. So the required probability is $28/110 = 14/55$.

(*d*) (*i*) The sample space has $11 \times 11 = 121$ points. Of these the 'one-red- one-green' combination belongs to $28 + 28 = 56$. So the probability is 56/121.

(*e*) (*ii*) The sample space has 121 points. Of these those that represent the 'first-red-second-green' combination are 28 in number. Hence the probability is $28 \times 1/121 = 28/121$.

EXAMPLE 2. *A fine arts association stages a play enacted by an amateur drama troupe. It allows five complimentary family passes to the members of the troupe who are 5 men, viz., a, b, c, d and e and 3 women, viz., x, y and z. What is the probability that the persons who get the passes finally include either the four men a, b, c, d or the three women x, y, z or the combination a, b, c, x, y?*

SOLUTION. Let us assume that when the passes are finally distributed, no preference or partiality is shown. Then the sample space contains $\binom{8}{5} = 56$ equally likely ways. Of these the number of ways which include *a, b, c, d* is the number of ways of choosing one person from the remaining four in the troupe. This is $\binom{4}{1} = 4$. In fact, these are *a b c d e, a b c d x, a b c d y, a b c d z*. The number of ways which include the 3 women is $\binom{5}{2} = 10$. These are *xyzab, xyzac, xyzad, xyzae, xyzbc, xyzbd, xyzbe, xyzcd, xyzce, xyzde*. The number of ways which include the combination *abcxy* is just 1. Call these three events *E, F, G*. We note they are *m.e.* So the required probability $P(E \cup F \cup G) = P(E) + P(F) + P(G) = 4 \times 1/56 + 10 \times 1/56 + 1/56 = 15/56$.

We shall now prove a few easy theroems on probability.

Theorem 1. If E^c is the event complementary to E, then $P(E^c) = 1 - P(E)$.

Proof. E and E^c (= not-E) are *m.e.* Therefore $P(E \cup E^c) = P(E) + P(E^c)$ *i.e.*, $1 = P(S) = P(E) + P(E^c)$ which gives $P(E^c) = 1 - P(E)$.

As an illustration, the probability of throwing a sum $\neq 5$ with two dice

$$= 1 - P\text{ (throw of a sum} = 5).$$

But P(throw of a sum = 5)

$$= P(\{1, 4\} \cup \{2, 3\} \cup \{3, 2\} \cup \{4, 1\})$$
$$= 1/36 + 1/36 + 1/36 + 1/36 = 4/36 = 1/9.$$

So $\quad P(\text{sum} \neq 5) = 1 - 1/9 = 8/9.$

Again, Prob (sum ≥ 6)

$$= 1 - \text{Prob (sum} = 2, 3, 4 \text{ or } 5)$$
$$= 1 - (1/36 + 2/36 + 3/36 + 4/36) = 26/36 = 13/18.$$ □

Corollary (to Theorem 1) $P(\varphi)$ is always zero. For axiom 3 gives $P(S) = 1$. But $\varphi = S^c$; so $P(\varphi) = P(S^c) = 1 - P(S)$

$$= 1 - 1 = 0.$$ □

Theorem 2. If $E \subset F$, then $P(E) \leq P(F)$.

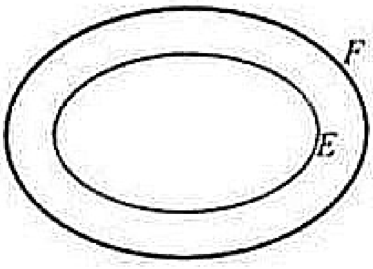

Fig. 13.1

Proof. See Fig. 13.1. Either with the help of the figure or otherwise one can prove that

$$F = E \cup (E^c \cap F) = E \cup E^cF.$$

Since E and E^cF are *m.e.* (= non-overlapping) we have

$$P(F) = P(E \cup E^cF) = P(E) + P(E^cF)$$

This shows that $P(F) \geq P(E)$ since $P(E^cF) \geq 0$. □

As illustrations note that

(*i*) Probability of drawing two kings from a pack of 52 cards is $\geq$ probability of drawing two aces, two kings, two queens and two jacks out of the pack.

The former is, $$\frac{\binom{4}{2}}{\binom{52}{2}}$$

while the latter is $$\frac{\binom{4}{2} \times \binom{4}{2} \times \binom{4}{2} \times \binom{4}{2}}{\binom{52}{8}};$$

(*ii*) In five tosses of an unbiased coin,
prob (head appearing at least twice)
≤ Prob (head appearing at least once) (*)

This is verified because,

Prob (head never appearing)

$= P(T, T, T, T, T) = 1/32$. (Consider all elementary events of the sample space).

Prob (head appearing precisely once)

$$= 5 \times 1/32 = 5/32$$

So Prob ($2H$'s or $3H$'s or $4H$'s or $5H$'s)

$$= 1 - (1/32 + 5/32) = 26/32$$

and Prob ($1H$ or $2H$'s or $3H$'s or $4H$'s or $5H$'s)

$= 1 - 1/32 = 31/32$. Hence the inequality (*)

Theorem 3. For any two events E and F, $P(E \cup F) = P(E) + P(F) - P(EF)$.

Proof. Look at the events as subsets of the sample space. Consult Fig. 13.2.

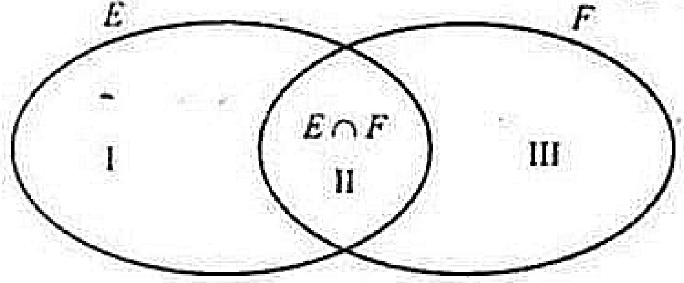

Fig. 13.2

The portions $E - (E \cap F)$, $E \cap F$ and $F - (E \cap F)$ are marked as sets I, II and III. These three are *m.e.* So

$$P(E \cup F) = P(\text{I}) + P(\text{II}) + P(\text{III})$$

But $$P(E) = P(\text{I}) + P(\text{II})$$

and $$P(F) = P(\text{III}) + P(\text{II})$$

So
$$\begin{aligned} P(E) + P(F) &= P(\text{I}) + P(\text{III}) + 2P(\text{II}) \\ &= P(\text{I}) + P(\text{III}) + P(\text{II}) + P(II) \\ &= P(E \cup F) + P(\text{II}) \\ &= P(E \cup F) + P(EF). \end{aligned}$$

So $$P(E \cup F) = P(E) + P(F) - P(EF).$$ ❑

As illustration, Prob (throwing either an even number or a sum of 2, 3, 4, with two dice) may be calculated as follows.

Let $E = \text{event } \{2, 4, 6\}$; $F = \text{event } \{2, 3, 4\}$.

Then $E \cup F = \{2, 3, 4, 6\}$. $P(E \cup F) = 11/36$ (Why ?)

$P(E) = 9/36$; $P(F) = 6/36$; $P(EF) = P(2, 4) = 4/36$.

$P(E) = 9/36$; $P(F) = 6/36$ $1/36 = P(E \cup F)$.

Independent events

Definition. Two events E and F are said to be **independent** if the probability of their simultaneous occurrence is the same as the product of the probabilities of their individual occurrences. In other words, E and F are independent if

$$P(EF) = P(E)\,.\,P(F).$$

Two events are said to be **dependent** if they are not independent.

As illustration, consider the experiment of drawing a card from a pack of 52 cards. Prob (card drawn is a diamond) = 13/52 = 1/4.

Prob (card drawn is a king) = 4/52 = 1/13.

Prob (card drawn is the King of Diamonds) = 1/52

If we call the first two events E and F, the third event is EF; and we have

$$P(EF) = \frac{1}{52} = \frac{1}{4} \times \frac{1}{13} = P(E)\,.\,P(F)$$

So E and F are independent.

Recall the discussion of independent events in Chapter 9. Note also the distinction between mutually exclusive (= *m.e.*) events and independent events. **The former corresponds to non-overlapping sets. The latter has no corresponding concept in set theory.**

In the sample space (d) consisting of the experiment of throwing two dice, let. event E be the throw of 10. This consists of the following points of the sample space

$$\{(4, 6), (5, 5), (6, 4)\}$$

and so has the probability 3/36 = 1/12. Let event F be the event that no die shows 5. The points of the sample space corresponding to this are $\{(x, y);\ x \neq 5,\ y \neq 5\}$. The probability of this event is 1 − 11/36 = 25/36. The simultaneous event $EF = E \cap F$ is $\{(4,6), (6,4)\}$ has probability 2/36 = 1/18. Clearly

$$P(EF) \neq P(E) : P(F).$$

So E and F are dependent events.

Warning. The *m.e.* sets correspond to *m.e.* events, *m.e.* sets are just non-overlapping sets. But one should guard against thinking of non-overlapping sets as corresponding to independent events. In fact, the opposite is true. Whenever there are two independent events, there should be an overlap of at least one point, between them in the sample space. This is proved in the following simple theorem.

Theorem 4. If E and F are independent events with non-zero probabilities, the corresponding sets of the sample space must have at least one common point.

Proof. Let A and B, be the sets of the sample space corresponding to the events E and F, which are such that

$P(EF) = P(E), P(F)$ with $P(E) \neq 0$, $P(F) \neq 0$. If $A \cap B = \varphi$ then $P(E \cap F) = P(\varphi) = 0$. So either $P(E)$ or $P(F)$ is zero. This contradicts the hypothesis that neither of $P(E)$, $P(F)$ is zero. Hence $A \cap B \neq \varphi$. Hence the Theorem.

Illustration. In the dealing of a pack of 52 cards, Prob (dealing an ace) = 4/52 = 1/13 and Prob (dealing a spade or a club) = 26/52 = 1/2. Now if we call these two events E and F, $P(EF)$ = Prob (ace of spades or ace of clubs) = 2/52 = 1/26.

So $P(EF) = P(E)\,.\,P(F)$.This means E and F are independent. The sets corresponding to E and F are

{4 cards of ace}, {all 13 spades and all 13 clubs}

Their intersection is {ace of spades, ace of clubs}.

EXAMPLE 3. *A deck of cards is dealt out. (a) What is the probability that the tenth card dealt is (i) a King (ii) a spade (iii) the king of spades? (b) What is the probability that (i) the first king (ii) the first spade (iii) the king of spades occurs on the tenth card?*

SOLUTION. (*a*) (*i*) There are four kings and there are 52 cards. So the probability of a king occurring is 4/52 = 1/13. This is the probability even for the tenth card, in fact, even for the last card–because, we are not given any information about what has happened in the previous dealings. The problem will be better understood if you keep tossing a coin and ask what is the probability of 'heads' at the 10th toss. It is again 1/2 as it always was.

In the dealing of cards, if on the other hand we are given such and such a thing happened in the first 9 deals, then the probability at the 10th deal might be different. Otherwise it is the same 4/52 (= 1/13) for all the deals!

(*ii*) Similarly the answer for the probability of a spade occuring at the 10th card is 13/52 = 1/4.

(*iii*) There is only one king of spades. So the required probability is 1/52.

(*b*) (*i*) Now we are told that the first 9 cards should not be a king and then the tenth should be a king. The probability of this happening is

$$\left(1-\frac{1}{13}\right)^9 \times \frac{4}{52}.$$

(*ii*) Similar to the above. The answer is

$$\left(1-\frac{1}{4}\right)^9 \times \frac{13}{52}.$$

(*iii*)

$$\left(1-\frac{1}{52}\right)^9 \times \frac{1}{52}.$$

EXAMPLE 4. Once upon a time there was a dictator. An astrologer forecast something bad for him; so the dictator awarded a death penalty to the astrologer. The latter pleaded for his life, so the dictator gave him a chance to save himself and decreed as follows: "I will allow you to put two white balls and two black balls in any manner you like in two urns without disclosing it to anybody. My executioner will choose one of the urns, dip his hand into it and take out a ball. If the ball is black, he will cut off your head. If the ball he picks is white, your life is saved. Try save yourself if you can". What would you advise the astrologer to do, in order to give himself the maximum probability of saving his life?

The different possibilities of distributions of the 4 balls into the two urns are pictorially depicted in Fig. 13.3.

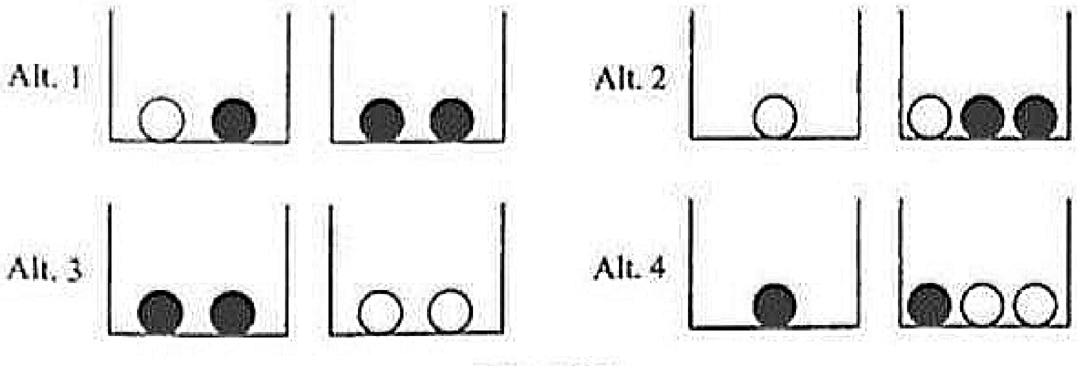

Fig. 13.3

In Alternative 1. Prob (white ball) = Prob (white ball in the first urn) + Prob (white ball in the 2nd urn)

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}.$$

In Alternative 2, it is $= \frac{1}{2} \times 1 + \frac{1}{2} \times 1/3 = 2/3.$

In Alternative 3, it is $= \frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2}.$

In Alternative 4, it is $= \frac{1}{2} \times 0 + \frac{1}{2} \times 2/3 = 1/3.$

Thus in order to maximize his probability of saving his life, the astrologer should be advised to go with alternative 2.

EXAMPLE 5. *Go back to the problem of n people checking in their umbrellas and at the end of the party none of them receiving their own umbrellas (Beginning of Chapter 12). What is the probability that none receives his umbrella?*

SOLUTION. Denoting as we did in the Example cited, the number of derangements of n objects by D_n, the required probability is $D_n/n!$

Important note on $D_n/n!$ This probability $p_n = D_n/n!$ that no person gets back his umbrella behaves in an interesting manner as n becomes large. Actually

$$p_n = \frac{1}{2!} + \frac{1}{3!} +, \ldots + (1)^n \frac{1}{n!}$$

The calculation of the value of p_n for the first few values of n is given below.

Numbere of umbrellas	Probability p_n that no person gets his own umbrella
2	0.500000
3	0.333333
4	0.375000
5	0.366667
6	0.368056
7	0.367857
8	0.367882

It can be proved with the help of higher Mathematics that these probabilities approach a number

$$1/e = .367879...$$

where $e = 2.718281 \ldots$

This number e, like the number π, plays a very major role in several branches of mathematics.

EXAMPLE 6. *Consider the following experiment. From a pack of 52 cards draw a card. If it is a spade throw a six faced die. If it is a club toss a coin. If it is diamonds or hearts, replace it in the pack and draw a card once again. Set up a sample space for this experiment and assign probabilities to each elementary event in the most reasonable way.*

Let spades, hearts, diamonds and clubs be denoted by the small letters s, h, d and c. Let Heads and Tails of tossing the coin be denoted by the capital letters H and T. Let the results of the throw of the single die be denoted by 1, 2, 3, 4, 5 and 6.

Let us assume that the elementary events associated with drawing a card are equally likely; those associated with the toss of the coin are equally likely and those associated with the throw of the die be equally likely.

Now the points of the sample space are:

$$\begin{array}{l} \{\, s1, s2, s3, s4, s5, s6\,; \\ cH, cT\,; \\ ds, dc, dh, dd\,; \\ hs, hc, hh, hd\,\} \end{array}$$

There are 16 of them. First let us note that these 16 are not equally likely. For, we know that $P(s) = 1/4 = P(c) = P(d) = P(h)$. But when the event s (= spades) itself breaks as 6 elementary events, *viz.*,

$$s1, s2, s3, s4, s5, s6$$

it is reasonable to expect

$$P(s1 \cup s2 \cup s3 \cup s4 \cup s5 \cup s6) = 1/4.$$

But $s1$, $s2$ etc., are *m.e.* Therefore

$$P(s1) + P(s2) + P(s3) + P(s4) + P(s5) + P(s6) = 1/4.$$

On the other hand,

$s1, s2, s3, s4, s5, s6$ are equally likely so each of them should have probability 1/24. By the same reason, cH, cT should have each a probability of 1/8 making a total of 1/4 for the probability of c, which is as it should be. And, similarly,

$$P(ds) = 1/6 = P(dc) = P(dh) = P(dd)$$

and $$P(hs) = 1/6 = P(hc) = P(hh) = P(hd)$$

Thus the 16 elementary events have probabilites ;

1/24, 1/24, 1/24, 1/24, 1/24, 1/24 ; 1/8, 1/8 ; 1/16, 1/16, 1/16, 1/16 ; 1/16, 1/16, 1/16, 1/16, totalling 1.

EXAMPLE 7. *A closet contains 12 pairs of shoes. If 8 shoes are randomly selected what is the probability that there will be*

(a) no complete pair, and (b) exactly one complete pair?

SOLUTION. (*a*) The number of elements in the sample space is $\binom{24}{8}$. The event that no complete pair is in the choice happens as follows. Choose 8 pairs out of the 12 pairs. This can be done in $\binom{12}{8}$ ways. For each such pair choose only one of pair, avoiding the other. This can be done 2^8 ways. Thus Prob (no complete pair)

$$= \frac{\binom{12}{8} \times 2^8}{\binom{24}{8}}.$$

(*b*) The sample space again has $\binom{24}{8}$ points. To choose 8 shoes with only one complete pair, first choose the complete pair, in $\binom{12}{1}$ ways.

Keep it aside. From the remaining 11 pairs choose 6 pairs from which you then choose 6 single (unmatching) shoes from each pair. This can be done in $\binom{11}{6} \times 2^6$ ways.

Thus we get the required probability as

$$\frac{12 \times \binom{11}{6} 2^6}{\binom{24}{8}}.$$

EXAMPLE 8. *A pair of dice is thrown until either a 4 or 6 appears. Find the probability that a 6 occurs first.*

SOLUTION. Let E_n denote the event that a 6 occurs in the nth throw and no 4 or 6 occurs in the first $(n-1)$ throws. The probability for this is calculated as follows.

Sample points corresponding to 4 are

$$(1, 3); (2, 2); (3, 1).$$

Sample points corresponding to 6 are

$$(1, 5); (2, 4); (3, 3); (4, 2); (5, 1).$$

So $$P(E_n) = (1 - 8/36)^{n-1}\, 5/36 = \binom{7}{9}^{n-1} \times \frac{5}{36}$$

The probability that 6 occurs first

$$= P(E_1 \cup E_2 \cup E_3 \cup \ldots.)$$
$$= P(E_1) + P(E_2) + P(E_3) + \ldots \text{ since } E_i\text{'s are } m.e.$$
$$= 5/36 + 7/9 \times 5/36 + (7/9)^2 \times 5/36 + \ldots.$$

This is an infinite geometric series with first term = 5/36 and common ratio = 7/9. Therefore, its sum, by the methods of Chapter 15, is

$$= \frac{5/36}{1 - 7/9} = \frac{5}{36} \times \frac{9}{2} = \frac{5}{8}.$$

This is the required probability.

EXAMPLE 9. *From the set of all permutations of* $\{1, 2, 3, \ldots, n\}$ *select a permutation at random, assuming equal likelihood of all permutations. What is the probability that (a) the cycle containing 1 has length k?; (b) 1 and 2 belong to the same cycle?*

SOLUTION. (*a*) Let us count the permutations in which 1 is contained in a cycle of length k.

There are $\binom{n-1}{k-1}$ possible ways of choosing the elements of this cycle.

There are $(k-1)!$ ways of writizng them as a cycle and $(n-k)!$ ways of permuting the rest of the numbers. Thus we get

$$\binom{n-1}{k-1}(k-1)!\ (n-k)! = (n-1)!$$

ways of having 1 in a cycle of length k. So the desired probability is

$$\frac{(n-1)!}{n!} = \frac{1}{n}.$$

Note that **the answer is independent of k**. This is an interesting surprise!

(*b*) Let us count the permutations in which 1 and 2 belong to distinct cycles. If the cycle containing 1 (but not 2) has length k, there are $\binom{n-2}{k-1}$ ways of choosing its elements, $(k-1)!$ ways of writing them as a cycle with 1 and $(n-k)!$ ways of permuting the rest (which includes 2). Summing this product

$$\binom{n-2}{k-1}\ (k-1)!(n-k)!$$

for values of k from $k = 1$ to $k = n - 1$, we get the total number of permutations in which 1 belongs to a cycle distinct from that of 2, as

$$(n-2)! \sum_{k=1}^{n-1} (n-k) = (n-2)! \times \frac{n(n-1)}{2} = \frac{n!}{2}.$$

Note here that the summation we have done uses methods from Chapter 15. Thus the number of permutations in which 1 and 2 belong to the same cycle is $n! - n!/2 = n!/2$. The desired probability is then $n!/2 \div n! = 1/2$.

EXAMPLE 10. *If F is the set of all onto functions from $A = \{a_1, a_2, ..., a_n\}$ to $B = \{x, y, z\}$ and $f \in F$ is chosen randomly what is the probability that*

(*a*) *$f^{-1}(x)$ has 2 elements in it?*

(*b*) *$f^{-1}(x)$ is a singleton?*

SOLUTION. First let us count the functions from A to B which are onto. Consider the three properties:

Range of the function omits x;
Range of the function omits y ; and
Range of the function omits z.

(Recall Example 4 of Sec. 12.1 for a similar strategy in the use of *IEP*).

Number of onto functions
= Number of those functions which have none of the three properties above
= $n(0)$, in the notation of *IEP* of Chapter 12
= Total number of all functions – $n(1) + n(2) - n(3)$.

Now, total number of all functions is 3^n.

Number of functions whose range omits x is 2^n. So $n(1) = 3.2^n$.

Number of functions whose range omits x and y is just 1. So $n(2) = 3$.

Hence, the number of onto functions = $3^n - 3.2^n + 3$.

(*a*) Among these we have now to count how many has 2 elements in $f^{-1}(x)$. Pick those 2 elements in the preimage of x. This can be done in $n(n-1)/2$ ways. Map the remaining $(n-2)$ elements of A to the two elements y, z of B. This has 2^{n-2} possibilities. Of these we have to omit the 2 functions whose range is just $\{y\}$ or $\{z\}$ in order to stay within the population of onto functions. Hence the desired probability is

$$= \frac{\frac{n(n-1)}{1.2}(2^{n-2}-2)}{3^n - 3.2^n + 3}$$

$$= \frac{n(n-1)(2^{n-1}-1)}{3^n - 3.2^n + 3}.$$

(*b*) We shall now count the onto functions which satisfy $f^{-1}(x)$ is a singleton. We can choose this singleton in n ways. The remaining $(n-1)$ elements of A can be mapped onto $\{y, z\}$ in $2^{n-1} - 2$ ways. Thus the desired probability is

$$\frac{n(2^{n-1}-2)}{3^n - 3.2^n + 3}.$$

PROBLEMS

1. Two numbers are selected at random from 1, 2, 3, ... 10. What is the probability that the sum of the two numbers is (*i*) odd? (*ii*) even?
2. A committee of 4 is to be chosen from a group of 16 people. What is the probability that a specified member of the group will be on the committee? What is the probability that this specified member will not be on the committee?
3. Consider the following experiment: Toss a coin. If it falls heads, throw a six-faced die. If it falls tails, toss it again. Set up a sample space for this experiment and assign reasonable values to the probabilities of the elementary events. Assume that the elementary events associated with tossing the coin are equally likely and so also are the elemctary events associated with throwing a die.
4. A committee of seven is to be selected from 10 men and 8 women. What is the probability that (*a*) the committee so formed has a majority of women? (*b*) the committee includes members of both sexes?
5. If a number x between 1 and 200 (both inclusive) is picked at random what is the probability that $gcd\,(x, 6) > 1$?
6. If x is any integer such that $1 \le x \le 100$, what is the probability that x is a prime?
7. If $A = p_n! + 1$ where p_n is the nth prime number, what is the probability that a number picked at random from the sequence

$$A+1, A+2, \dots, A+n$$

is a prime number?

8. In a high school public examination 15% of the students failed in Mathematics and 12% of the students failed in English. Also 3% of the students failed in both Mathematics and English. In an experiment of random selection of students from the school consider the two events; "failure in Maths" and "failure in English". Are these two events independent?
9. In a residential university a survey revealed that 60% of the students read magazines in the regional language, 50% read English magazines, 30% read magazines in a foreign language. Also 30% read ‘magazines in English as well as in the regional language, 20% read English magazines as well as foreign language magazines and 16% read regional language magazines as well as foreign language magazines, while a bare 5% read all three kinds of magazines. If a student is randomly selected from the rolls of the University what is the probability that he does not read any magazine?

10. F is the set of all functions from an n-set A to a 3-set $B = \{a, b, c\}$. In a random experiment of selection of functions from f assume that every $f \in F$ is equally likely. What is the probability that such a function has a in its range?
11. A deck of 52 cards is dealt to four players in a game of Bridge. What is the probability that one of the players receives 13 spades?
12. Do Example 4 with three white balls and three black balls distributed in (*a*) two urns; (*b*) three urns. Generalise to n white balls and n black balls and n urns.
13. Generalise Example 7 to n pairs of shoes from which $2m$ shoes are randomly selected. Assume $n > 2m$.
14. A group of 8 men and 8 women is randomly divided into two groups of size 8 each. What is the probability that both groups will have the same number of women? Generalise by replacing 8 by $2n$.
15. A couple has 2 children.
    (*a*) If the elder one is a girl, what is the probability that the other child is a girl?
    (*b*) If one is a boy, what is the probability that the other is a girl?
16. One picks two cards from a standard 52-card deck. What is the probability that the first card is a spade and the second is not a king?
17. Suppose that each child born to a couple is equally likely to be a boy or a girl irrespective of the sex composition of the other children in the family. For a couple having 6 children in the family, compute the probabilities of the following events:
    (*a*) All children are of the same sex.
    (*b*) The three eldest are boys, and the others are girls.
    (*c*) Exactly 3 are girls.
    (*d*) The three youngest are girls.
    (*e*) The first, third and fifth are girls.
    (*f*) There is at least one boy.

CHA

14

BEGINNINGS OF NUMBER THEORY

14.1 CONGRUENCES

We saw in Chapter 2 that divisibility plays a very important role in the Arithmetic of Integers. In this section we introduce the notion of congruences which enables us to describe divisibility and related properties of **Z** in a compact form, at times leading to some beautiful theorems in the Theory of Numbers. The theory of congruences was first developed scientifically by Gauss (1777-1855) who gave coherence to a miscellaneous aggregation of disconnected special results about numbers of the earlier centuries.

Definition 1. Let $n \neq 0$ be any integer. We say that $a \equiv b \pmod{n}$ (read as *a is congruent to b modulus n*) if n divides $a - b$.

For example $3 \equiv 1 \pmod{2}$, $9 \equiv 0 \pmod{3}$,

$15 \equiv 39 \pmod{4}$, $-4 \equiv -16 \pmod{6}$

$17 \equiv 5 \pmod{-12}$ $-4 \equiv 10 \pmod{7}$.

Note. Congruence modulo n is not actually a new idea, $a \equiv b \pmod{n}$ is the same thing as $n \mid a - b$. It is therefore only a different notation for a particular case of divisibility. But each notation has its advantages.

Congruences are of great practical importance in everyday life. For instance, 'Today is Thursday' is a congruence property (mod 7) of the number of days which have passed since a fixed date. What day will it be 10 days from now if it is Thursday today? In answering such questions we usually throw out multiples of 7 and take only the remainder for calculation. The remainder here is 3. So we count three days from now thus (Thursday-1, Friday-2 and Saturday-3). Thus the answer is Saturday. Expressed mathematically, this is nothing but the recognition and application of the congruence $10 \equiv 3 \pmod{7}$.

Proposition 1. For any integer $n \neq 0$, we have

(1) $a \equiv a \pmod{n}$ for every $a \in \mathbf{Z}$.

(2) $a \equiv b \pmod{n}$ iff $b \equiv a \pmod{n}$ for a, b in $\mathbf{Z}$.

(3) $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ implies that $a \equiv c \pmod{n}$ for any three integers a, b and c

(4) $a \equiv b \pmod{n}$ iff $a \equiv b \pmod{-n}$.

Proof.

(1) For any $a \in \mathbf{Z}$ we have $a - a = 0$ and n divides 0.

$\therefore \quad a \equiv a \pmod{n}$ for every $a \in \mathbf{Z}$.

(2) $a \equiv b \pmod{n}$ implies that $a - b = kn$ for some $k \in \mathbf{Z}$ which means that $b - a = (-k)\,n$; and hence $b \equiv a \pmod{n}$ whenever $a \equiv b \pmod{n}$

(3) $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ implies that there exist integers k and l such that $a - b = kn$ and $b - c = ln$. Therefore, $a - c = (a - b) + (b - c) = (k + l)n$. In other words $a \equiv c \pmod{n}$.

(4) $a \equiv b \pmod{n}$ implies that there exists an integer k such that $a - b = kn$. This means that $a - b = (-k)(-n)$ or $a \equiv b \pmod{-n}$. □

Note (1), (2) and (3) mean that congruences (to the same modulus n) are just like equalities. In other words they are (1) *reflexive*, (2) *symmetric* and (3) *transitive*.

Proposition 2. If $a_1 \equiv a_2 \pmod{n}$ and $b_1 \equiv b_2 \pmod{n}$, then $a_1 \pm b_1 \equiv a_2 \pm b_2 \pmod{n}$. Also we have $a_1 b_1 \equiv a_2 b_2 \pmod{n}$. In other words, congruences (to the same modulus) can be 'added', 'subtracted' and also 'multiplied', provided the modulus is the same in all, to give other congruences.

Proof. If $a_1 \equiv a_2 \pmod{n}$ and $b_1 \equiv b_2 \pmod{n}$ then there exist integers k_1, k_2 such that $a_1 - a_2 = k_1 n$ and $b_1 - b_2 = k_2 n$.

Therefore,

$$(a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2) = (k_1 + k_2)n$$

or

$$a_1 + b_1 \equiv a_2 + b_2 \pmod{n}.$$

$$(a_1 - b_1) - (a_2 - b_2) = (a_1 - a_2) - (b_1 - b_2) = (k_1 - k_2)n$$

or

$$a_1 - b_1 \equiv a_2 - b_2 \pmod{n}.$$

Thus

$$a_1 \pm b_1 \equiv a_2 \pm b_2 \pmod{n}$$

Also

$$a_1 b_1 - a_2 b_2 = (a_1 - a_2)b_1 + a_2(b_1 - b_2) = (k_1 b_1 + k_2 a_2)n.$$

Hence

$$a_1 b_1 \equiv a_2 b_2 \pmod{n}$$ □

Caution. But congruences cannot be divided. For instance,

$$8 \equiv 12 \text{ (mod) } 2$$

and

$$2 \equiv 4 \text{ (mod) } 2.$$

However,

$$(8/2) \not\equiv (12/4) \text{ (mod) } 2.$$

Again,

$$4 \equiv 8 \text{ (mod) } 4;$$

However,

$$2 \not\equiv 4 \text{ (mod) } 4.$$

Thus

$$ka \equiv ka'$$

does not imply

$$a \equiv a'$$

So also

$$a_1 \equiv a_2 \text{ and } b_1 \equiv b_2$$

with $b_1 \neq 0$, $b_2 \neq 0$, do not imply

$$(a_1/b_1) \equiv (a_2/b_2).$$

How much of the division process can be redeemed is shown by propositions 3 and 5 below.

Proposition 3. If $a \equiv b \pmod{n}$ and d is a common divisor of a and b such that $(d, n) = 1$ then $a/d \equiv (b/d) \pmod{n}$.

Proof. $a \equiv b \pmod{n}$ implies that there exists an integer k such that $a - b = kn$. Suppose d is a common divisor of a and b with $a = md$ and $b = ld$. Then we have $a - b = (m - l)\,d = kn$. Therefore $(d, n) = 1$ implies that $m - l$ divides k. This means that $m - l = (k/d).n$ = a multiple of n or $m \equiv l \pmod{n}$. In other words $a/d \equiv b/d \pmod{n}$. □

Proposition 4. Let f be a polynomial with integral coefficients. (*i.e.*, $f(x) \in \mathbf{Z}[x]$). If $a \equiv b \pmod{n}$ then $f(a) \equiv f(b) \pmod{n}$.

Proof. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k$ with $a_i \in Z$.

Then $f(a) = a_0 + a_1a + a_2a^2 + a_3a^3 \ldots + a_ka^k$ and

$f(b) = a_0 + a_1b + a_2b^2 + a_3b^3 \ldots + a_kb^k$.

Now, by Proposition 2, $a_j\, a^j \equiv a_jb^j \pmod{n}$ for $j = 1, 2, 3, \ldots, k$ and $a_0 \equiv a_0 \pmod{n}$. Again using proposition 2, we get

$$a_0 + a_1a + a_2a^2 + \ldots + a_ka^k \equiv a_0 + a_1b + a_2b^2 + \ldots + a_kb^k \pmod{n}.$$

i.e., $f(a) = f(b) \pmod{n}$ □

Proposition 3 has the following generalisation:

Proposition 5. (1) $kx \equiv ky \pmod{n}$ if $x \equiv y \pmod{n/(n, k)}$

(2) $x \equiv y \pmod{n_i}$ for $i = 1, 2$ iff $x = y \pmod{[n_1, n_2]}$

Proof. (1) $kx \equiv ky \pmod{n}$ implies that there exists $m \in \mathbf{Z}$ such that

$$k(x - y) = mn. \text{ Therefore } \frac{k}{(k, n)}(x - y) = \frac{mn}{(k, n)}.$$

This means that $\frac{n}{(k, n)}$ divides $\frac{k}{(k, n)} \cdot (x - y)$.

But $$\left(\frac{n}{(k, n)}, \frac{k}{(k, n)}\right) = 1$$

and hence $\frac{n}{(k, n)}$ must divide (x, y). In other words,

$$x \equiv y \bmod \left(\frac{n}{(k, n)}\right).$$

The converse part is left as an exercise.

(2) $x \equiv y \pmod{n_i}$ $i = 1, 2$ implies that $x - y$ is a multiple of n_1 and n_2. Hence $x - y$ must be a multiple of l.c.m $(n_1, n_2) = [n_1, n_2]$. In other words $x \equiv y \pmod{[n_1, n_2]}$. Conversely, $x \equiv y \pmod{[n_1, n_2]}$ implies that $x - y = P[n_1, n_2]$ for some integer P. Hence $x - y$ is a common multiple of n or $x \equiv y \pmod{n_i}$ $i = 1, 2$. □

EXAMPLE 1. *Find integers x such that* $7x \equiv 4 \pmod{5}$.

SOLUTION. We have $x = 2, 12$ satisfy $7x \equiv 4 \pmod{5}$. In fact, $7x \equiv 4 \pmod{5}$ iff $2x \equiv 4 \pmod{5}$ (since $7 \equiv 2 \pmod{5}$). Now $2x \equiv 4 \pmod{5}$ iff $x \equiv 2 \pmod{5}$ since $(2, 5) = 1$. Hence, the required solution set is $\{\ldots, -8, -3, 2, 7, 12, 17, \ldots\}$.

EXAMPLE 2. *Do there exist integers x such that* $12x \equiv 5 \pmod{8}$?

SOLUTION. All numbers of the form $12x$ are even while numbers of the form 5 (mod 8) are odd. Hence we do not have any solution.

EXAMPLE 3. *If* $|a| < n/2$ *and* $|b| < n/2$ *and* $a \equiv b \pmod{n}$ *then a must be equal to b.*

SOLUTION. Now $|a| < n/2$ and $|b| < n/2$ imply that $-n/2 < a, b < n/2$. Thus a, b both belong to the interval $(-n/2, n/2)$ which is of length n. This means that $|a - b| < n$ and hence the assumption $a \equiv b \pmod{n}$ implies $a - b = 0$.

EXAMPLE 4. *Write a single congruence that is equivalent to the pair of congruences* $x \equiv 1 \pmod{4}$, $x \equiv 2 \pmod{3}$.

SOLUTION. $x \equiv 1 \pmod 4$ iff $x = 4n + 1$ for some $n \in \mathbf{Z}$ and $x \equiv 2 \pmod 3$ iff $x = 3m + 2$ for some $m \in \mathbf{Z}$. We note that $5 \equiv 1 \pmod 4$ and $5 \equiv 2 \pmod 3$. Therefore any x of the form $12k + 5$ satisfies $x \equiv 1 \pmod 4$ and $x \equiv 2 \pmod 3$; also $x \equiv 1 \pmod 4$ and $x \equiv 2 \pmod 3$ implies that $x - 5$ is a common multiple of 3 and 4, or $x \equiv 5 \pmod{12}$ (since $[3, 4] = 12$). Thus the single congruence $x \equiv 5 \pmod{12}$ is equivalent to the system $x \equiv 1 \pmod 4$, $x \equiv 2 \pmod 3$.

EXAMPLE 5. *Give a test as to the divisibility of a number by 7, 11 or 13.*

SOLUTION. Let n be a positive integer, then we may write n as $n = a_0 + a_1(1000) + a_2(1000)^2 + \ldots + a_k(1000)^k$ where $0 \le a_i \le 1000$ for $i = 0, 1, 2, \ldots k$. We note that $1001 = 7.11.13$ and therefore

$$1000 \equiv -1 \pmod 7$$
$$1000 \equiv -1 \pmod{11} \text{ and}$$
$$1000 \equiv -1 \pmod{13}.$$

This gives $n \equiv a_0 + a_2 - \ldots + (-1)^k a_k \pmod n$ for $n = 7, 11$ or 13.

$\therefore$ 7, 11 or 13 divides n iff $a_0 - a_1 + a_2 - \ldots + (-1)^k a_k \equiv 0 \pmod n$ or $0 \pmod{11}$ or $0 \pmod{13}$ respectively.

For example, consider $n = 1278465413$. Then we have

$$\begin{aligned} n &\equiv 413 + 465(1000) + (278)(1000)^2 + 1(1000)^3 \\ &\equiv (413 - 465 + 278 - 1) \pmod 7 \\ &\equiv 0 - 3 + 5 - 1 \equiv 1 \pmod 7 \\ n &\equiv (413 - 465 + 278 - 1) \pmod{11} \\ &\equiv 6 - 3 + 3 - 1 \equiv 5 \pmod{11} \\ n &\equiv (413 - 465 + 278 - 1) \pmod{13} \\ &\equiv 10 - 10 + 5 - 1 \equiv 4 \pmod{13}. \end{aligned}$$

EXAMPLE 6. *Solve* $17x \equiv 1 \pmod{180}$.

SOLUTION. We observe that $180 = 4.5.9$. We search for solutions of the system:

$$17x \equiv 1 \bmod 4,\ 17x \equiv 1 \bmod 5 \text{ and } 17x \equiv 1 \pmod 9.$$

$17x \equiv 1 \pmod 4$ implies that $x \equiv 1 \pmod 4$ as $17 \equiv 1 \pmod 4$ and $(17, 4) = 1$. Similarly $17x \equiv 1 \pmod 5$ is equivalent to $x \equiv 3 \pmod 5$ and $17x \equiv 1 \pmod 9$ is equivalent to $x \equiv 8 \pmod 9$. The system reduces to

$$x \equiv 1 \pmod 4,\ x \equiv 3 \pmod 5 \text{ and } x \equiv 8 \pmod 9.$$

$x \equiv 1 \pmod 4$ implies that $x = 4n + 1$ for some $n \in \mathbf{Z}$. Now $4n + 1 = x \equiv 3 \pmod 5$ implies that $4n \equiv 2 \pmod 5$; which implies that $4n \equiv 12 \pmod 5$ or $n \equiv 3 \pmod 5$. This gives $x = 4(5m + 3) + 1 = 20m + 13$ for some $m \in \mathbf{Z}$. Again, $20m + 13 = x \equiv 8 \pmod 9$ implies that $20m \equiv -5 \pmod 9$. Thus

$$x = 20(9k + 2) + 13 = 180k + 53,\ k \in \mathbf{Z}$$

gives the required solution.

EXERCISE 14.1

1. List all the integers between 100 and 300 which are 11 (mod 17).
2. If P is a prime and $a^2 \equiv b^2 \pmod P$ then prove that $a \equiv b \pmod P$ or $a \equiv -b \pmod P$.
3. *If* $f(x)$ is a polynomial with integral coefficients and if $f(\alpha) \equiv k \pmod n$ then $f(\alpha + nm) \equiv k$ and n for every integer m.

4. If n is a perfect square and $n^2 \equiv k \pmod{10}$ with $0 \le k < 9$, find the possible values of k.
5. If $n = a^4$ where $a \in \mathbf{Z}$ then prove that $n \equiv 0, 1, 5$ or $6 \pmod{10}$.
6. Prove that $4n^2 + 4 \equiv 0 \pmod{19}$ for any n.
7. Solve for n, $5n \equiv 3 \pmod{8}$. 8.
8. Solve for n, $8n = 10 \pmod{30}$.
9. If $x \equiv y \pmod{n}$ then prove that $(x, n) = (y, n)$.

14.2 THE THEOREMS OF FORMAT AND WILSON

We saw in section (14.1) that the notion of 'relatively prime integers' plays a very important role in congruences and related problems. In fact, the number of positive integers less than a given positive integer n and prime to n defines a function $\phi : \mathbf{N} \to \mathbf{N}$, having many interesting and useful properties.

This function is called *Euler's ϕ–function*, as we recall from Example 10 of Section 12.1. In addition we stipulate that $\phi(1) = 1$. From the definition we have $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, $\phi(6) = 2$, $\phi(7) = 6$, $\phi(8) = 4$, $\phi(9) = 6$, $\phi(10) = 4$, $\phi(11) = 10$ and $\phi(12) = 4$. These can be readily checked by enumeration. For instance, the set of positive numbers less than 12 and prime to 12 is $\{1, 5, 7, 11\}$; and hence $\phi(12) = 4$. We observe that a positive integer $P > 1$ is a prime iff $\phi(P) = P - 1$.

Proposition 6. If d is prime to n then in any set $S = \{a, a + d, a + 2d, \ldots, a + (n-1)d\}$ the number of numbers prime to n is $\phi(n)$.

Proof. If $a + kd \equiv a + ld \pmod{n}$ then $(k - l)\,d$ is a multiple of n. But $(d, n) = 1$ and therefore n should divide $(k - l)$. Now for any two elements $a + kd$, $a + ld$ in S we have $0 \le k, l \le (n - 1)$. Therefore $|k - l| < n$; and so n divides $(k - l)$ implies that $k - l = 0$ or $k = l$. In other words all the n numbers in S are mutually incongruent modulo n. This means that $\{a, a + d, a + 2d, \ldots, a + (n-1)\,d\} = \{0, 1, 2, \ldots, (n-1)\}$ modulo n. Further if $a + jd \equiv k \pmod{n}$ with $0 \le k < n - 1$, then $(a + j\,d, n) = 1$ iff $(k, n) = 1$. Hence the number of integers in S which are prime to n is precisely $\phi(n)$. ☐

EXAMPLE 1. Let $S = \{13, 18, 23, 28, 33, 38, 43\}$. Then S has $n = 7$ elements and the common difference d here is given by $d = 5$. We have $(5, 7) = 1$. Also $13 \equiv 6 \pmod{7}$, $18 \equiv 4 \pmod{7}$, $23 \equiv 2 \pmod{7}$, $28 \equiv 0 \pmod{7}$, $33 \equiv 5 \pmod{7}$, $38 \equiv 3 \pmod{7}$ and $43 \equiv 1 \pmod{7}$. The elements of S which are prime to 7 are 13, 18, 23, 33, 38, 43; They are six in number and thus $\phi(7) = 6$.

EXAMPLE 2. Let $S = \{-3, 0, 3, 6, 9, 12, 15, 18, 21, 24\}$. Then S has $n = 10$ elements and the common difference here is $d = 3$. We have $(d, n) = (3, 10) = 1$. the integers in S which are prime to 10 are $-3, 3, 9, 21$; and therefore the number of integers in S prime to 10 is $4 = \phi(10)$.

Proposition 7. If $(m, n) = 1$, then $\phi(mn) = \phi(m)\,\phi(n)$.

Proof. Let $1 \le x \le mn$. Then $(x, mn) = 1$ iff $(x, m) = 1 = (x, n)$ since $(m, n) = 1$. Let us write the mn numbers from 1 to mn as a $(m \times n)$ matrix

$$A = \begin{pmatrix} 1 & m+1 & 2m+1 & 3m+1 & \cdots & (n-1)m+1 \\ 2 & m+2 & 2m+2 & 3m+2 & \cdots & (n-1)m+2 \\ 3 & m+3 & 2m+3 & 3m+3 & \cdots & (n-1)m+3 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ k & m+k & 2m+k & 3m+k & \cdots & (n-1)m+k \\ m & 2m & 3m & 4m & \cdots & nm \\ \cdot & \cdot & \cdot & \cdot & & \cdot \end{pmatrix}$$

We note that if $(k, m) = 1$ then every entry in the kth row is prime to m. Again the kth row has $\phi(n)$ elements which are prime to n (Proposition 6). Thus if we pick all the entries which are prime to both m and n then we have them in the $\phi(m)$ rows namely $k_1, k_2, \ldots, k$ $\phi(m)$-th rows where $(k_i, m) = 1$; each such row contains $\phi(n)$ numbers prime to n. Hence the number of positive integers less than mn and prime to mn is precisely $\phi(m)\,\phi(n)$. Then $\phi(mn) = \phi(m)\,\phi(n)$ whenever $(m, n) = 1$. ❑

Corollary. If $n_1, n_2, \ldots, n_k$ are mutually prime then

$$\phi(n_1 n_2 \ldots n_k) = \phi(n_1)\,\phi(n_2) \ldots \phi(n_k).$$

Proof. Follows immediately by induction. ❑

EXAMPLE 3.

$$\begin{aligned} \phi(2431) &= \phi(11.13.17) = \phi(11)\,\phi(13)\,\phi(17) \\ &= 10.12.16 = 1920. \end{aligned}$$

In the above example, 2431 is a product of distinct primes. Suppose we take $n = 676 = 2^2.13^2$. Then $\phi(n) = \phi(2^2)\,\phi(13^2)$. Now $k < 13^2$ and $(k, 13^2) = 1$ iff k is not a multiple of 13. The number of multiples of 13 which are less than 169 is 12. Hence $\phi(13^2) = 168 - 12 = 169 - 13 = 156$. The same reasoning tells us that $\phi(p^2) = p^2 - p$.

Proposition 8. If p is a prime number, then $\phi(p^k) = p^k(1 - 1/p)$ for $k \in \mathbf{N}$.

Proof. If $k = 1$, then $t\phi(p) = p - 1$ since p is a prime; and the proposition is true for $k = 1$. If $k > 1$, the numbers in $\{1, 2, 3, \ldots, p^k\}$ which are not prime to p^k are precisely p, $2p, 3p, \ldots, (p^k - 1).p$. These are $p^k - 1$ in number.

Therefore $\qquad \phi(p^k) = p^k - p^{k-1} = p^k(1 - 1/p)$ for all $k \in N$ ❑

Proposition 9. *If* $n = p_1^{a_1}\ p_2^{a_2} \ldots p_k^{a_k}$ is the unique prime factorisation of n with $p_1 < p_2 < \ldots < p_k$ then $\phi(n) = n(1 - 1/p_1)(1 - 1/p_2) \ldots (1 - 1/p_k)$.

Proof. For $i \neq j$ we have $p_i^{a_i}\ p_j^{a_j} = 1$.

Therefore,

$$\begin{aligned} \phi(n) &= \phi\left(p_1^{a_1}\right)\phi\left(p_2^{a_2}\right)\ldots\phi\left(p_k^{a_k}\right) \quad \text{(by Corollary to proposition 7)} \\ &= p_1^{a_1}(1 - 1/p_1)\,p_2^{a_2}(1 - 1/p_2) \ldots p_k^{a_k}(1 - 1/p_k) \\ &\qquad \text{(by proposition 8)} \\ &= n(1 - 1/p_1)(1 - 1/p_2) \ldots (1 - 1/p_k). \end{aligned}$$ ❑

Note. Thus we have a second proof, in preposition 9 of the evaluation of Euler's ϕ–function, done already in Example 10 of Section 12.1.

EXAMPLE 8. (*i*) $\phi(24) = \phi(2^3.3) = 24\,(1 - 1/2)\,(1 - 1/3)$

$= 24\,(1/2)\,(2/3) = 8$

(*ii*) $\phi(3072) = \phi(2^{10}.3) = 3072\,(1 - 1/2)\,(1 - 1/3)$

$= 3072\,(1/2)\,(2/3) = 1024.$

EXAMPLE 9. Consider $n = 2^6 = 64$. The divisors of n are 1, 2, 4, 8, 16, 32 and 64. Now $\phi(4) + \phi(2) + \phi(8) + \phi(16) + \phi(32) + \phi(64)$

$$= 1 + (2-1) + (4-2) + (8-4) + (16-8) + (32-16) + (64-32) = 64$$

In other words $$\sum_{d|64} \phi(d) = 64.$$

In general, the divisors of p^k for any prime p are $1, p, p^1, \ldots p^{k-1}$ and p^k.

Therefore $$\sum_{d|p^k} \phi(d) = \phi(1) + \phi(p) + \phi(p^2) + \ldots + \phi(p^k)$$

$$= 1 + (p-1) + (p^2 - p) + \ldots + (p^k - p^{k-1})$$

$$= p^k$$

Example 9 suggests that $\sum\limits_{d|n} \phi(d) = n$ may be true for every positive integer.

In fact it is true and we have the following.

Proposition 10. For any positive integer n we have

$$\sum_{d|n} \phi(d) = n$$

Proof. Any divisor d of the positive integer n with prime factorisation

$n = p_1^{a_1}\ p_2^{a_2}\ p_3^{a_3} \ldots p_k^{a_k}$ is of the form $d = p_1^{b_1}\ p_2^{b_2} \ldots p_k^{b_k}$

where $0 \le b_i \le a_i$ for $i = 1, 2, \ldots, k$. This gives

$$\sum_{d|n} \phi(d) = \sum_{0 \le b_i \le a_i} \phi\left(p_1^{b_1}\ p_2^{b_2} \ldots p_k^{b_k}\right) = \sum_{0 \le b_i \le a_i} \phi\left(p_1^{b_1}\right) \phi\left(p_2^{b_2}\right) \ldots \phi\left(p_k^{b_k}\right)$$

Any typical term of the above summation is a term of the product

$$(1 + \phi(p_1) + \phi(p_1^2) + \ldots \phi\left(p_1^{a_1}\right)\left((1 + \phi(p_2) + \phi(p_2^2) + \ldots + \phi\left(p_2^{(a_2)}\right)\right)$$

$$\times \left(1 + \phi(p_k) + \phi(p_k^2) + \ldots + \phi\left(p_k^{a_k}\right)\right)$$

Therefore $$\sum_{d|n} \phi(d) = p_1^{a_1}\ p_2^{a_2} \ldots p_k^{a_k} = n.$$ ☐

EXAMPLE 10. For any positive integer n we always see that n^5 and n have the same last digit. In other words 10 always divides $n^5 - n$. To prove this we must prove that $5 \mid (n^5 - n)$ and $2 \mid (n^5 - n)$ for all $n \in \mathbb{N}$. We have $n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1)$. If $n \equiv 0$ or $n \equiv \pm 1 \pmod 5$. It is clear that $5 \mid n^5 - n$. If $n \equiv \pm 2 \pmod 5$ then $n^2 \equiv 2^2 + 1 \equiv 0 \pmod 5$ and hence $5 \mid n^5 - n$. Thus 5 always divides $n^5 - n$. It is easily seen that $2 \mid (n^5 - n)$ for all $n \in \mathbb{N}$. Thus $10 \mid (n^5 - n)$ for all $n \in \mathbb{N}$.

In fact, for any prime P we always have $P \mid (n^k - n)$ for all $n \in \mathbb{N}$. This follows from the following theorem due to Fermat.

Theorem 1. (Fermat's Theorem)

If P is a prime and a is any integer prime to P then $a^{P-1} \equiv 1 \pmod p$.

Proof. Let $S = \{a, 2a, 3a, 4a, \ldots, (p-1)a\}$. Then by Proposition 6, each $ka \in S$ is congruent to some $n \in \{1, 2, 3, \ldots, p-1\}$ modulo p. Therefore $a.2a.3a \ldots (p-1)a = a^{p-1}(p-1)! \equiv (p-1)! \pmod p$. Now $(p-1)!$ is prime to p and hence we may cancel it to get $a^{p-1} \equiv 1 \pmod p$. ☐

Corollary. If p is a prime and n is an integer then $n^p - n = 0 \pmod p$.

Proof. If $(n, p) = 1$ then by Fermat's theorem, $n^{p-1} \equiv 1 \pmod{p}$ and hence $n^p \equiv n \pmod{p}$ or $n^p - n \equiv 0 \pmod{p}$. If on the other hand $(n, p) \neq 1$ then $n \equiv 0 \pmod{p}$ as p is a prime, and therefore $n^p - n \equiv 0 \pmod{p}$.

When p is a prime, we have $\phi(p) = p - 1$. Fermat's theorem says that $a^{p-1} \equiv 1 \pmod{p}$ whenever $(a, p) = 1$. If we replace p by a positive integer n and $(p - 1)$ by $\phi(n)$, Fermat's theorem becomes $a \phi(n) \equiv 1 \pmod{n}$ whenever $(a, n) = 1$. This is in fact true. The proof is very much similar to that of Fermat's theorem.

Theorem 2. (Euler's theorem) ☐

If n is any positive integer and a is prime to n then $a \phi(n) = 1 \pmod{n}$.

Proof. Let $1 = a_1 < a_2 < a_3 < \dots < a \phi(n)$ be the positive integers less than and prime to n. Consider $S = \{a, a_1 a, a_2 a, \dots a \phi(n) a\}$. If $aa_j \equiv aa_k \pmod{n}$ then $a_j \equiv a_k \pmod{n}$ since $(a, n) = 1$. This is possible iff $j = k$. Further each $(aa_j, n) = 1$ since a and a_j are prime to n. This implies that

$$S = \{a, a_1 a, a_2 a \dots a^{\phi(n)} a\} = \{a_1, a_2, \dots a^{\phi(n)}\} \pmod{n}$$

$$\therefore \quad a, a_1 a, a_2 a \dots a^{\phi(n)} a \equiv a_1 a_2 a_3 \dots a^{\phi(n)} \pmod{n}$$

$$\text{i.e.,} \quad a^{\phi(n)} a_1 a_2 \dots a^{\phi(n)} \equiv a_1 a_2 a_3 \dots a^{\phi(n)} \pmod{n}$$

$$\text{Now} \quad (a_1, a_2, \dots a^{\phi(n)}, n) = 1 \text{ since } (a_j, n) = 1 \text{ for } j = 1, 2, \dots \phi(n).$$

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

☐

Remark. Fermat's theorem is a special case of Euler's theorem, because whenever p is a prime $\phi(p) = (p - 1)$.

EXAMPLE 11. Let $S = \{2, 3, 4, 9\}$ we see that

$1 \equiv 2.6 \equiv 3.4 \equiv 5.9 \equiv 7.8 \pmod{11}$. Therefore $1.2.3.4.5.6.7.8.9 \equiv 1 \pmod{11}$ and $10! \equiv 10 \pmod{11} \equiv -1 \pmod{11}$ Thus $10! + 1$ is a multiple of 11. If we now replace 11 by a prime $p > 3$ and take $S = \{2, 3, 4, \dots, p - 2\}$, we observe that the $(p - 3)$ elements in S can be paired of such that the product of the elements in each pair is congruent to 1 modulo p. Fix any $a \in S$ and consider $a, 2a, 3a, 4a, \dots (p - 2)a, (p - 1)a$. If $ja \equiv ka \pmod{p}$ then $j \equiv k \pmod{p}$ since $(a, p) = 1$. This is possible only if $j = k$. Thus $\{a, 2a, 3a, \dots, (p - 1)a\} = \{1, 2, 3, \dots, p - 1\}$ modulo p. Therefore, there exists a unique $b \in \{1, 2, 3 \dots, p - 1\}$ such that $ab \equiv 1 \pmod{p}$. But $a \neq 1$ or $p - 1$ implies that $b \neq 1$ or $p - 1$. In other words for every $a \in S = \{2, 3, 4, \dots, p - 2\}$ there exists a unique $b \in S$ such that $ab \equiv 1 \pmod{p}$. This proves our earlier observation that the elements of S can be paired of such that the product of the elements in each pair is congruent to 1 modulo p. This fact leads us to the following theorem.

Theorem 3. (Wilson's Theorem) If p is a prime number, then p divides $(p - 1)! + 1$.

Proof If $p = 2$ or 3 the theorem is readily verified. Assume now that p is a prime bigger than 3. Then by our observation just preceding the theorem we see that $2.3.4.5 \dots (p - 2) \equiv 1 \pmod{p}$ and hence $(p - 1)! \equiv (p - 1) \pmod{p} \equiv -1 \pmod{p}$. ☐

Theorem 4. If $(p - 1)! + 1 \equiv 0 \pmod{p}$ then p is a prime.

Proof This theorem is the converse of Wilson's theorem. Let $(p - 1)! + 1 = np$ for some integer n. If k is a divisor of p different from 1 or p then k divides $(p - 1)!$ as well as np; which means that k divides 1. This is impossible since by our choice $1 < k < p$. Hence p must be a prime. ☐

Theorem 5. If $d = (a, n)$ then $ax \equiv b \pmod{n}$ has a solution iff d divides b, When d divides b we have d mutually incongruent solutions.

Proof. Suppose $ax \equiv b \pmod{n}$ then $ax - b = ny$ for some integer y. This gives $ax + (-n)y = b$. This equation has a solution for x, y in $\mathbf{Z}$ iff $d = (a, n)$ divides b. Therefore $ax \equiv b \pmod{n}$ has a solution iff d divides b. When d divides b let $x = x_0$, $y = y_0$ be a particular solution of $ax + (-n)y = b$, guaranteed by the corollary to Theorem 3 of Chapter 2, Section 2. Also, by Example 12 of Section 2.2, any other solution is of the form $x = x_0 + nl$, $y = y_0 + al/d$ with $l \in \mathbf{Z}$. The solutions x_0, $x_0 + n/d$, $x_0 + 2n/d$, ... $x_0 + (d-1)n/d$ are all mutually incongruent solutions of $ax \equiv b \pmod{n}$. Using Euclid's division algorithm, we see that these are all the mutually incongruent solutions. (The details are left to the reader as a simple exercise). ☐

EXAMPLE 12. *When a particular set of n objects is put into bags each containing three we are left with one object; when put into bags each containing four we are left with two; and when put into bags each containing five we are left with three. Find n.*

SOLUTION. This problem is equivalent to solving the following system of congruences $n \equiv 1 \pmod{3}$, $n \equiv 2 \pmod{4}$, $n \equiv 3 \pmod{5}$. By trial and error one finds that $n = 58$ is a solution. Now, we are faced with the following questions.

1. Do we have a solution for such a system of congruences?
2. How to find all the solutions of such a system, when solution exists?

The answers to these questions are given by the following theorem.

Theorem 6. (Chinese Remainder Theorem) Let $n_1, n_2, n_3, \ldots, n_k$ be k positive integers which are pairwise relatively prime. If $a_1, a_2, \ldots a_k$ are such that $(a_j, n_j) = 1$ for $j = 1, 2, \ldots, k$ then the congruences

$$a_1x \equiv b_1 \pmod{n_1},\ a_2x \equiv b_2 \pmod{n_2} \ldots a_kx \equiv b_k \pmod{n_k}$$

have a common solution which is unique modulo $[n_1, n_2, \ldots n_k]$.

Proof. Consider $a_jx \equiv b_j \pmod{n_j}$. Since $(a_j, n_j) = 1$, we always have a solution for $a_jx \equiv b_j \pmod{n_j}$ whatever be b_j (Theorem 5). Choose a solution C_j for $a_jx \equiv b_j \pmod{n_j}$ for $j = 1, 2, \ldots k$. Then $a_jC_j \equiv b_j \pmod{n_j}$ for $j = 1, 2, \ldots, k$. We have $[n_1, n_2, \ldots, n_k] = n_1n_2 \ldots n_k$ since $n_1, n_2 \ldots, n_k$ are pairwise relatively prime. Call this number M. If $m_j = M/n_j$ we see that $(m_j, n_j) = 1$ (Why?). Solving $m_jx \equiv 1 \pmod{n_j}$ using Theorem 5 we have a unique solution $x \equiv m_j \pmod{n_j}$. This gives $m_jm_j' \equiv 1 \pmod{n_j}$. Take $x_0 = c_1m_1m_1' + c_2m_2m_2' + \ldots + c_km_km_k'$. For $i \neq j$, n_i divides $m_j = n_1n_2 \ldots n_k/n_j$. Therefore

$$\begin{aligned} a_jx_0 &= \sum_{i=1}^{k} a_ic_im_im_i' \\ &\equiv a_ic_jm_jm_j' \pmod{n_j} \\ &\equiv a_jc_j \pmod{n_j} && \text{since } m_jm_j' \equiv 1 \pmod{n_j} \\ &\equiv b_j \pmod{n_j} && \text{for } j = 1, 2, .. k. \end{aligned}$$

Thus x_0 is a common solution to our system of congruences. If x is any other solution of the same system then $x_0 \equiv c_j \equiv x \pmod{n_j}$ (by Theorem 5). This means that $x_0 - x$ is a common multiple of $n_1, n_2 \ldots n_k$ and hence $x_0 - x$ is a multiple of $[n_1, n_2, \ldots, n_k] = M$. Therefore $x \equiv x_0$ (mod) $[n_1, n_2, \ldots, n_k]$. ☐

EXAMPLE 13. *Given that $25x \equiv 2 \pmod{9}$ and $55x \equiv 4 \pmod{7}$ find the general value of x.*

SOLUTION. $25x \equiv 7x \equiv 2 \pmod{9}$

This has $C_1 \equiv 8 \pmod{9}$ as a solution.

Again, $55x \equiv 6x \equiv 4 \pmod 7$ has $C_2 = 3$ as a solution.

We have $m_1 = (9 \times 7)/9 = 7$ and $m_2 = (9 \times 7)/7 = 9$ and $m_1' = 4$, $m_2' = 4$ such that $m_1 m_1' \equiv 1 \pmod 9$ and $m_1 m_2' = 1 \pmod 7$.

Therefore $x_0 = c_1 m_1 m_1' + c_2 m_2 m_2' = 8.7.4 + 3.9.4 = 332$ satisfies $25x_0 = 25.332 \equiv 7.8 \equiv 2 \pmod 9$ and $55x_0 = 55.332 = 6.3 \equiv 4 \pmod 7$.

Now $332 \equiv 17$ (mod $[9, 7] = 63$) and therefore a general solution for the given congruences is

$$x = 17 + 63k \text{ where } k \in Z.$$

EXERCISE 14.2

1. If $\phi(n) \mid (n - 1)$, then prove that there is no prime p such that $p^2 \mid n$.
2. Prove that $\phi(n)$ is even if $n > 2$.
3. If n has k distinct prime factors, then prove that $\phi(n) \geq n\, 2^{-k}$.
4. Find all integers for which $\phi(n) = 12$.
5. Let g.c.d $(m, n) = 1$; $A = \{x \mid 0 \leq x \leq n - 1$ and x is prime to $m\}$ and $B = \{x \mid 0 \leq x \leq n - 1$ and x is prime to $n\}$. If $C = \{na + mb \mid a \in A, b \in B\}$ then prove that C assumes all the values x, $0 \leq x \leq mn - 1$, x is prime to mn, read modulo mn.
6. Use problem 5 to prove that $\phi(mn) = \phi(m)\phi(n)$ if $(m, n) = 1$.
7. Find all m, n such that $\phi(mn) = \phi(n)$.
8. If $A = \{n \in N \mid 10 \text{ divides } \phi(n)\}$ prove that A is infinite.
9. Prove that there are infinitely many n for which $\phi(n)$ is a perfect square.
10. Prove that for any given $n \in N$, $\phi(x) = n$ has only finitely many solutions.
11. Prove that if 5 does not divide $n - 1$, n, $n + 1$ then 5 divides $n^2 + 1$.
12. Find the smallest prime that divides $(p - 1)! + 1$, where p is a prime.
13. Find all n for which $10 \mid (n - 1)! + 1$.
14. Let p be a prime. Then prove that $x^2 \equiv -1 \pmod p$ has solutions if and only if $p = 2$ or $p \equiv 1 \pmod 4$.
15. Prove that $n^7 - n$ is divisible by 42 for all $n \in$ N.
16. Prove that $n^{12} - a^{12}$ is divisible by 91 if n and a are prime to 91.
17. What is the last digit of 3^{1992} in the decimal representation?
18. If n is composite and $n > 4$, prove that $(n - 1)! \equiv 0 \pmod n$.
19. For a prime p, if $x^p \equiv y^p \pmod p$ then prove that $x^p \equiv y^p \pmod{p^2}$.
20. Prove that $(p - 1)! \equiv p - 1 \pmod m$ where $m = 1 + 2 + 3 + \ldots + (p - 1)$ and p is a prime.
21. What is the least positive integer x such that $x \equiv 2 \bmod 3$, $x \equiv 3 \bmod 5$, $x \equiv 2 \bmod 7$?
22. Solve $3x \equiv 11 \bmod 25$, $3x \equiv 11 \bmod 7$, $3x \equiv 11 \bmod 13$.

PROBLEMS

1. Prove that if n is not a prime and $\phi(n) \mid (n - 1)$ then n has at least three distinct prime factors. (use Problem 1 of Ex. 14.2).
2. If n is not a prime and $\phi(n) \mid (n - 1)$ then prove that n has at least four distinct prime factors.
3. If $d(n)$ denotes the number of divisiors of n then prove that $d(n) < 2\sqrt{n}$.

4. n is a *perfect number* if $\sigma(n) = 2n$. ($\sigma(n)$ stands for the sum of the divisors of n). If n is a perfect number prove that $\Sigma\{1/d \mid d \text{ divides } n\} = 2$.
5. Prove that for any given $n \in N$, we can find n_1, n_2 in N such that $d(n_1) + d(n_2) = n$.
6. Prove that $\Pi\ \{d \backslash d \text{ divides } n\} = n^{d(n)/2}$.
7. If $n = p_1^{a_1}\, p_2^{a_2} \dots p_k^{a_k}$ is the prime factorization of n then prove that

$$\sigma(n)\phi(n) = n^2\left(1 - p_1^{-a_1-1}\right)\left(1 - p_2^{-a_2-1}\right)\dots\left(1 - p_k^{-a_k-1}\right)$$

$$\sigma(n)\phi(n) > n^2(1 - 1/p_1^2)\,(1 - 1/p_2^2)\,(1 - 1/p_3^2)\dots(1 - 1/p_k^2).$$

8. Prove that $1 + a + a^2 + a^3 + \dots + a^{\phi(n)-1} = 0 \bmod n$ if $(a,n) = 1$ and $(a - 1, n) = 1$.
9. Solve $x^3 \equiv -1 \bmod 13$.
10. For each prime $p \neq 2$, and $0 \le a \le p - 1$ prove that $\binom{p-1}{a} = (-1)^a \bmod p$.
11. For a prime p, $n < p < 2n$ prove that $\binom{2n}{n} = 0 \bmod p$ but $\binom{2n}{n} \neq 0 \bmod p^2$.
12. If p is a prime different from 2, 5 prove that
    (*i*) p divides infinitely many of the integers 9,99,999. ...
    (*ii*) p divides infinitely many of the integers 1,11,111, ...
13. If $p \equiv 3 \bmod 4$, then prove that $2.4.6.8\dots(p-1) \equiv \pm 1 \bmod p$.
14. Let m be an integer grater than 2. Show that there exists an $a \in \{0, 1, 2, \dots, m-1\}$ such that $x^2 = a \bmod m$ has no solution in $\{1, 2, \dots m\}$.
15. Prove that $(a + b)^p = a^p + b^p \bmod p$ where p is a prime.
16. Prove that there exist $2^n - 2$ numbers that have n digits made up of the digits 1,2 and contain each at least once.
17. Find all positive integers for which $2^n + 1$ is divisible by 3.
18. Prove that $2903^n - 803^n - 464^n + 261^n$ is divisible by 1897 for every $n \in N$.
19. Given integers a,b,c,d with $d \neq 0 \pmod 5$ and m an integer for which $am^3 + bm^2 + cm + d = 0 \bmod 5$, prove that there exists an integer n for which $dn^3 + cn^2 + bn + a = 0$ (mod) 5.
20. If $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime number; find $\phi(n)$.
21. If $a \in N$, show that the number of positive integral solutions of $x_1 + 2x_2 + 3x_3 + \dots + nx_n = a$ is equal to the number of nonnegative integral solutions of $y_1 + 2y_2 + 3y_3 + \dots + ny_n = a - n(n+1)/2$. {A solution of the above means a set $\{x_1, x_2, \dots, x_n\}$ satisfying $x_1 + 2x_2 + 3x_3 + \dots + nx_n = a$ etc.}
22. Determine all three digit numbers n having the property that n is divisible by 11 and $n/11$ is equal to the sum of the squares of the digits of n.
23. Solve $x + y + z = a$; $x^2 + y^2 + z^2 = b^2$; $xy = z^2$ where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z are distinct positive numbers.
24. (*a*) Find all $n \in N$ for which $2^n - 1$ is divisible by 7.
    (*b*) Prove that there is no positive integer n for which $2^n + 1$ is divisible by 7.
25. Let a,b,c be natural numbers such that $a + b + 1$ is a prime greater than $c + 1$. If $k_n = n(n+1)$, prove that $(k_{b+1} - k_n)\,(k_{b+2} - k_a)\dots(k_{b+c} - k_a)$ is divisible by $k_1k_2k_3\dots k_c$.
26. Prove that there are infinitely many $x \in N$ for which $y = n^4 + x$ is not prime for any $n \in N$.
27. Let $A = \{2^k - 3 \mid k = 2, 3, 4, \dots\}$. Prove that A has an infinite subset B in which any two elements are coprime.

28. Let m, n be in N. Prove that $(2m)!\,(2n)!\,/(m!n!(m+n)!)$ is an integer.

29. Prove that $\Sigma\left\{\binom{2n+1}{2k+1}2^{3k} \mid k=0,1,2,\ldots,n\right\}$ is not divisible by 5 for any integer $n \geq 0$.

30. Let a, b, c, d be integers relatively prime to $k = ad - bc$; prove that the pairs of integers (x, y) for which $ax + by = 0 \bmod k$ are identical with those for which $cx + dy = 0 \bmod k$.

31. m and n are natural numbers with $1 \leq m < n$. If the last three digits of 1978^m are equal respectively to the last three digits of 1978^n, find m and n such that $m + n$ has its least value.

32. p and q are in N and $p/q = 1 - 1/2 + 1/3 - 1/4 + \ldots - 1/1318 + 1/1319$. Prove that p is divisible by 1979.

33. Factor the number $5^{1985} - 1$ into a product of three integers each of which is bigger than 5^{100}.

34. There are n boxes each containing some balls in it. Let $m < n$; choose m boxes and put one ball in each of them. Repeat this operation. If $(m,n) = 1$, prove that it is possible that after a finite number of operations, we arrive at a situation in which all boxes contain the same number of balls. If $(m,n) \neq 1$, then there may be an initial distribution for which an equal distribution is not possible to be achieved by the above process.

35. Prove that for any set of n integers there is a subset of them whose sum is divisible by n.

36. Prove that if $2n + 1$ and $3n + 1$ are both perfect squares then $40 \mid n$.

37. Let $n \in N$. Can you find n consecutive integers each of which contains a repeated prime factor?

38. Let A be a set of primes such that x, y are in A implies that $xy + 4$ is also in A. Show that $A = \phi$.

39. Prove that every integer $k > 1$ has a multiple which is $< k^4$ and can be written in the decimal system with at most four different digits.

40. Let n be a composite natural number and p be a divisor of n such that $1 < p < n$. Find the binary representation of the smallest natural number N for which $(1 + 2^p + 2^{n-p})\,N - 1$ is a multiple of 2^n.

41. Find all positive integers n such that $(2^n + 1)/n^2$ is an integer.

42. Prove that for any positive integer n there exist infinitely many pairs (x,y) of integers such that (*i*) g.c.d $(x,y) = 1$ (*ii*) $y|(x^2 + n)$ (*iii*) $x|(y^2 n)$.

43. Find all $(p,q,r) \in N \times N \times N$ such that $1 < p < q < r$ and $(p-1)(q-1)(r-1)$ is a divisor of $pqr - 1$.

44. Prove that $(5^{125} - 1)/(5^{25} - 1)$ is a composite number.

45. Let a_n be the last non-zero digit in the decimal representation of $n!$. Does the sequence of $a_1, a_2, \ldots, a_m, \ldots$ become periodic after a finite number of terms?

CHAPTER

15

FINITE SERIES

15.1 INTRODUCTION

Suppose we are asked to find the sum of the numbers from 1 to 10. One way is to go on adding these numbers one by one. Another way is to single out certain basic properties of these numbers and use them to find the sum more easily. For example, we have $1 + 10 = 11$, $2 + 9 = 11$ and so on. Thus the sum can be written as

$$(1 + 10) + (2 + 9) + (3 + 8) + (4 + 7) + (5 + 6).$$

The sum of the numbers in each grouping is equal to 11, and there are 5 such groups. Thus the sum is $5 \times 11 = 55$. We have used the fundamental property of integers, *viz.*,

$$m + n = (m - 1) + (n + 1)$$

for all integers m and n, and commutativity of addition in **Z**. The same reasoning can be used to find the sum of n consecutive integers. The great Mathematician, Gauss used the above reasoning when he was 10 years old! He took only, one minute to do an addition problem for which his classmates needed one whole hour to complete, and that too, wrongly!

Theorem 1. The sum of the first n consecutive natural numbers is given by

$$\frac{n(n+1)}{2}.$$

Proof. Let us denote sum by S_n:

$$S_n = 1 + 2 + \ldots + n. \tag{1}$$

We can write S_n in the form

$$S_n = n + (n - 1) + \ldots + 1. \tag{2}$$

Adding (1) and (2) and rearranging the terms, we get

$$2S_n = [\,1 + n] + [2 + (n - 1)] + \ldots + [n + 1]. \tag{3}$$

We observe that any term on R.H.S. of (3) is of the form $k + (n + 1 - k)$ which is equal to $n + 1$. There are totally n such terms. Hence the sum $2S_n$ is given by

$$2S_n = n(n + 1).$$

This in turn gives $$S_n = n\,\frac{(n+1)}{2}. \tag{4}$$

REMARK. We can add any n consecutive integers. Let $p + 1, p + 2, \ldots, p + n$ be n consecutive integers. Then

$$(p + 1) + (p + 2) + \ldots + (p + n) = np + (1 + 2 + \ldots + n).$$

$$= np + \frac{n(n+1)}{2} = \frac{n(2p+n+1)}{2}.$$

Theorem 1 gives the sum of all elements in a special subset of **N**; namely, the set $\{1, 2, \ldots, n\}$. We can consider this set as the range of the mapping f from $(1, 2, \ldots, n)$ into **N** defined by $f(k) = k$. Similarly the set $\{1, 1/2, 1/3, \ldots 1/n)$ can be realised as the range of the mapping f from $\{1, 2, \ldots, n\}$ into **Q**, defined by $f(k) = 1/k$, $k = 1, 2, 3, \ldots, n$. Consider the set $\{1, 3, 5, \ldots, 23\}$, the set of all odd integers between 1 and 23. This again can be realised as the range of the mapping $f: \{1, 2, \ldots, 12\} \to \mathbf{N}$, given by $f(k) = 2k - 1$, $k = 1, 2, \ldots, 12$.

Definition. A real valued function u defined on the subset $\{1, 2, \ldots, n\}$ of **N** is called a *finite real sequence*. Here n may be any fixed natural number. We normally denote the function u here by its range $\{u(1), u(2), \ldots, u(n)\}$ and call this set itself a finite sequence. It is convenient to work with the range of u rather than u itself. It is customary to write u_i for $u(i)$. We say u_k is the kth term of the sequence u. Thus we consider $(u_1, u_2, \ldots, u_n)$ itself as a finite sequence.

EXAMPLE 1. Let $u(k) = k^2$, $k = 1, 2, \ldots, n$. Then we get a finite sequence $\{1, 4, 9, \ldots, n^2)$.

EXAMPLE 2. Let $u(k) = 2k$, $k = 1, 2, \ldots, n$. We get the finite sequence $\{2, 4, 6, \ldots, 2n\}$. This is the set of all even integers between 1 and $2n$.

EXAMPLE 3. Let $u(k) = k/(k + 1)$, $k = 1, 2, \ldots, n$. Then we get a finite sequence fractions $\{1/2, 2/3, 3/4, \ldots, n/(n + 1)\}$.

A real valued function u defined on the set of all natural numbers **N** is called an infinite (real) sequence or a (real) *sequence* for short. We denote this by (u_n). For example $u(k) = k$ defines a sequence, the sequence of natural numbers itself. Similarly $u(k) = 2k$ defines a sequence, the sequence of even natural numbers. However, we shall have no occasion to use infinite sequences in this book. Thus, hereafter when we refer to a sequence, we assume that the sequence in question is finite.

We shall now introduce special classes of sequences called progressions. An *Arithmetic Progression* (A.P. for short) is a sequence u defined by

$$u(k) = a + (k - 1)d, \; k = 1, 2, \ldots, n. \tag{5}$$

where a and d are fixed real numbers. Here a is called the *initial term* and d is called the *common difference* of the A.P. (u_k). We observe that, given a and d, (u_k) is completely determined. In fact, given a and d the corresponding A.P. is

$$a, a + d, a + 2d, \ldots, a + (n - 1)\, d.$$

Since $u_k = a + (k - 1)d$, we also observe that

$$u_{k+1} - u_k = d \text{ for all } k \in \{1, 2, \ldots, n\}. \tag{6}$$

Thus starting from $u_1 = a$, the successive terms of the sequence are obtained by adding the common difference d to the corresponding previous terms.

EXAMPLE 4. Let us consider the sequence defined by $u(k) = k$, $k = 1, 2, \ldots, n$. This is an A.P. with $a = 1$, $d = 1$ and $u_k = 1 + (k - 1)1$.

EXAMPLE 5. Let us take $a = 1$ and $d = 2$ in (5). We get

$$u_k = 1 + (k - 1)2 = 2k - 1.$$

Thus the A.P. is the sequence of odd integers between 1 and $2n$.

EXAMPLE 6. *If A, B, C are the angles of a triangle, show that cot* $\frac{A}{2}$, *cot* $\frac{B}{2}$, *cot* $\frac{C}{2}$ *are in A.P. iff sin A, sin B, sin C are in A.P.*

SOLUTION. Recalling formulae from 6.9B we see that the ratios cot $(A/2)$, cot $(B/2)$, cot $(C/2)$ are in A.P.

iff $\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}, \sqrt{\frac{s(s-b)}{(s-c)(s-a)}}, \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$ are in *A.P.*

i.e., iff $s-a, s-b, s-c$ are in *A.P.*,

i.e., iff $-a, -b, -c$ are in *A.P.*

i.e., iff $-2R \sin A, -2R \sin B, -2R \sin C$ are in *A.P.*,

i.e., iff $\sin A, \sin B, \sin C$ are in *A.P.*

EXAMPLE 7. With the usual notations for a triangle ABC, prove that if a, b, c are in *A.P.*, then r_1, r_2, r_3 are in H.P. and conversely. Note that three terms are in H.P if their reciprocals are in A.P.

The ex-radii r_1, r_2, r_3 are in H.P.,

iff $\frac{\Delta}{s-a}, \frac{\Delta}{s-b}, \frac{\Delta}{s-c}$ are in H.P.,

i.e., iff $\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}$ are in H.P.,

i.e., iff $s-a, s-b, s-c$ are in A.P.,

i.e., iff $-a, -b, -c$ are in A.P.

i.e., iff a, b, c are in A.P.

Note. H.P. means Harmonic Progression. See Note under Definition 7 in Section 3.5.

We note that (5) may be written as

$$u(k) = a + d + d + \dots + d \tag{5'}$$

where d is added $(k-1)$ times. Replacing + in (5′) by multiplication and d by r we can define a new sequence $u(k) = a \,.\, r^{k-1}$ for $k = 1, 2, \dots, n$. We define a *Geometric Progression* (G.P.) as a sequence

$$u(k) = ar^{k-1}, k = 1, 2, \dots, n \tag{7}$$

where a and r are fixed real numbers. Thus the above G.P. is given by

$$(a, ar, ar^2, \dots, ar^{n-1}),$$

and this is determined once we know a and r. Again a is called the *initial term* of the progression and r is called the *common ratio* of the progression.

If $a = 0$, then $u(k) = 0$ for all k in (7). Thus the sequence (7) reduces to the constant sequence 0, 0, ..., 0. If $r = 1$, we get the constant sequence (a), we also observe that

$$\frac{u_{k+1}}{u_k} = r \quad \text{for } k = 1, 2, \dots, n-1. \tag{8}$$

Thus any particular term of a G.P. is obtained by multiplying the previous term by r.

EXAMPLE 8. The sequence $(1, r, r^2, \dots, r^{n-1})$ is a G.P. with initial term 1 and common ratio r.

EXAMPLE 9. If we take $a = 1$ and $r = 1/10$, we get the G.P.

$$1, \frac{1}{10}, \frac{1}{10^2} \dots \frac{1}{10^{n-1}}.$$

This can also be written in the form

$$1, 0.1, 0.01, \dots, 0.000\dots01,$$

where the kth term is $0.00\dots01$ with $(k-2)$ zeros after the decimal point.

Our interest in this chapter is not the sequences themselves, but the sums defined by some special sequences. If we have a sequence $(u_1, u_2, \dots, u_n)$, we are interested in computing $u_1 + u_2 + \dots + u_n$. Such a sum is called a **finite series**. Thus a finite series is the sum of a given finite sequence. If the sequence is $(u_1, u_2, \dots, u_n)$, the finite series is denoted by $\sum_{k=1}^{n} u_k$. We consider some special finite series in the next few sections.

EXERCISE 15.1

1. Find the G.P. whose initial term is 1/6, the fifth term is 81/6 and the second term is a positive rational.
2. If ab, b^2 and c^2 are successive terms of an A.P., prove that b, c and $2b - a$ are successive terms of a G.P.
3. Find all sequences which are simultaneously an A.P. and a G.P.
4. Find the G.P. with positive terms having 1 as the initial term and 256 as the ninth term and a positive integer as common ratio.
5. Three positive numbers form a G.P. If the second term is increased by 8, the resulting sequence is an A.P. In turn, if we increase the last term of this A.P. by 64, we get a G.P. Find the progression.
6. Find four numbers forming a G.P. in which the third term is greater than the first by 9 and the second term is greater than the fourth by 18.
7. If we subtract 2, 7, 9 and 5 respectively from the four terms of a G.P., we get an A.P. Find the A.P.
8. Let $(u_1, u_2, \dots, u_{15})$ be an A.P. such that the arithmetic mean of u_1 and u_{15} is 15. If u_7, is given to be 12, find the A.P.
9. If (a^2, b^2, c^2) is in A.P., prove that $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ is in A.P.
10. Given $(u_1, u_2, \dots, u_n)$ is an A.P. and $(k_1, k_2, \dots, k_m)$ is an A.P. of natural numbers with $1 \le k_1 < k_2 < \dots < k_m \le n$, prove that $(u_{k_1}, u_{k_2}, \dots, u_{k_m})$ is also an A.P.
11. Given two terms of an A.P. $(u_1, u_2, \dots, u_{50})$ namely $u_7 = 4.9$ and $u_{17} = 10.9$, find the number of terms of the A.P. each of which is smaller than 20.
12. If a, b, c are three successive terms of an A.P., prove that
$$a^2 + 8bc = (2b + c)^2.$$
13. Let $(u_1, u_2 \dots, u_n)$ be an A.P. having common difference $d > 0$. Suppose $u_n^2 = n^2 d^2$ and u_1 is negative. If $n = 15$, find the A.P.
14. How many three-term A.P's can be obtained from $1, 2, 3, \dots, n$?
15. If a, b, c are in A.P., then so are $\cos A \cot A/2$, $\cos B \cot B/2$, $\cos C \cot C/2$.

16. If the sides of a triangle are in A.P. and the greatest and the smallest angles are θ and ϕ, then $4(1 - \cos\theta).\ (1 - \cos\phi)) = \cos\theta + \cos\phi$.

17. If the sides of a triangle are in A.P. and the greatest angle exceeds the smallest by α, show that the sides are in the ratio $1 - x : 1 : 1 + x$, where $x = \sqrt{(1 - \cos\alpha)\,(7 - \cos\alpha)}$.

15.2 SUM OF AN ARITHMETIC PROGRESSION

An A.P. is defined by an initial term 'a', a common difference d and a positive integer n;

$$u_k = a + (k - 1)d, \qquad k = 1, 2, \ldots, n.$$

(We observe that u_k may be defined for any natural number k, by the above equation, giving rise to an infinite sequence (u_k). In general an arithmetic progression may be defined as an infinite sequence. But we consider here only a finite part of such progressions). We are interested in finding the sum $\sum_{k=1}^{n} u_k$, where $(u_1, u_2, \ldots, u_n)$ is an arithmetic progression. This sum can be elegantly expressed in a formula involving only a, d and n.

Theorem 2. Consider an A.P. defined by an initial term a, a common difference d, and a natural number n;

$$u_k = a + (k - 1)d, \qquad k = 1, 2, \ldots, n. \tag{1}$$

Then the sum of the finite series $\sum_{k=1}^{n} u_k$ is given by.

$$\sum_{k=1}^{n} u_k = n\left\{a + \frac{(n-1)}{2}d\right\}. \tag{2}$$

Proof. Let us write

$$S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \ldots + u_n. \tag{3}$$

If we write the same sum in the reverse order, we get

$$S_n = u_n + u_{n-1} + \ldots + u_1 \tag{4}$$

we can arrange the sum in two ways;

$$S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$$

$$S_n = a + (n - 1)d + (a + (n - 2)d) + \ldots + a$$

$$2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + \ldots + (2a + (n - 1)d). \tag{5}$$

Hence each term in (5) is a constant $2a + (n - 1)d$, and there are n such terms. Adding these n terms we get

$$2S_n = n[2a + (n - 1)d]$$

or

$$S_n = n\left[a + \frac{(n-1)}{2}d\right].$$

□

REMARK. The formula (2) can also be expressed in the following form

$$\sum_{k=1}^{n} u_k = (n/2)\{2a + (n-1)d\}$$
$$= (n/2)\,(\text{first term} + \text{last term})$$
$$= (n/2)\,(u_1 + u_n).$$

The quantity $\frac{(u_1 + u_n)}{2}$ is the arithmetic mean of u_1 and u_n. Thus the sum of an A.P. $(u_1, u_2, \ldots, u_n)$ is also equal to n times the arithmetic mean of its first and the last terms.

EXAMPLE 1. *Sum the finite series*

$$2 + 5 + 8 + 11 + \ldots + 47 + 50.$$

SOLUTION. First we observe that the sequence (2, 5, 8, 11, ..., 47, 50) is an A.P. with common difference 3 and initial value 2. Since

$$u_n = u_1 + (n-1)d,$$

we have $$(n-1) = \frac{(u_n - u_1)}{d} = \frac{50-2}{3} = 16.$$

Thus the given A.P. has $n = 17$ terms. We can use the remark made earlier to get the sum

$$S = \sum_{k=1}^{17} u_k = \frac{17(u_1 + u_{17})}{2} = \frac{17(2+50)}{2} = 442.$$

EXAMPLE 2. *Find the sum of all even integers between 20 and 40 (20 and 40 being included).*

SOLUTION. Since the sequence of even integers is an A.P. with common difference 2, the number of even integers between 20 and 40 is given by

$$n - 1 = \frac{40-20}{2} = 10.$$

Thus $n = 11$. There are 11 even integers between 20 and 40. Their sum is given by

$$S = \frac{n(20+40)}{2} = 11 \times 30 = 330.$$

EXAMPLE 3. *An A.P. has 30 terms, the sum of the first 15 terms is equal to 450 and the sum of the first 20 terms is equal to 800. Find the last term of the A.P.*

SOLUTION. Let us denote by d the common difference of the given A.P. Let a be the initial term of the A.P. Then if we make use of (2) the given conditions imply that

$$15(a + 7d) = 450$$
$$20(a + (19/2)d) = 800.$$

Solving these we get $a = 2$, $d = 4$. Hence the last term of the given A.P. is

$$u_{30} = a + (30-1)d = 2 + (29 \times 4) = 118.$$

EXAMPLE 4. *The 7th term of an A.P. is 10 and the sum of the first 7 terms is equal to 7. It the A.P. has 15 terms, find its sum.*

SOLUTION. Let $(u_1, u_2, \ldots, u_{15})$ be the given A.P. with common difference d. Now we know that $u_7 = 10$ and

$$u_1 + u_2 + \dots + u_7 = 7$$

But $$u_7 = u_1 + 6d$$

and $$u_1 + \dots + u_7 = 7\,\frac{(u_1 + u_7)}{2} = 7\,(u_1 + 3d).$$

Hence we get the following system of equations for u and d;

$$u_1 + 6d = 10;\ 7(u_1 + 3d) = 7.$$

i.e., $$u_1 + 6d = 10;\ u_1 + 3d = 1.$$

$\therefore$ $$u_1 = -8,\ d = 3.$$

Since the A.P. has 15 terms, we have

$$\sum_{k=1}^{15} u_k = 15(u_1 + (14/2)d) = 15(-8 + 21) = 195.$$

One of the important facts we have often used in the preceding examples is that any set of consecutive terms of an A.P. is again an A.P. with the same common difference as that of the given A.P.

EXERCISE 15.2

1. Use induction to prove the formula (2) for the sum S_n of an A.P.
2. Find the sum of an A.P. having 25 terms, given that its initial term is -25 and common difference is 3.
3. An A.P. has common difference 5 and contains 51 terms. If its sum is 1275, find the 25th term of the A.P.
4. An A.P. has 20 terms, its initial term is 20 and the sum is also 20. Write down the A.P.
5. Is the sequence defined by $u(k) = k^2$, $k = 1, 2, \dots, n$, an A.P.?
6. Given that 24, 21, 18, 15, 12, ..., u_n is an A.P. and the sum is zero, find n and u_n.
7. Find the sum of $81297 + 81495 + 81693 + \dots + 100899$.
8. Suppose $(u_1, u_2, \dots, u_n)$ is an A.P., the sum of the first 12 terms is -108 and the sum of the first 24 terms is 72. If $n = 30$, find u_n.
9. If $(u_1, u_2, \dots, u_n)$ and $(v_1, v_2, \dots, v_n)$ are in A.P., show that $(u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$ is also an A.P.
10. Find the sum of all natural numbers with 2 digits.
11. Find the sum of all natural numbers with 3 digits and which are divisible by 3.
12. Find the sum of an A.P., given that its first term is -10, the last term is 20, and the sum of the 3rd, 4th and 6th terms is zero.
13. Suppose $(u_1, u_2, \dots, u_{10})$ is an A.P. having all positive terms and common difference 2. If the product of u_1 and u_{10} is equal to 40, find the A.P.
14. Find the sum of even integers between -150 and 250.
15. Let $(u_1, u_2, \dots, u_n)$ be an A.P. of positive terms with common difference d. Prove that
$$\sum_{k=1}^{n} u_k \ge n\sqrt{u_1^2 + (n-1)du_1}\,.$$
16. Find the sum of all two-digit natural numbers which are not divisible by 2 or 3.

15.3 SUM OF A GEOMETRIC PROGRESSION

In section 15.1, we have introduced another type of sequences, *viz.*, geometric progressions. As in the case of an A.P., we have a finite series associated with a G.P. If $(u_1, u_2, ..., u_n)$ is a G.P., then

$$u_k = u_1 r^{k-1} \tag{1}$$

and we have the series $\sum_{k=1}^{n} u_k$. In this section we find a formula for such a sum.

Theorem 3. Let $(u_1, u_2, ..., u_n)$ be a G.P. with common ratio r. If $r \neq 1$, then the sum of the G.P. is given by

$$S_n = \sum_{k=1}^{n} u_k = \frac{u_1(1-r^n)}{(1-r)}. \tag{2}$$

If $r = 1$, then the sum is obviously, $n\,u_1$.

Proof. We have

$$S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n} u_1 r^{k-1}$$
$$= u_1(1 + r + r^2 + ... + r^{n-1}).$$

Multiplying by r on both sides, we get

$$rS_n = u_1(r + r^2 + ... + r^n)$$
$$= u_1(1 + r + r^2 + ... + r^{n-1}) + u_1 r^n - u_1$$
$$= S_n + u_1(r^n - 1).$$

This implies that

$$S_n(r-1) = u_1(r^n - 1). \tag{3}$$

If $r \neq 1$, we can divide both sides of (3) by $1 - r$ to get

$$S_n = u_1 \frac{(1-r^n)}{(1-r)}.$$

If $r = 1$, then $S_n = u_1 + u_1 + ... + u_1$ (n terms)

$$= nu_1.$$

☐

EXAMPLE 1. *Find the sum of the sequence (1, 2, 4, 8, 16, ..., 1024).*

SOLUTION. The given sequence is a G.P. with common ratio 2. Hence, we can use theorem 1 for finding the sum.

$$1 + 2 + 4 + 8 + ... + 1024 = 1 + 2 + 2^2 + 2^3 + ... + 2^{10}$$
$$= \frac{1(1-2^{11})}{(1-2)} = 2^{11} - 1 = 2047.$$

EXAMPLE 2. *Suppose a G.P. begins with 3, and ends with 96. It has the sum 189. Find the number of terms in the G.P.*

SOLUTION. Let $(u_1, u_2, ..., u_n)$ be the given G.P. Then $u_1 = 3$, $u_n = 96$ and

$$\sum_{k=1}^{n} u_k = 189.$$

Since the sequence is not a constant sequence, the common ratio r is different from 1. Hence

$$\sum_{k=1}^{n} u_k = \frac{u_1(1-r^n)}{(1-r)}.$$

We get $$\frac{3(1-r^n)}{(1-r)} = 189$$

i.e., $$\frac{(1-r^n)}{(1-r)} = 63. \tag{4}$$

But we are also given

$$96 = u_n = u_1 r^{n-1} = 3r^{n-1}$$

so that $$r^{n-1} = 32. \tag{5}$$

Combining (4) and (5), we get

$$\frac{1-32r}{1-r} = 63,$$

which gives $1 - 32r = 63(1-r)$, leading to $r = 2$. But then (5) gives $2^{n-1} = 32$ which means $n = 6$. Hence there are 6 terms in the given G.P. The given G.P. is (3, 6, 12, 24, 48, 96).

We have found the sum of a G.P. of real numbers. We note that the result is true for a G.P. of complex numbers. We use this observation in some of the problems.

EXERCISE 15.3

1. Suppose the first term of a G.P. is 5 and its last term is 3645. If the sum of the first three terms is 65, find the G.P.
2. For the series $1 + 22 + 333 + \ldots + \underbrace{999\ldots9}_{9\text{terms}} + \ldots$, prove that, if S_n is the sum to n terms then $9(S_n - S_{n-1}) = n \times 10^n - 1$.
3. Given that the eighth term of a G.P. is 2.56 and the common ratio is 2, find the sum of the first 16 terms.
4. A sequence (u_n) is defined by $u_1 = 2$ and $u_k = 3\,u_{k-1} + 1$.
   Find the sum $u_1 + u_2 + \ldots + u_n$.
5. The sum of an A.P. with three terms is equal to 21. If we reduce the second term by 1 and increase the last term by 1, we get a G.P. Find these numbers.
6. The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.
7. Find all arithmetic progressions of natural numbers with initial term 3 and whose sum is a three-digit number whose digits form a non constant G.P.
8. Let $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime. Show that the sum of all positive divisors of n is equal to $2n$.
9. Show that for any n, the number $1 + 10^4 + \ldots + 10^{4n}$ is a composite number.
10. For what values of n is the polynomial $1 + x^2 + x^4 + \ldots + x^{2n-2}$ divisible by $1 + x + x^2 + \ldots + x^{n-1}$?

15.4 SOME SPECIAL FINITE SERIES

In sections 15.2 and 15.3, we have found the sums of finite series whose terms are either A.P. or G.P. Suppose we have been asked to find the sum of the squares of first n natural numbers. Obviously, it does not fall into any of the two kinds of series that we have studied in the previous sections. In order to find a method of summing this series, we go back to the problem of finding the sum of the first n natural numbers, (see Chapter 2) We approach the latter problem in different way. For any natural number k, we have the identity

$$(k+1)^2 - k^2 = 2k + 1. \tag{1}$$

Giving values 1, 2, ..., n to k, we get

$$2^2 - 1^2 = 2.1 + 1$$
$$3^2 - 2^2 = 2.2 + 1$$
$$. \quad . \quad . \quad .$$
$$(n+1)^2 - n^2 = 2.n + 1.$$

Adding all these equalities, we get

$$(n+1)^2 - 1^2 = 2(1 + 2 + ... + n) + n.$$

$$\therefore \quad 2(1 + 2 + ... + n) = n^2 + 2n + 1 - 1 - n = n(n+1)$$

$$\therefore \quad (1 + 2 + ... + n) = \frac{n(n+1)}{2}.$$

We can adopt this technique for finding the sum of the squares of the first n natural numbers.

Theorem 4. The sum of the squares of the first n natural numbers is given by

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2}$$

Proof. We begin with the identity

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1 \tag{3}$$

which is valid for any natural number k. Giving the values 1, 2, ..., n for k, we get a system of equalities

$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$
$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$
$$.$$
$$.$$
$$.$$
$$(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$$

Adding all these equalities, we get

$$(n+1)^3 - 1^3 = 3(1^2 + 2^2 + ... + n^2) + 3(1 + 2 + ... + n) + \underbrace{(1 + 1 + ... + 1)}_{n \text{ terms}}$$

Hence,
$$3\left(\sum_{k=1}^{n} k^2\right) = (n+1)^3 - 1 - 3\left(\sum_{k=1}^{n} k\right) - n$$
$$= n^3 + 3n^2 + 3n + 1 - 1 - 3\frac{n(n+1)}{2} - n$$
$$= n^3 + (3/2)n^2 + (n/2)$$

$$\therefore \quad \sum_{k=1}^{n} k^2 = \frac{n}{6}(2n^2 + 3n + 1)$$
$$= \frac{n(n+1)(2n+1)}{6}. \quad \square$$

We have used a fundamental idea in Theorem 1. While summing the n equalities obtained by giving the values 1, 2, ..., n to k, we obtained a sum of the form $(2^3 - 1^3) + (3^3 - 2^3) + ... + ((n+1)^3 - n^3)$. In this sum, the first term in each pair cancels the second term in the successive pair, resulting in an elimination of all but two terms. This idea is called the **telescoping technique**. In many summations, each term of the given series can be written as a difference of two terms ending with the telescoping property; if $\sum_{k=1}^{n} u_k$ is the given series, and if it is possible to write $u_k = v_k - v_{k-1}$, $1 \le k \le n$, then

$$\sum_{k=1}^{n} u_k = (v_1 - v_0) + (v_2 - v_1) + ...+ (v_n - v_{n-1}) = v_n - v_0.$$

In such cases, the sum can be evaluated easily. We will consider these ideas in succeeding examples.

EXAMPLE 1. *Find the sum of the series $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n+1)$.*

SOLUTION. Any term of the series is of the form $k(k+1) = k^2 + k$. Hence the sum is given by

$$\begin{aligned} S_n &= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{2}\left[\frac{(2n+1)}{3} + 1\right] \\ &= \frac{n(n+1)(n+2)}{3}. \end{aligned}$$

We can also directly employ the telescoping technique. We have the identity

$$k^2 + k = 1/3\ \{(k+1)^3 - k^3 - 1\}.$$

Summing over k from 1 to n, we get

$$\begin{aligned} \sum_{k=1}^{n} k(k+1) &= (1/3)\ \{(n+1)^3 - 1^3\} - (1/3)\ \{1 + 1 + ...+ 1\} \\ &= (1/3)\ \{n^3 + 3n^2 + 3n - n\} \\ &= (1/3n\ \{n^2 + 3n + 2\} \\ &= \frac{n(n+1)(n+2)}{3}. \end{aligned}$$

EXAMPLE 2. *Find the sum $1^2 + 3^2 + 5^2 + ... + (2n-1)^2$.*

SOLUTION. We can write the sum in the form

$$\begin{aligned} S_n &= \{1^2 + 2^2 + 3^2 + 4^2 + ...+ (2n-1)^2 + (2n)^2\} - \{2^2 + 4^2 + ...+ (2n)^2\} \\ &= \sum_{k=1}^{2n} k^2 - 4 \sum_{k=1}^{n} k^2 \\ &= \frac{2n(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{6} \\ &= \frac{n(4n^2 - 1)}{3}. \end{aligned}$$

Alternatively, we can also write the sum in the form

$$S_n = \sum_{k=1}^{n} (2k-1)^2 = \sum_{k=1}^{n} (4k^2 - 4k + 1)$$

$$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + n$$

$$= 4\,\frac{n\,(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} + n$$

$$= (n/3)\ \{2(n+1)\,(2n+1) - 6(n+1) + 3\}$$

$$= (n/3)\ \{2(n+1)\,[2n+1-3] + 3\}$$

$$= (n/3)\ \{4(n+1)\,(n-1) + 3\}$$

$$= \frac{n(4n^2-1)}{3}.$$

We can adopt the same "telescoping sum" technique to find the sum of cubes of the first n natural numbers which is given by

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2. \tag{4}$$

(See excercise at the end of this section).

EXAMPLE 3. *Find the sum of the series*

$$1\cdot 2\cdot 4 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 6 + \ldots + n(n+1)\,(n+3).$$

SOLUTION. The general term of the series is $k(k+1)\,(k+3) = k^3 + 4k^2 + 3k$. Hence the sum S_n is given by

$$S_n = \sum_{k=1}^{n} k^3 + 4\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + 4\,\frac{n\,(n+1)\,(2n+1)}{6} + 3\,\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{4(2n+1)}{3} + 3\right]$$

$$= \frac{n(n+1)}{12}\,(3n^2 + 19n + 26\}$$

$$= \frac{n(n+1)\,(n+2)\,(3n+13)}{12}.$$

EXAMPLE 4. *Find the sum of the series*

$$1^2 - 2^3 + 3^2 - 4^2 + \ldots + (-1)^{n+1}\,n^2.$$

SOLUTION. We consider the two cases, n odd and n even. Suppose n is odd so that $n = 2m + 1$ for some m. In this case the sum S_n is given by

$$S_n = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2m+1)^2$$

$$= \{1^2 + 2^2 + 3^2 + 4^2 + \ldots + (2m)^2 + (2m+1)^2\}$$

$$- 2\{2^2 + 4^2 + \ldots + (2m)^2\}$$

$$= \sum_{k=1}^{2m+1} k^2 - 8 \sum_{k=1}^{m} k^2$$

$$= \frac{(2m+1)(2m+2)(4m+3)}{6} - \frac{8m(m+1)(2m+1)}{6}$$

$$\therefore \qquad S_n = \frac{(2m+1)(2m+2)}{2} = \frac{n(n+1)}{2}.$$

If n even, then $n = 2m$ for some m. Hence the sum S_n is given by

$$S_n = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2m-1)^2 - (2m)^2$$

$$= \{1^2 + 2^2 + 3^2 + 4^2 + \ldots + (2m-1)^2 (2m)^2\} - 2\{2^2 + 4^2 + \ldots + (2m)^2\}$$

$$= \sum_{k=1}^{2m} k^2 - 8 \sum_{k=1}^{m} k^2$$

$$= \frac{2m(2m+1)(4m+1)}{6} - \frac{8m(m+1)(2m+1)}{6}$$

$$= -m(2m+1) = \frac{-n(n+1)}{2}.$$

Thus for any n, $\qquad S_n = \dfrac{(-1)^{n+1} n(n+1)}{2}.$

EXAMPLE 5. *Find the sum of the series*

$$\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)(2n+3)}$$

SOLUTION. We observe that

$$\frac{1}{(2k-1)(2k+1)(2k+3)} = \frac{1}{4}\left[\frac{1}{(2k-1)(2k+1)} - \frac{1}{(2k+1)(2k+3)}\right].$$

Hence the sum S_n is given by

$$S_n = \frac{1}{4}\left[\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5}\right] + \frac{1}{4}\left[\frac{1}{3 \cdot 5} - \frac{1}{5 \cdot 7}\right] + \ldots + \frac{1}{4}\left[\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right]$$

$$= \frac{1}{4}\left[\frac{1}{1 \cdot 3} - \frac{1}{(2n+1)(2n+3)}\right].$$

EXERCISE 15.4

Find the sum of the following series to n terms

1. $1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + \ldots$
2. $2 \cdot 1 + 5 \cdot 3 + 8 \cdot 5 + ..$
3. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots$
4. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \ldots$
5. $\dfrac{1}{1 \cdot 2 \cdot 3} + \dfrac{1}{2 \cdot 3 \cdot 4} + \dfrac{1}{3 \cdot 4 \cdot 5} + \ldots$
6. $\dfrac{1}{1 \cdot 5} + \dfrac{1}{3 \cdot 7} + \dfrac{1}{5 \cdot 9} = \ldots$

7. $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{3}{2\cdot 3\cdot 4}+\dfrac{5}{3\cdot 4\cdot 5}+\ldots$

8. $2\cdot 2^0+3\cdot 2^1+4\cdot 2^2+\ldots$

9. $\dfrac{1}{2}+\dfrac{2}{2^2}+\dfrac{3}{2^3}+\ldots$

10. $\dfrac{1}{2}+\dfrac{3}{2^2}+\dfrac{5}{2^3}+\ldots$

11. $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\ldots$

12. $\dfrac{1}{1\cdot 10}+\dfrac{1}{10\cdot 19}+\dfrac{1}{19\cdot 28}+\ldots$

13. $\dfrac{1}{5\cdot 11}+\dfrac{1}{11\cdot 17}+\dfrac{1}{17\cdot 23}+\ldots$

14. $\dfrac{1}{1\cdot 3}+\dfrac{2^2}{3\cdot 5}+\dfrac{3^2}{5\cdot 7}+\ldots$

15. $\dfrac{1}{1\cdot 3\cdot 5}+\dfrac{2}{3\cdot 5\cdot 7}+\dfrac{3}{5\cdot 7\cdot 9}+\ldots$

16. Show that the sum of the cubes of the first n natural numbers is

$$\left[\frac{n(n+1)}{2}\right]^2.$$

Find the sum to n terms of the following series:

17. $2\cdot 3+3\cdot 6+4\cdot 11+\ldots$

18. $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\ldots$

19. $1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\ldots$

20. $\dfrac{1}{2\cdot 5\cdot 8}+\dfrac{1}{5\cdot 8\cdot 11}+\dfrac{1}{8\cdot 11\cdot 14}+\ldots$

21. $\dfrac{1}{1\cdot 2\cdot 3\cdot 4}+\dfrac{1}{2\cdot 3\cdot 4\cdot 5}+\dfrac{1}{3\cdot 4\cdot 5\cdot 6}+\ldots.$

22. $\dfrac{3}{1\cdot 2\cdot 4}+\dfrac{4}{2\cdot 3\cdot 5}+\dfrac{5}{3\cdot 4\cdot 6}+\ldots$

23. Find the sum $11^2+12^2+\ldots+21^2$.

24. Find the sum $11^2-12^2+13^2-\ldots-20^2+21^2$.

25. The *Fibonacci sequence* is defined recursively by

$$F_1=1,\ F_2=1,\ F_n=F_{n-1}+F_{n-2} \text{ for } n\geq 3.$$

Prove the following

(*a*) $F_1+F_2+\ldots+F_n=F_{n+2}-1$.

(*b*) $F_1+F_3+\ldots+F_{2n-1}=F_{2n}$.

(*c*) $F_1^2+F_2^2+\ldots+F_n^2=F_n\,F_{n+1}$.

(*d*) $\dfrac{1}{F_1F_3}+\dfrac{1}{F_2F_4}+\ldots+\dfrac{1}{F_{n-1}F_{n+1}}=1-\dfrac{1}{F_nF_{n+1}}$.

26. Let d_n be the sum

$$d_n=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}.$$

Prove that

$$d_n=1-(1/2)+(1/3)-(1/4)+\ldots+(1/(2n-1))-(1/2^n).$$

27. Find the sum $\displaystyle\sum_{k=1}^{n}\frac{1}{\sqrt{k+1}+\sqrt{k}}$.

28. Using induction prove that

$$\sum_{k=1}^{n}\frac{k}{k^4+k^2+1}=\frac{n(n+1)}{2(n^2+n+1)}.$$

29. If $f(t) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{t}$ prove that

$$\sum_{k=1}^{n} (2k+1)\, f(k) = (n+1)^2 f(n) - \frac{n(n+1)}{2}.$$

30. Sum the series

$$\frac{a_1}{1+a_1} + \frac{a_2}{(1+a_1)(1+a_2)} + \ldots + \frac{a_n}{(1+a_1)(1+a_2)\ldots(1+a_n)}.$$

15.5 SUMMATION OF FINITE TRIGONOMETRICAL SERIES

In this section, we shall find the sum of certain series of simple trigonometric functions of angles which are in A.P.

EXAMPLE 1. *Show that*

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin[\alpha+(n-1)\beta]$$
$$= \sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\frac{n\beta}{2} / \sin(\beta/2).$$

SOLUTION. Let Q denote the left hand side.

Then $2Q\sin(\beta/2) = 2\sin\alpha\sin(\beta/2) + 2\sin(\alpha+\beta)\sin(\beta/2) + 2\sin(\alpha+2\beta)\sin(\beta/2)$
$+ \ldots + 2\sin[\alpha+(n-1)\beta]\sin\beta/2$

$$= (\cos(\alpha-\beta/2) - \cos(\alpha+\beta/2)) + (\cos(\alpha+\beta/2) - \cos(\alpha+3\beta/2)) + (\cos(\alpha+3\beta/2) - \cos(\alpha+5\beta/2)) + \ldots$$
$$+ \cos\left[\alpha + \left(n-\frac{3}{2}\right)\beta\right] - \cos\left[\alpha + \left(n-\frac{1}{2}\right)\beta\right]$$
$$= \cos(\alpha-\beta/2) - \cos\left[\alpha + \left(n-\frac{1}{2}\right)\beta\right]$$
$$= 2\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right).$$

Hence $$Q = \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin(n\beta/2)}{\sin(\beta/2)},$$ which is what is required.

Remark 1. Similarly one proves that

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(n-1)\beta]$$
$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin(n\beta/2)}{\sin(\beta/2)}.$$

EXAMPLE 2. *Sum the series*

$$\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \cos(\alpha+3\beta) + \ldots \text{to } n \text{ terms.}$$

SOLUTION. The given alternating series can be converted into a series with all positive signs. Indeed, the given series is equal to

$$\cos\alpha + \cos(\alpha+\beta+\pi) + \cos(\alpha+2\beta+2\pi) + \cos(\alpha+3\beta+3\pi) + \ldots \text{ to } n \text{ terms}$$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)(\beta+\pi)\right]\sin[n(\pi+\beta)/2]}{\sin[(\pi+\beta)/2]}$$

$$= \frac{\cos[\alpha + ((n-1)/2)\,(\pi+\beta)]\,\sin[n\,(\pi+\beta)/2]}{\cos(\beta/2)}.$$

EXAMPLE 3. *Find the sum to n terms of the following series:*

$$\sin^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha + \dots$$

SOLUTION. We have $\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$.

Hence $\sum_{r=1}^{n} \sin^3 r\alpha = \frac{1}{4}\sum_{r=1}^{n}(3\sin r\alpha - \sin 3r\alpha)$

$$= \frac{1}{4}\frac{3\sin\frac{(n+1)}{2}\alpha\sin\frac{n\alpha}{2}}{\sin(\alpha/2)} - \frac{\sin\frac{3(n+1)\alpha}{2}\sin\frac{3n\alpha}{2}}{\sin 3\alpha/2}$$

EXAMPLE 4. *Let O be any point on the circumference of a circle circumscribing a regular polygon $A_1, A_2, A_3, \dots, A_{2n+1}$ such that O lies on the arc A_1A_{2n+1}. Show that $OA_1 + OA_3 + \dots OA_{2n+1} = OA_2 + OA_4 + \dots + OA_{2n}$.*

SOLUTION. If P is the circle, with radius r, and $\angle OPA_1 = \theta$, then the angles OPA_1, OPA_2. $OPA_3, \dots, OPA_{2n}, OPA_{2n+1}$, are respectively

$$\theta,\ \theta + \frac{2\pi}{2n+1},\ \theta + \frac{4\pi}{2\pi+1}, \dots, \theta + \frac{2(2n-1)\pi}{2n+1},\ \theta + \frac{2(2n)\pi}{2n+1}.$$

Hence the lengths of $OA_1, OA_2, OA_3, \dots$ are respectively

$$2r\sin\frac{\theta}{2},\ 2r\sin\left(\frac{\theta}{2} + \frac{\pi}{2n+1}\right),\ 2r\sin\left(\frac{\theta}{2} + \frac{2\pi}{2n+1}\right), \dots$$

Therefore $(1/2r)\,(OA_1 + OA_3 + OA_5 + \dots + OA_{2n+1})$

$$= \sin\frac{\theta}{2} + \sin\left(\frac{\theta}{2} + \frac{2\pi}{2n+1}\right) + \sin\left(\frac{\theta}{2} + \frac{4\pi}{2n+1}\right) \dots \text{ to } (n+1) \text{ terms}$$

$$= \frac{\sin\left[\frac{\theta}{2} + \frac{n}{2}\cdot\frac{2\pi}{2n+1}\right]\sin\frac{(n+1)\pi}{2n+1}}{\sin\frac{\pi}{2n+1}}.$$

and $(1/2r)\,(OA_2 + OA_4 + OA_6 + \dots + OA_{2n})$

$$= \sin\left(\frac{\theta}{2} + \frac{\pi}{2n+1}\right) + \sin\left(\frac{\theta}{2} + \frac{3\pi}{2n+1}\right) + \sin\left(\frac{\theta}{2} + \frac{5\pi}{2n+1}\right) + \dots \text{ to } n \text{ terms}$$

$$= \frac{\sin\left[\frac{\theta}{2} + \frac{n\pi}{2n+1}\right].\sin\frac{n\pi}{2n+1}}{\sin\frac{\pi}{2n+1}}.$$

Since $\sin\frac{n\pi}{2n+1} = \sin\frac{(n+1)\pi}{2n+1}$, we have the desired inequality.

EXERCISE 15.5

Find the sum of the following series ((1) – (6))

1. $\sin\theta + \sin 3\theta + \sin 5\theta + \ldots$ to n terms.
2. $\sin^2\theta + \sin^2 3\theta + \sin^2 5\theta + \ldots$ to n terms.
3. $\cos^4\theta + \cos^4 3\theta + \cos^4 5\theta + \ldots$ to n terms.
4. $\sin\alpha \sin 3\alpha + \sin 3\alpha \sin 5\alpha + \sin 5\alpha \sin 7\alpha + \ldots$ to n terms.
5. $\sin\alpha \cos 2\alpha + \sin 2\alpha \cos 3\alpha + \sin 3\alpha \cos 4\alpha + \ldots$ to n terms.
6. $\cos^2\theta + \cos^2(\theta + \varphi) + \cos^2(\theta + 2\varphi) + \ldots$ to n terms.
7. Suppose $A_1 A_2 \ldots A_n$ is a regular polygon inscribed in a circle centre O, radius a, and P any point on arc $A_1 A_n$. If perpendiculars $p_1, p_2, \ldots, p_n$ are drawn from P on the sides of the polygon, then prove that

   (a) $\sum_{i=1}^{n} p_i^2 = \frac{3}{2} na^2$

   (b) $\sum_{i=1}^{n} p_i^3 = \frac{5}{2} na^3$

8. Show that

$$\frac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin(2n-1)\alpha}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots + \cos(2n-1)\alpha} = \tan(n\alpha).$$

9. Show that

$$\frac{\sin\theta + \sin 2\theta + \sin 3\theta + \ldots \text{ to } n \text{ terms}}{\cos\theta + \cos 2\theta + \cos 3\theta + \ldots \text{ to } n \text{ terms}} = \tan\left(\frac{n+1}{2}\right)\alpha.$$

10. Show that the sum of the sines (cosines) of n angles in A.P. with common difference equal to an integral multiple of $2\pi/n$ is zero.
11. If $\theta = \frac{2\pi}{17}$, then show that $\cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta$ and

    $\cos 3\theta + \cos 5\theta + \cos 6\theta + \cos 7\theta$ are the roots of $2x^2 + x - 2 = 0$.

12. Sum to n terms:

$$\sin\theta + \sin\frac{n-4}{n-2}\theta + \sin\frac{n-6}{n-2}\theta + \ldots .$$

15.6 SUMMATION INVOLVING BINOMIAL COEFFICIENTS

Recall, from Chapter 9 the Binomial Theorem. We shall start this section by giving a proof by Mathematical Induction of the Theorem.

Theorem 5. (Binomial Theorem for a Positive Integral Index)

For any positive integer n and real (or complex) x,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad (1)$$

Proof. We prove the theorem by induction on n. For $n = 2$, we have $(1+x)^2 = 1 + 2x + x^2$ so that (1) is true. Suppose the expansion (1) is true for some positive integer m;

$$(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k .$$

Then, we have,

$$(1+x)^{m+1} = (1+x)\,(1+x)^m$$

$$= (1+x)\left[\sum_{k=0}^{m}\binom{m}{k}x^k\right] = \sum_{k=0}^{m}\binom{m}{k}x^k + \sum_{k=0}^{m}\binom{m}{k}x^{k+1}.$$

Hence,

$$(1+x)^{m+1} = \binom{m}{0} + \left[\binom{m}{1}+\binom{m}{0}\right]x + \left[\binom{m}{2}+\binom{m}{1}\right]x^2$$

$$+\ldots+\left[\binom{m}{m}+\binom{m}{m-1}\right]x^m + \binom{m}{m}x^{m+1}.$$

But $\binom{m}{k}+\binom{m}{k-1} = \binom{m+1}{k}$ by Example 8. Chapter 9, Section 2.

Similarly, $\binom{m}{0} = 1 = \binom{m+1}{0}$ and $\binom{m}{m} = 1 = \binom{m+1}{m+1}$.

Hence we can write now

$$(1+x)^{m+1} = \binom{m+1}{0} + \binom{m+1}{1}x + \binom{m+1}{2}x^2 + \ldots + \binom{m+1}{m}x^m + \binom{m+1}{m+1}x^{m+1}$$

$$= \sum_{k=0}^{m+1}\binom{m+1}{k}x^k.$$

This is precisely the expansion (1) for $n = m + 1$. Thus, we have proved that whenever (1) is true for $n = m$, it is also true for $n = m + 1$. By induction, (1) is valid for all n in N.

Thus for any a and b and natural number n we have

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$ ❑

It is possible to find an expansion for $(1+x)^\alpha$ for any real number α with the condition that $|x| < 1$. However, it involves an infinite series and brings in questions about its convergence. We shall not pursue it further.

The Binomial Theorem can be used to find the sum of certain finite series involving binomial coefficients $\binom{n}{k}$. Recall Examples 4, 5, 6 of Section 9.3 of Chapter 9.

EXAMPLE 1. *Sum the series*

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2.$$

SOLUTION. We have the identity, $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$.

But we also have $\binom{n}{k} = \binom{n}{n-k}$.

so that the binomial expansion can also be written in the form (Recall the different statements of the Binomial Theorem from Chapter 9.3)

$$(1+x)^n = \binom{n}{n} + \binom{n}{n-1}x + \binom{n}{n-2}x^2 + \binom{n}{0} + \ldots + x^n.$$

We multiply both the expansions for $(1+x)^n$ and collect the coefficients of x^n. We observe that x^n appears in the product when we multiply a term involving x^k in the first expansion with a term involving x^{n-k} in the second expansion. Hence for each k, we get the coefficient $\binom{n}{k}\binom{n}{n-k}$ for x^n in the product of two expansions.

$$(1+x)^n\,(1+x)^n = \left[\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \ldots + \binom{n}{n}\binom{n}{0}\right]x^n$$

$$+ \text{ terms involving other powers of } x.$$

$$\therefore \qquad (1+x)^{2n} = \left[\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2\right]x^n$$

$$+ \text{ terms with other power of } x.$$

But, expanding $(1+x)^{2n}$, we get

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \ldots + \binom{2n}{n}x^n + \ldots + \binom{2n}{2n}x^{2n}.$$

Comparing the coefficients of x^n in two expansions for $(1+x)^{2n}$, we get

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n} \qquad (2)$$

EXAMPLE 2. *Find the sum of the cubes of the first n natural numbers. We have already noted in Section 15.4 that the sum is equal to*

$$\left[\frac{n(n+1)}{2}\right]^2.$$

SOLUTION. Here we prove this using some properties of the binomial coefficients. For any natural number k, we write

$$k^3 = a\binom{k}{3} + a\binom{k}{2} + c\binom{k}{1}. \qquad (3)$$

where a, b and c are constants to be determined. Now (3) can be written as

$$k^3 = a\,\frac{k(k-1)(k-2)}{6} + b\,\frac{k(k-1)}{2} + ck$$

$$= (a/6)k^3 + ((-a/2) + (b/2))k^2 + ((a/3) - (b/2) + c)k.$$

Since (3) is an identity in k comparing the coefficients of various powers of k on both sides, we get

$$a/6 = 1,\ (-a/2) + (b/2) = 0,\ (a/3) - (b/2) + c = 0.$$

Solving for a, b and c, we get

$$a = 6,\ b = 6 \text{ and } c = 1.$$

Thus we get an identity

$$k^3 = 6\binom{k}{3} + 6\binom{k}{2} + \binom{k}{1}. \tag{4}$$

We observe that (4) reduces to

$$1 = \binom{1}{1}$$

$$2^3 = 8 = 6\binom{2}{2} + \binom{2}{1}$$

and (4) is valid for $k \geq 3$. Now, giving the values $k = 1, 2, \ldots, n$ in (4), we get equalities

$$1^3 = \binom{1}{1}$$

$$2^3 = 6\binom{2}{2} + \binom{2}{1}$$

$$3^3 = 6\binom{3}{3} + 6\binom{3}{2} + \binom{n}{1}$$

$$\vdots$$

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1}.$$

Adding these, we get

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = 6\sum_{k=3}^{n}\binom{k}{3} + 6\sum_{k=2}^{n}\binom{k}{2} + \sum_{k=1}^{n}\binom{k}{1}.$$

We now write $\displaystyle\sum_{k=3}^{n}\binom{k}{3} = \binom{3}{3} + \binom{4}{3} + \ldots + \binom{n}{3}$

$$= \binom{3}{0} + \binom{4}{1} + \ldots + \binom{n}{n-3}.$$

However, for any positive integers n and k

$$\binom{n}{0} + \binom{n+1}{1} + \ldots + \binom{n+k}{k} = \binom{n+k+1}{k}. \tag{5}$$

(See exercises at the end of this section)

Hence, we get
$$\sum_{k=3}^{n}\binom{k}{3} = \binom{n+1}{n-3}. \tag{6}$$

Similarly, we can prove that

$$\sum_{k=2}^{n}\binom{k}{2} = \binom{n+1}{n-2}. \tag{7}$$

Moreover
$$\sum_{k=1}^{n}\binom{k}{1} = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\binom{n+1}{n-1}.$$

Thus the desired sum can be reduced to

$$\sum_{k=1}^{n} k^3 = 6\binom{n+1}{n-3} + 6\binom{n+1}{n-2} + \binom{n+1}{n-1}$$

$$= 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2}.$$

A further simplification gives

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2. \tag{8}$$

EXAMPLE 4. *Sum the series*

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{(n+1)}\binom{n}{n}.$$

SOLUTION. We begin with the observation that

$$\binom{n+1}{k+1} = \frac{(n+1)!}{(n-k)!(k+1)!}$$

$$= \frac{n+1}{k+1}\,\frac{n!}{(n-k)!k!} = \frac{n+1}{k+1}\binom{n}{k}$$

for any positive integer $k \le n$. If S_n is the sum of the given series, then

$$S_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\sum_{k=0}^{n} \frac{n+1}{k+1}\binom{n}{k}$$

$$= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1} = \frac{1}{n+1}\left[\sum_{k=0}^{n+1}\binom{n+1}{k} - 1\right]$$

$$= \frac{1}{n+1}[2^{n+1} - 1].$$

EXERCISE 15.6

1. Prove that for any positive integers n and k,

$$\binom{n}{0} + \binom{n+1}{1} + \ldots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

Sum the following series:

2. $1.2\binom{n}{2} + 2.3\binom{n}{3} + \ldots + (n-1)\,n\binom{n}{n}$

3. $\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} + \ldots + (n+1)\binom{n}{n}$

4. $\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \ldots + n^2\binom{n}{n}$

5. $\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \ldots + \binom{n}{n-1}\binom{n}{n}$

6. $\binom{n}{1} - 2^2\binom{n}{2} + 3^2\binom{n}{3} + \ldots + (-1)^{n+1}n^2\binom{n}{n}$

7. $\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\ldots+(-1)^n\frac{1}{n+1}\binom{n}{n}$

8. $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots+n\binom{n}{n}$

9. $\binom{n}{0}-2\binom{n}{1}+3\binom{n}{2}+\ldots+(-1)^n(n+1)\binom{n}{n}$

10. $\binom{n}{2}+2\binom{n}{3}+\ldots+(n-1)\binom{n}{n}$

11. $\binom{n}{0}\binom{n}{r}+\binom{n}{1}\binom{n}{r+1}+\ldots+\binom{n}{n-r}\binom{n}{n}$ for $r<n$

12. $\frac{\binom{n}{1}}{\binom{n}{0}}+2\frac{\binom{n}{2}}{\binom{n}{1}}+\ldots+n\frac{\binom{n}{n}}{\binom{n}{n-1}}$

13. Show that

$$\binom{n}{1}^2+2\binom{n}{2}^2+\ldots+n\binom{n}{n}^2=\frac{(2n-1)!}{((n-1)!)^2}$$

14. Prove that

$$\binom{n}{0}^2-\binom{n}{2}^2+\binom{n}{2}^2+\ldots+(-1)^n\binom{n}{n}^2$$

$$=\begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{n!}{[(n/2!]^2} & \text{if } n \text{ is even} \end{cases}$$

15. Find the sum of the products of $\binom{n}{0},\binom{n}{1},\ldots\binom{n}{n}$ taken two at a time.

16. Find the sum

$$\frac{1}{1!(n-1)!}+\frac{1}{2!(n-2)!}+\ldots+\frac{1}{(n-1)!1!}.$$

17. Find the sum of the fourth powers of the first n natural numbers.

18. If $(1+x+x^2)^n=C_0+C_1x+C_2x^2+\ldots+C_{2n}x^{2n}$

prove that

(*i*) $C_k=C_{2n-k}$ for $0\le k\le 2n$

(*ii*) $C_0^2-C_1^2+C_2^2-\ldots+C_{2n}^2=C_n$

(*iii*) $C_0+C_2+C_4+\ldots+C_{2n}=1+C_1+C_3+\ldots+C_{2n-1}$.

19. Sum the series

$$\binom{n}{1}-\frac{1}{2}\binom{n}{2}+\frac{1}{3}\binom{n}{3}\ldots+(-1)^{n-1}\frac{1}{n}\binom{n}{n}.$$

PROBLEMS

1. For any positive integer n, let $\sigma(n)$ denote the sum of all positive divisors of n. Prove that

$$\sigma(n) = \frac{(p_1^{k_1+1}-1)(p_2^{k_2+1}-1)\ldots(p_m^{k_m+1}-1)}{(p_1-1)(p_2-1)\ldots(p_m-1)}$$

where $n = p_1^{k_1}\, p_2^{k_2} \ldots p_m^{k_m}$ is the prime decomposition of n.

2. For any positive integer n, prove that

$$\frac{n}{(n+1)(2n+1)} < \sum_{k=n+1}^{2n} \frac{1}{k^2} < \frac{n}{(n+1)(2n+1)} + \frac{3n+1}{4n(n+1)(2n+1)}.$$

3. Define a sequence (u_k) by

$$u_0 = 1,\ u_k = 2u_{k-1} + 1 \text{ for } k > 0.$$

Find a closed expression for u_n.

4. Define a sequence (u_k) by

$$u_0 = 0,\ u_k = \left(\frac{k+2}{k}\right) u_{k-1} + (1/k) \ \text{ for } k > 0.$$

Prove that $u_n = \dfrac{n(n+3)}{4}$.

5. Find the sum of all the products taken two at a time of the first n natural numbers.

6. Find the sum of all the products taken two at a time of the numbers 1, 4, 7 ...., $(3n-2)$.

7. Find the sum of the series

$$\frac{n}{1.2.3} + \frac{n-1}{2.3.4} + \ldots + \frac{1}{n(n+1)(n+2)}.$$

8. Prove that

$$\sum_{k=1}^{n-1} k(n-k)\binom{n}{k}^2 = n^2\binom{2n-2}{n}.$$

9. Prove the identity

$$\binom{2n}{n} - \binom{n}{1}\binom{2n-2}{n} + \binom{n}{2}\binom{2n-4}{n} - \ldots = \quad 2^n,$$

10. $$\sum_{k=0}^{n} (-1)^k \binom{m}{n-k}\binom{m}{k} = \begin{cases} (-1)^{n/2}\dbinom{m}{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

11. If $(1 + 2x + 2x^2)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_{2n} x^{2n}$,
prove that

$$C_2 = 4n^2 - 2^{2-n}\, n\left[1 + 3\binom{n-1}{2} + \ldots + (2n-1)\binom{n-1}{n-1}\right]$$

12. If n is a positive multiple of 6, show that

(*a*) $\dbinom{n}{1} - 3\dbinom{n}{3} + 3^2\dbinom{n}{5} - \ldots = 0$

(*b*) $\dbinom{n}{1} - \dfrac{1}{3}\dbinom{n}{3} + \dfrac{1}{3^2}\dbinom{n}{5} - \ldots = 0.$

13. For any integer $k > 1$ and $n \ge 1$, prove that n^k is a sum of n consecutive odd numbers.

14. Prove the identity

$$\cos\theta + \binom{n}{1}\cos 2\theta + \binom{n}{2}\cos 3\theta + \ldots + \binom{n}{n-1}\cos n\theta + \cos(n+1)\theta$$
$$= 2^n \cos^n(\theta/2)\cos\left(\frac{n+2}{2}\theta\right).$$

15. Prove that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} < \frac{7}{4} \text{ for any } n \ge 1.$$

16. Find constant $a_0, a_1, a_2, \ldots, a_{10}$ such that $\cos^{10}\theta = \sum_{k=1}^{10} a_k \cos k\theta$.

17. Prove that

(*a*) $\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \ldots = 2^{n/2}\cos\frac{n\pi}{4}$

(*b*) $\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots = 2^{n/2}\sin n\frac{\pi}{4}$.

18. Prove that

$$\sum_{k=0}^{n}\left[\frac{n-2k}{n}\binom{n}{k}\right]^2 = \frac{2}{n}\binom{2n-2}{n-1}.$$

19. Find the sum

$$\sum_{k=0}^{n}\left(\frac{k}{n} - \alpha\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

20. The cubic $x^3 + ax^2 + bx + c$ has three distinct zeros in G.P. Suppose the reciprocals of these zeros are in A.P. Prove that

$$2b^2 + 3ac = 0.$$

21. Find all arithmetic progressions of natural numbers such that the sum of n terms of A.P. is a perfect square for all natural numbers n.

22. If n is a positive integer, then

$$\sum_{r=1}^{n} \frac{1}{\sin 2^r x} = \cot x - \cot 2^n x.$$

23. Let $a_1, a_2, a_3, \ldots, a_n$ be n real numbers and a real function f be defined by

$f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \ldots + \frac{1}{2^{n-1}}\cos(a_n + x)$, for all real x. If $f(x_1) = 0$ and $f(x_2) = 0$ for some real x_1, x_2, show that $x_2 - x_1 = m\pi$ for some integer m.

24. Find the minimum integral value of n for which

$$\sin x_1 + \sin x_2 + \ldots + \sin x_n = 0,$$

and $$\sin x_1 + 2\sin x_2 + \ldots + n\sin x_n = 0$$

simultaneously hold good for some real $x_1, x_2, \ldots x_n$.

25. Show that

$$\sum_{r=0}^{n} \cos(2r\pi/(2n+1) = 1/2.$$

26. Evaluate

(*a*) $\prod_{r=1}^{n} \sin\,[r\pi/2\,(2n+1)]$,

(*b*) $\prod_{r=1}^{n-1} \sin\,[r\pi/2n]$.

27. If $a_1, a_2, \ldots, a_n$ are positive numbers in A.P. prove that

$$(a_1\,a_n)^{n/2} < a_1\,a_2 \ldots a_n < \left(\frac{a_1 + a_n}{2}\right)^n.$$

CHAPTE

16

De Moivre's Theorem and its Applications

16.1 De MOIVRE'S THEOREM

Recall from Chapter 1 that cos θ + i sin θ is a complex number whose argument is θ. The modulus of this number is $(\cos^2\theta + i\sin^2\theta)^{1/2}$ which is 1. So the number lies on the unit circle $|z| = 1$. The Theorem of De Moivre proved in 1730 A.D. tells us that the powers of a complex number cos θ + i sin θ is again a complex number of the same form namely, cos ϕ + i sin ϕ where ϕ is only a suitable multiple of θ. This leads to the following interesting geometrical interpretation. When a complex number on the unit circle is raised to a power say, the nth power, it is again a number on the same unit circle. De Moivre's Theorem says more than this. It says raising a complex number of modulus 1 to the nth power simply multiplies the argument by n. The manipulatory uses of this fact are many. We shall see some of the easy ones in this chapter.

Theorem 1. If n is an integer, positive or negative,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

and if n is a rational fraction, $\cos n\theta + i\sin n\theta$ is one of the values of

$$(\cos\theta + i\ \sin\theta)^n.$$

Proof. In order to prove the Theorem we first note an elementary Lemma.

Lemma

$$\begin{aligned}&(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta).\end{aligned}$$

Consequently,

$$\begin{aligned}&(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)\\ &= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos\gamma + i\sin\gamma]\\ &= \cos(\alpha+\beta+\gamma) + i\sin(a+\beta+\gamma)\end{aligned}$$

and so on. This can be extended to any number of quantities by induction. Now let us take the main theorem.

Case I. Let n be a positive integer. Consider n quantities $\alpha_1, \alpha_2, ..., \alpha_n$.

By Lemma, we have,

$$\begin{aligned}&(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)\dots(\cos\alpha_n + i\sin\alpha_n)\\ &= \cos(\alpha_1+\alpha_2+\dots+\alpha_n) + i\sin(\alpha_1+\alpha_2+\dots.+\alpha_n).\end{aligned}$$

Putting $\alpha_1 = \alpha_2 = \dots \alpha_n = \theta.$

we get $(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)\dots n$ times

26. Evaluate

(*a*) $\prod_{r=1}^{n} \sin[r\pi/2\,(2n+1)]$.

(*b*) $\prod_{r=1}^{n-1} \sin[r\pi/2n]$.

27. If $a_1, a_2, \ldots, a_n$ are positive numbers in A.P. prove that

$$(a_1\, a_n)^{n/2} < a_1\, a_2 \ldots a_n < \left(\frac{a_1 + a_n}{2}\right)^n.$$

CHAPTER

16

De Moivre's Theorem and its Applications

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Proof In order to prove the Theorem we first note an elementary Lemma.

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$$\begin{aligned}&(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta).\end{aligned}$$

Consequently,
$$\begin{aligned}&(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)\\ &= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)][\cos\gamma + i\sin\gamma]\\ &= \cos(\alpha+\beta+\gamma) + i\sin(a+\beta+\gamma)\end{aligned}$$

and so on. This can be extended to any number of quantities by induction. Now let us take the main theorem.

(Let n be a positive integer. Consider n quantities $\alpha_1, \alpha_2, \ldots, \alpha_n$.

By Lemma, we have,

$$\begin{aligned}&(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)\ldots(\cos\alpha_n + i\sin\alpha_n)\\ &= \cos(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + i\sin(\alpha_1 + \alpha_2 + \ldots. + \alpha_n).\end{aligned}$$

Putting $\alpha_1 = \alpha_2 = \ldots\ \alpha_n = \theta,$

we get $(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)\ \ldots\ n$ times

$$= \cos(\theta + \theta + \ldots n \text{ times}) + i \sin(\theta + \theta + \ldots n \text{ times}).$$

Hence $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Case 2. Let n be a negative integer and equal to $-m$, say, where m is a positive integer.

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m}\\ &= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{(\cos m\theta + i\sin m\theta)} \qquad \text{by Case 1.}\\ &= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}\\ &= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}\\ &= \cos m\theta - i\sin m\theta = \cos(-m\theta) + i\sin(-m\theta)\\ &= \cos n\theta + i\sin n\theta.\end{aligned}$$

Case 3. Let n be a rational fraction and equal to p/q where p and q are integers.

By Cases I and 2, we have,

$$\begin{aligned}&[\cos(p/q)\theta + i\sin(p/q)\theta]^q\\ &= [\cos(q.(p/q)\theta) + i\sin(q.(p/q)\theta)]\\ &= \cos p\theta + i\sin p\theta\\ &= (\cos\theta + i\sin\theta)^p \qquad \text{again, since } p \text{ is an integer.}\end{aligned}$$

Thus, $(\cos(p/q)\theta + i\sin(p/q)\theta)$ is one of the qth roots of

$$(\cos\theta + i\sin\theta)^p$$

that is, one of the value of

$$((\cos\theta + i\sin\theta)^p)^{1/q}$$

i.e., of $$(\cos\theta + i\sin\theta)^{p/q},$$

which means that

$$\cos n\theta + i\sin n\theta \text{ is a value of } (\cos\theta + i\sin\theta)^n. \qquad \square$$

Remark Using the notation of cis θ for $\cos\theta + i\sin\theta$, the result of the theorem can be written as

$$\text{cis } n\theta = \text{one of the values of } (\text{cis } \theta)^n.$$

The result of the lemma will be

$$\text{cis}\alpha_1 . \text{cis}\alpha_2 \ldots \text{cis}\alpha_n = \text{cis}(\alpha_1 + \alpha_2 + \ldots + \alpha_n).$$

Corollary
$$\begin{aligned}\cos\theta - i\sin\theta &= \cos(-\theta) + i\sin(-\theta)\\ &= \text{cis}(-\theta) = [\text{cis }\theta]^{-1}\\ &= \frac{1}{\cos\theta + i\sin\theta}.\end{aligned}$$

Thus the conjugate numbers $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$ are reciprocals of each other. This is also shown by the working:

$$\begin{aligned}&(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)\\ &= \cos^2\theta - (i\sin\theta)^2\\ &= \cos^2\theta + \sin^2\theta = 1.\end{aligned}$$

Remark If $$z = \cos\theta + i\sin^2\theta$$

then $$(1/z) = \cos\theta - i\sin\theta.$$

so that $z + (1/z) = 2\cos\theta$ and $z - (1/z) = 2i\sin\theta$.

De Moivre's theorem will now give

$$z^n = \cos n\theta + i\sin n\theta$$

and $$(1/z)^n = \cos n\theta - i\sin n\theta.$$

so that $$z^n + (1/z)^n = 2\cos n\theta$$

and $$z^n - (1/z)^n = 2i\sin n\theta.$$

These results will be of immense use in what follows.

EXAMPLE 1. *If* $x = cis\,\alpha$, $y = cis\,\beta$, *prove that*

$$(x^m/y^n) - (y^n/x^m) = 2i\sin(m\alpha - n\beta).$$

SOLUTION. We have, by De Moivre's theorem,

$$x^m = \text{cis}\, m\alpha \quad\text{and}\quad y^n = \text{cis}\, n\beta.$$

Therefore

$$\begin{aligned}(x^m/y^n) - (y^n/x^m) &= \frac{\text{cis}\, m\alpha}{\text{cis}\, n\beta} - \frac{\text{cis}\, n\beta}{\text{cis}\, m\alpha}\\ &= \text{cis}\, m\alpha\,.\,\text{cis}\,(-n\beta) - \text{cis}\, n\beta\,.\,\text{cis}\,(-m\alpha)\\ &= \text{cis}(m\alpha - n\beta) - \text{cis}(n\beta - m\alpha)\\ &= \text{cis}\,\theta - \text{cis}\,(-\theta) \quad \text{where } \theta = m\alpha - n\beta\\ &= 2i\sin\theta = 2i\sin(m\alpha - n\beta).\end{aligned}$$

EXAMPLE 2. *Show that the real and imaginary part of* $(1 + xi)^n$, *where* x *is real and* n *is a positive integer, will be equal if*

$$x = \tan((4r + 1)\pi/4n),$$

r being zero or any integer.

SOLUTION. $(1 + ix)^n = \left[\sqrt{(1 + x^2)}\,\text{cis}\,\theta\right]^n$

where $$\cos\theta = \frac{1}{\sqrt{(1 + x^2)}} \text{ and } \sin\theta = \frac{x}{\sqrt{(1 + x^2)}}$$

So $$(1 + xi)^n = (1 + x^2)^{n/2}\,\text{cis}\, n\theta.$$

Hence real part $= (1 + x^2)^{n/2}\cos n\theta$

and imaginary part

$$= (1 + x^2)^{n/2}\sin n\theta = (1 + x^2)^{n/2}\cos((1/2)\pi - n\theta).$$

These are equal if

$$\cos n\theta = \cos((1/2)\pi - n\theta).$$

$$\therefore \quad n\theta = 2r\pi \pm ((1/2)\pi - n\theta).$$

The lower sign gives no value for θ.

$$\therefore \quad n\theta = 2r\pi + (1/2)\pi - n\theta,$$

i.e., $$2n\theta = \frac{(4r + 1)\pi}{2}, \quad \textit{i.e.,} \quad \theta = \frac{(4r + 1)\pi}{4n}$$

$$\therefore \quad x = \tan\theta = \tan\frac{(4r + 1)\pi}{4n}.$$

EXAMPLE 3. *Find the sum of the finite series*

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta \text{ for } 0 < \theta < 2\pi.$$

SOLUTION. Consider the complex number $\cos\theta + i\sin\theta$.

We have $\qquad (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta,$

for any integer k. If we put $z = \cos\theta + i\sin\theta$, then $\cos k\theta$ = Real part of z^k.

Hence we can write the given series in the form

$$\begin{aligned} S_n &= \cos\theta + \cos 2\theta + \ldots + \cos n\theta \\ &= \text{Real part of } (z + z^2 + \ldots z^n) \\ &= \text{Re}\,(1 + z + \ldots z^n - 1). \end{aligned}$$

If $0 < \theta < 2\pi$, then $z \neq 1$. Hence, we get

$$S_n = \text{Re}\left(\frac{z^{n+1} - 1}{z - 1} - 1\right).$$

Thus $\qquad S = \text{Re}\left[\dfrac{\cos(n+1)\theta + i\sin(n+1)\theta - 1}{\cos\theta + i\sin\theta - 1}\right] - 1$

$$= \text{Re}\left[\frac{[(\cos\theta - 1) - i\sin\theta][\cos(n+1)\theta - 1 + i\sin(n+1)\theta]}{[(\cos\theta - 1) + i\sin\theta][(\cos\theta - 1) - i\sin\theta]}\right] - 1$$

$$= \frac{(\cos\theta - 1)(\cos(n+1)\theta - 1) + \sin\theta\sin(n+1)\theta}{(\cos\theta - 1)^2 + \sin^2\theta} - 1$$

$$= \frac{\cos\theta\cos(n+1)\theta + \sin\theta\sin(n+1)\theta - \cos(n+1)\theta - \cos\theta + 1}{(\cos\theta - 1)^2 + \sin^2\theta} - 1.$$

$$= \frac{\cos n\theta - \cos(n+1)\theta - \cos\theta + 1}{2(1 - \cos\theta)} - 1$$

$$= \frac{\cos n\theta - \cos(n+1)\theta}{2(1 - \cos\theta)} - \frac{1}{2}.$$

This can be further simplified to give

$$S_n = \frac{2\sin\left(n + \frac{1}{2}\right)\theta\sin(\theta/2)}{4\sin^2(\theta/2)} - \frac{1}{2} = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{2\sin(\theta/2)} - \frac{1}{2}.$$

EXERCISE 16.1

1. Simplify $\dfrac{(\cos 5\alpha + i\sin 5\alpha)^3(\cos 2\alpha + i\sin 2\alpha)^5}{(\cos 2\alpha - i\sin 2\alpha)^8(\cos 3\alpha + i\sin 3\alpha)^9}$

2. Prove that $-\dfrac{1 + \sin(1/8)\pi + i\cos(1/8)\pi}{1 + \sin(1/8)\pi - i\cos(1/8)\pi} = -1$

3. Find the modulus and amplitude of

$$[(i - (\cos\theta - i\sin\theta))/(1 + \cos\theta - i\sin\theta)]^3.$$

4. If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$, prove that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$$

and

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma).$$

5. Show that

$$\sum_{k=1}^{n} \sin(2k-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$$

6. Let $n = 2m$ where m is an odd integer greater than 1. Let $z = \text{cis}(2\pi)n$. Show that

$$\frac{1}{(1-z)} = 1 + z^2 + z^4 + \ldots + z^{m+1}$$

7. Find the sums

(*a*) $\sin\theta + \sin 2\theta + \ldots + \sin n\theta$

(*b*) $\cos^2\theta + \cos^2 2\theta + \ldots + \cos^2 n\theta$

(*c*) $\sin\theta + a\sin(\theta + \delta) + a^2\sin(\theta + 2\delta) + \ldots + a^{n-1}\sin(\theta + (n-1)\delta)$.

8. If $x + \frac{1}{x} = 2\cos\theta$, prove that

$$x^n + \frac{1}{x^n} = 2\cos n\theta.$$

16.2 n^{th} ROOTS OF A COMPLEX NUMBER

Suppose $z = r \text{ cis } \theta$. Let an n^{th} root of this be $\rho \text{ cis } \phi$.

Then $(\rho \text{ cis } \phi)^n = r \text{ cis } \theta$. *i.e.*, $\rho^n \text{ cis } n\phi = r \text{ cis } \theta$.

Hence $\rho^n = r$, *i.e.*, $\rho = r^{1/n}$ and $n\phi = \theta + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

So $$\phi = \frac{\theta + 2k\pi}{n}$$

Thus the n^{th} roots of $r \text{ cis } \theta$ are

$$r^{1/n} \text{ cis } \frac{\theta + 2k\pi}{n}, \qquad k \text{ being zero or any integer.}$$

Actually there are only n n^{th} roots of z, the others being repetitions. We shall illustrate this by taking specific examples.

EXAMPLE 1. *Find all the 5th roots of 1 + i.*

SOLUTION. Now $1 + i = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$

$$= \sqrt{2}\ \text{cis}(\pi/4).$$

$$(1+i)^{1/5} = 2^{1/10}\text{cis}((\pi/4)/5 + (2k\pi/5))$$

$$= 2^{1/10}\text{cis}((\pi/20) + (2k\pi/5)), \quad k = 0, 1, 2, 3, 4.$$

$$(1+i)^{1/5} = 2^{1/10}\text{cis}((\pi/20) + (2k\pi/5)), \quad k = 0, 1, 2, 3, 4.$$

Writing these in detail, we have the roots as

$$2^{1/10}\text{cis}(\pi/20) = \alpha_0,\ 2^{1/10}\text{cis}((\pi/20) + (2\pi/5)) = \alpha_1$$

$$2^{1/10}\text{cis}((\pi/20) + (4\pi/5)) = \alpha_2,\ 2^{1/10}\text{cis}((\pi/20) + (6\pi/5)) = \alpha_3$$

$$2^{1/10}\text{cis}((\pi/20) + (8\pi/5)) = \alpha_4.$$

The fact that these are the only roots is shown by continuing with the substitution of the values $k = 5, 6$, etc., in (*) we see that

$$2^{1/10}\,\text{cis}((\pi/20) + (10\pi/5)) = 2^{1/10}\,\text{cis}(\pi/20) = \alpha_0$$
$$2^{1/10}\,\text{cis}((\pi/20) + (12\pi/5)) = 2^{1/10}\,\text{cis}((\pi/20) + (2\pi/5)) = \alpha_1$$

and so on. Thus we don't get any new roots, if we go beyond $k = 4$. Nor do we get any new roots if we substitute $k = -1, -2, -3 \ldots$.

EXAMPLE 2. *Find all the n^{th} roots of unity.*

$$z = cis\ 0 = cis\ 2k\pi.$$

SOLUTION. Here, $z^{1/n} = \text{cis}(2k\pi/n)$, $\quad k = 0, 1, 2, 3, \ldots n-1$.

They are $\qquad$ cis 0, $\text{cis}(2\pi/n)$, $\text{cis}(4\pi/n)$, ..., $\text{cis}(2(n-1)\pi/n)$.

Write ω for cis $(2\pi/n)$.

With this notation the n n^{th} roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

EXAMPLE 3. *Prove that the sum of the n n^{th} roots of unity is zero.*

SOLUTION. Use the following fact from Higher Algebra (See Note below.):

If the roots of

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$$

are $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = -(a_1/a_0)$$

and
$$\alpha_1\alpha_2\ldots\alpha_n = (-1)^n\,(a_n/a_0).$$

Thus since the n^{th} roots of unity are the roots of the equation

$$x^n - 1 = 0,$$

we have the sum of the roots $\quad = 1 + \omega + \omega^2 + \ldots + \omega^{n-1}$

$$= -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n} = 0$$

and the product of the roots

$$= 1\omega\omega^2 \ldots\ldots \omega^{n-1}$$
$$= \omega^{((n-1)/2)(1+n-1)} = \omega^{(n(n-1)/2)}$$
$$= (-1)^n\,\frac{\text{coefficient of } x^0}{\text{coefficient of } x^n} = \frac{(-1)^n(-1)}{1}$$
$$= \frac{(-1)^{n+1}}{1}.$$

In particular if ω is a cube root of unity then $\omega = \cos(2\pi/3)$ and

$$1 + \omega + \omega^2 = 0$$

and
$$1\omega\omega^2 = \omega^3 = (-1)^4 = 1.$$

Note. This is a generalisation of the familiar result for quadratic equations, namely : If α and β are the roots of $ax^2 + bx + c = 0$ then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

EXAMPLE 4. *On the unit circle in the Argand Diagram represent*

(i) *The three cube roots of unity.*

(ii) *The five fifth roots of unity.*

SOLUTION. Note that the n n^{th} roots of unity form the vertices of a regular polygon of n sides inscribed in the unit circle. See Figures 16.1 and 16.2.

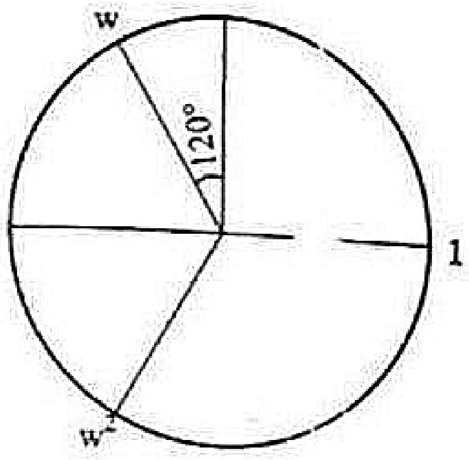

Fig. 16.1

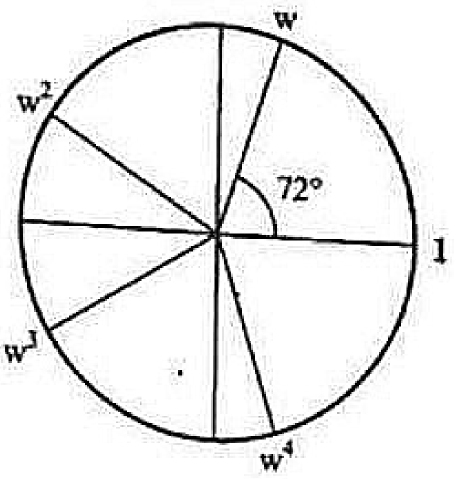

Fig. 16.2

EXAMPLE 5. *Expand cos 6θ in terms of powers of cos θ.*

$$\cos 6\theta = \text{Real part of } \cos 6\theta + i \sin 6\theta.$$
$$= \text{Real part of } (\cos\theta + i\sin\theta)^6.$$

SOLUTION. Expanding the expression $(\cos\theta + i\sin\theta)^6$ by Binomial theorem we get

$$\cos^6\theta + 6\cos^5\theta \,.\, i\sin\theta + \binom{6}{2}\cos^4\theta \,.\, i^2\sin^2\theta + \binom{6}{3}\cos^3\theta \,.\, i^3\sin^3\theta$$
$$+ \binom{6}{4}\cos^2\theta \,.\, i^4\sin^4\theta + \binom{6}{5}\cos\theta \,.\, i^5\sin^5\theta + i^6\sin^6\theta.$$

Since $i^2 = -1,\ i^3 = -i,\ i^4 = +1$, etc., the required Real part

$$= \cos^6\theta - \binom{6}{2}\cos^4\theta\sin^2\theta + \binom{6}{4}\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$= \cos^6\theta - 15\cos^4\theta\,(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$
$$= 32\cos^6\theta - 48\cos^4\theta + 16\cos^2\theta - 1.$$

EXERCISE 16.2

1. Find all the values of

(*a*) $\left(-1+\sqrt{3}i\right)^{1/6}$ (*b*) $(1+i)^{1/3}$

(*c*) $\left(-\sqrt{3}-i\right)^{3/2}$ (*d*) $64^{1/6}$

2. If $1, \omega, \omega^2$ are the cube roots of unity, prove that

$$\frac{3}{x^3-1} = \frac{1}{x-1} + \frac{1}{x\omega-1} + \frac{1}{x\omega^2-1}$$

3. If $\omega = \text{cis}\,(2\pi/n)$ prove that
$1 + \omega^p + \omega^{2p} + \ldots + \omega^{(n-1)p}$ is n or zero according as p is an integer which is or which is not a multiple of n.

4. If $\alpha = \text{cis}(2\pi/7)$ and $f(x) = A_0 + {}^{14}\Sigma_{n=1} A_n x^n$ then prove that

$$f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^6 x)$$
$$= 7(A_0 + A_7 x^7 + A_{14} x^{14}).$$

5. Expand $\sin 6\theta$ in terms of powers of $\sin \theta$.
6. Expand $\cos 5\theta$ in terms of powers of $\cos \theta$.
7. Expand $\sin 7\theta$ in terms of powers of $\sin \theta$.
8. Prove that

$$\cos(\alpha_1 + \alpha_2 + \ldots + \alpha_n)$$
$$= \cos\alpha_1 \cos\alpha_2 \ldots \cos\alpha_n\,[1 - S_2 - S_4 - S_6 + \ldots]$$

and $\sin(\alpha_1 + \alpha_2 + \ldots + \alpha_n)$

$$= \cos\alpha_1 \cos\alpha_2 \ldots \cos\alpha_n\,[S_1 - S_3 + S_5 - S_7 \ldots]$$

where $S_1 = \Sigma \tan\alpha_1$, $S_2 = \Sigma \tan\alpha_1 \tan\alpha_2$,

$S_3 = \Sigma \tan\alpha_1 \tan\alpha_2 \tan\alpha_3$ and so on.

Hence prove that

$$\tan n\theta = \frac{\binom{n}{1}t - \binom{n}{3}t^3 + \binom{n}{5}t^5 - \ldots}{1 - \binom{n}{2}t^2 + \binom{n}{4}t^4 - \ldots}.$$

where $t = \tan\theta$.

PROBLEMS

1. Prove that

(a) $\dfrac{\sin 9\theta}{\sin\theta} = 256 \sin^8\theta - 576 \sin^5\theta + 432 \sin^4\theta - 120 \sin^2\theta + 9$

(b) $128 \cos^3\theta \sin^5\theta = 6 \sin 2\theta - 2 \sin 4\theta - 2 \sin 6\theta + \sin 3\theta$

2. If $a = \cos\alpha - i\sin\alpha$ and $b = \cos\beta - i\sin\beta$

prove that

$$\frac{(a+b)(1-ab)}{(a-b)(1+ab)} = \frac{(\sin\alpha + \sin\beta)}{(\sin\alpha - \sin\beta)}$$

3. Sum of n terms of following:

(a) $\tan\theta \sec 2\theta + \tan 2\theta \sec^2\theta + \ldots + \tan 2^{n-1}\theta \sec 2^n\theta$

(b) $\cos\alpha + 2\cos 2\alpha + 2^2 \cos 3\alpha + \ldots$ to n terms.

4. Prove that

$$\frac{3\sin x - \sin 3x}{\cos 3x} + \frac{3\sin 3x - \sin 3^2 x}{3\cos 3^2 x} + \ldots$$
$$+ \frac{3\sin 3^{n-1}x - \sin 3^n x}{3^{-1}\cos 3^n x} = \frac{3}{2}\left(\frac{1}{3^n}\tan 3^n x - \tan x\right).$$

5. From the equation whose roots are

$$\cos\frac{1}{9}\pi, \cos\frac{3}{9}\pi, \cos\frac{5}{9}\pi, \cos\frac{7}{9}\pi$$

and hence prove

(*a*) $8\cos\frac{\pi}{9}.\cos\frac{5\pi}{9}.\cos\frac{7\pi}{9} = 1$

$$= 8\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}.$$

(*b*) $\sec^4\frac{\pi}{9} + \sec^4\frac{2\pi}{9} + \sec^4\frac{4\pi}{9} = 1104.$

Hint:

If $y = \text{cis}\,\frac{(2k+1)\pi}{9}, k = 0, 1, 2, ..., 8,$

then $\frac{y^9+1}{y+1} = 0$ has roots $\text{cis}\,\frac{(2k+1)\pi}{9}, k = 0, ... 3, 5, ... 8$

Put $y + \frac{1}{y} = 2x.$

6. Solve for x, y, z:

$$x + y + z = a$$
$$x + \omega y + \omega^2 z = b$$
$$x + \omega^2 y + \omega z = c$$

where ω is a cube root of unity.

MISCELLANEOUS PROBLEMS

1. Let α and β be the roots of

$$z + \frac{1}{z} = 2\cos(\phi + i\sin\phi) \qquad 0 < \phi < \pi$$

(*a*) Show that $\alpha + i$ and $\beta + i$ have the same argument and $|\alpha - i| = |\beta - i|$.

(*b*) Find the locus of the roots α and β in **C** as φ varies from 0 to 2π.

2. Suppose α, β, γ and δ are real numbers such that

$$\alpha + \beta + \gamma + \delta = \alpha^7 + \beta^7 + \gamma^7 + \delta^7 = 0$$

Prove that

$$\alpha(\alpha + \beta)(\alpha + \gamma)(\alpha + \delta) = 0.$$

3. Let α and β be the zeros of $x^2 - 6x + 1 = 0$. Show that $\alpha^n + \beta^n$ is an integer for any n and it is not divisible by 5.

4. A square matrix A is said to be *orthogonal* if $AA^T = I$. Show that the following matrices are orthogonal:

(*a*) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (*b*) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(*c*) The $n \times n$ matrix obtained by interchanging any two rows of the identity matrix I_n.

5. Show that, if

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

then

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2.$$

6. Find all polynomials $p(x)$ such that

$$p(x^2) = \{p(x)\}^2.$$

7. Let a_1, a_2, a_3, a_4 and b be real numbers such that

$$b + \sum_{k=1}^{4} a_k = 8; \; b^2 + \sum_{k=1}^{4} a_k^2 = 16$$

Find the maximum value of b.

8. For any natural number n prove that

$$2\sqrt{n} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{K}} < 2\sqrt{n} - 1.$$

9. Factorize:

(*a*) $\Sigma(bc - a^2)(ca - b^2)$ (*b*) $\Sigma(bc - a^2)^2(ca - b^2)^2$.

10. If $\tan A$ and $\tan B$ are the roots of $ax^2 + bx + c = 0$, evaluate

$$a \sin^2 (A + B) + b \sin (A + B) \cos (A + B) + c \cos^2 (A + B).$$

11. Show that there exists a convex hexagon in the plane such that

(*i*) all its interior angles are equal;

(*ii*) the lengths of its sides are 1, 2, 3, 4, 5, 6 in some order.

12. Find the remainder when 19^{92} is divided by 92.

13. Determine all pairs (m, n) of positive integers m, n for which $2^m + 3^n$ is a perfect square.

14. Find the number of positive integers $n \leq 1991$ such that 6 is a factor of $n^2 + 3n + 2$.

15. Let a, b, c be three real numbers with $0 < a < 1, 0 < b < 1, 0 < c < 1$ and $a + b + c = 2$. Prove that

$$\frac{a}{1-a} \frac{b}{1-b} \frac{c}{1-c} \geq 8.$$

16. Solve for real numbers x, y, z:

$$x + y - z = 4$$
$$x^2 - y^2 + z^2 = -4$$
$$xyz = 6.$$

17. Six generals propose locking a safe containing top secret with a number of different locks. Each general will be given keys to certain of these locks. How many locks are required and how many keys must each general have so that, unless at least four generals are present, the safe cannot be opened?

18. For any positive integer n, let $s(n)$ denote the number of ordered pairs (x, y) of positive integers for which $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ (For instance, $s(2) = 3$). Determine the set of positive integers n for which $s(n) = 5$.

19. If

$$D_n = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ -1 & a_2 & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & a_3 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & a_n \end{vmatrix}$$

Show that $D_n = a_n D_{n-1} + D_{n-2}$.

20. There are ten objects with total weight 20, each of the weights being a positive integer. Given that none of the weights exceeds 10, prove that the ten objects can be divided into two groups that balance each other in weight.

21. If $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ are two sets of real numbers such that either both are increasing or both are decreasing. Prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \le n\sum_{k=1}^{n} a_k b_k .$$

(This is known as Chebyshev's inequality).

22. (*a*) If $p = (x_1, y_1)$ and $Q = (x_2, y_2)$ with O as origin and $OPRQ$ is the □gm with OP, OQ as adjacent sides, then show that

Area of the □gm $OPRQ$ is $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

(*b*) If $A = (x_0, y_0)$, $P = (x_1, y_1)$ & $Q = (x_2, y_2)$ then show that the area of the parallelogram with adjacent sides AP, AQ is

$$\begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}$$

(*c*) Generalise this result to three dimensions.

23. Find all integers n such that

$$nx^2 + (n + 1)x + (n + 2) = 0$$

has only rational roots.

24. Find all integers n and a such that $t^2 - t + 1$ divides $t^n + t + a$.

25. Show that there is no polynomial $p(x)$ in $Z[x]$ for which $p(k)$ is a prime for all integers $k \ge 0$.

26. Let $A = \{1, 2, 3, ... 100\}$ and B be a subset of A having 48 elements. Show that B has two distinct elements x and y whose sum is divisible by 11.

27. If a, b, c, d are four nonnegative real numbers and $a + b + c + d = 1$. Show that $ab + bc + cd \le 1/4$.

28. Given a triangle ABC, define

$$x = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$$

$$y = \tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$$

$$z = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$$

Prove that $x + y + z + xyz = 0$.

29. Show that there do not exist four distinct real numbers a, b, c, d such that

$$a^3 + b^3 = c^3 + d^3$$

and $\quad a + b = c + d.$

30. Determine the largest number in the infinite sequence:

$$1^2, \sqrt[2]{2}, \sqrt[3]{3}, ... \sqrt[n]{n}$$

31. Prove that if a, b, c are odd integers then $ax^2 + bx + c = 0$ cannot have a rational number as a root.

32. Let P be any point inside a triangle ABC. If AP, BP and CP meet the sides BC, CA, AB at D, E, F respectively. Prove that

$$PD + PE + PF < \max(a, b, c).$$

33. Construct a quadrilateral which is not a ▭gm and in which a pair of opposite angles and a pair of opposite sides are equal.

34. Find all polynomials $p(x)$ such that

$$xp(x-1) = (x-15)p(x).$$

35. Find all natural numbers n for which the product of its digits is $n^2 - 13n - 25$.

36. Let $\Delta(x, y)$ be the numerical area of a triangle whose vertices are $(0, 0)$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Show that

$$\Delta(x, y) = \Delta(y, x);$$
$$\Delta(\alpha x, y) = |\alpha|\,\Delta(x, y);$$
$$\Delta(x + \alpha y, y) = \Delta(x, y).$$

37. With the notation of Qn. 36 above,

(*a*) If $x = (1, 2)$, $y = (-1, 4)$, $z = (1, -3)$ show that $\Delta(x + y, z) = \Delta(x, z) + \Delta(y, z)$

(*b*) If $x = (2, 4)$, $y = (2, 1)$, $z = (3, 5)$ show that $\Delta(x + y, z) = -\Delta(x, z) + \Delta(y, z)$;

(*c*) If $x = (2, 5)$, $y = (-1, 2)$, $z = (-1, 3)$ show that $\Delta(x + y, z) = \Delta(x, z) - \Delta(y, z)$.

38. If $\Delta'(x, y)$ be defined as the (signed) area of the triangle whose vertices are $(0, 0)$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ show that

$$\Delta'(x + y, z) = \Delta'(x, z) + \Delta'(y, z)$$
$$\Delta'(x, y) = -\Delta'(y, x)$$
$$\Delta'(\alpha x, y) = \alpha\Delta'(x, y)$$

and $$\Delta'(x + \alpha y, y) = \Delta'(x, y)$$

Calculate $\Delta'(x + y, z)$ in all three cases (*a*), (*b*) and (*c*) of Qn. 37 and contrast with the behaviour of $\Delta(x + y, z)$. Can you explain why $\Delta(x + y, z)$ has three different expressions in (*a*) (*b*) (*c*) of Qn. 37?

39. Show that

$$\Sigma \sin^3\alpha \sin(\beta - \gamma) = -\sin(\beta - \gamma)\sin(\gamma - \alpha)\sin(\alpha - \beta) \times \sin(\alpha + \beta + \gamma).$$

40. Find the value of the positive integer n for which the equation

$$\Sigma_{i=1}^{n}(x + i - 1)(x + i) = 10n.$$

41. Determine the set of all positive integers n for which 3^{n+1} divides $2^{2^n} + 1$.

42. Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

43. Show that if R and r are the circumradius and the in-radius of a nonobtuse-angled triangle, and h be the greatest altitude then

$$R + r \leq h.$$

44. A real number α is said to be algebratic if α is a zero of a polynomial in $\mathbb{Z}[x]$. *e.g.*, $\sqrt{2}$ is algebraic since it satisfies $x^2 - 2 = 0$. If α and β are algebraic show that $\alpha + \beta$ and $\alpha\beta$ are algebraic.

45. Eliminate x, y from the equations:

(*a*) $x^2 + xy = a^2$
$x^3 + xy = b^2$
$x^2 + y^2 = c^2$

(*b*) $(b - x)(c - y) = a^2$
$(c - x)(a - y) = b^2$
$(a - x)(b - y) = c^2$

(*c*) $\dfrac{1}{x-a} + \dfrac{1}{y-a} = \dfrac{1}{b}$
$\dfrac{1}{x-b} + \dfrac{1}{y-b} = \dfrac{1}{b}$
$x^2 + y^2 = 2(a^2 + b^2)$

(*d*) $x^2 - y^2 = px - qy$
$4xy = py + qx$
$x^2 + y^2 = 1$

(*e*) $4(x^2 + y^2) = ax + by$
$2(x^2 - y^2) = ax - by$
$xy = c^2$

(*f*) $ax^2 + by^2 = ax + by = \dfrac{xy}{x+y} = c$

(*g*) $\dfrac{x}{a^2 - y^2} + \dfrac{y}{a^2 - x^2} = \dfrac{1}{b}$
$xy = c^2 \ (a \neq 0)$

(*h*) $x + y = a$
$x^2 + y^2 = b^2$
$x^3 + y^3 = c^3$

46. Consider the collection of all three-element subsets drawn from the set {1, 2, 3 ..., 300}. Determine the number of those subsets for which the sum of the elements is a multiple of 3.

47. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incentre and the circumcentre of the triangle and the common point O are collinear.

48. Find all possible values of x, y, z such that $AA^T = I$ where

$$A = \begin{pmatrix} 1/\sqrt{2} & 2/3 & x \\ 1/\sqrt{2} & -2/3 & y \\ 0 & 1/3 & z \end{pmatrix}.$$

49. If
$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

Calculate $\lambda I - A$ and solve $|\lambda I - A| = 0$ for λ. Check that $A^3 - 4A + 3I$ is the zero matrix.

50. Solve for x:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1-x & 1 & \cdots & 1 \\ 1 & 1 & 2-x & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & 1 & \cdots & n-x \end{vmatrix} = 0$$

51. Without expanding, prove that

(a) $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

(b) $$\begin{vmatrix} 2yz-x^2 & z^2 & y^2 \\ z^2 & 2zx-y^2 & x^2 \\ y^2 & x^2 & 2xy-z^2 \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2$$

(c) $$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}^2 = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

where A_{ij} is the cofactor of α_{ij} in the matrix (α_{ij}).

52. Consider the system

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

where, the coefficients satisfy:

(i) a_1, b_2, c_3 are all > 0

(ii) The other coefficients are all < 0 and

(iii) in each equation the sum of the coefficients is positive.

Prove that the system has only one solution, *viz.* $x = 0 = y = z$.

53. On a given circle, six points A, B, C, D, E and F are chosen at random independently and uniformly with respect to arc length. Determine the probability that the two triangles ABC and DEF are disjoint, *i.e.*, have no common points.

54. A teacher distributes 7 books to 7 children (each child one book). On the next day she collects the books back & redistributes them in such a way that every child gets a new book. How many options are available for the teacher to do this?

55. If θ is expressed in radians, prove that

$$\cos(\sin\theta) > \sin(\cos\theta).$$

56. Find the number of permutations $(a_1, a_2, a_3, a_4, a_5, a_6)$ of $(1, 2, 3, 4, 5, 6)$ such that for any k, $1 \le k \le 5$, $(a_1, a_2, \ldots a_k)$ is not a permutation of $1, 2 \ldots, k$. (*e.g.*) $a_1 \ne 1$, (a_1, a_2) is not a permutation of $(1, 2)$ etc.

57. Let $A_1, A_2 \ldots A_n$ be a regular polygon of n sides. If

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$

determine n.

58. Find the number of ways in which one can place the numbers $1, 2, 3, \ldots, n^2$ on the n^2 squares of an $n \times n$ chessboard, one on each, such that the numbers in each row and column are in *A.P.*

59. Eliminate x, y, z from the equations:

(*a*) $\dfrac{x}{a}(y+z-x)=\dfrac{y}{b}(z+x-y)=\dfrac{z}{c}(x+y-z);$

$(ax+by+cz)=0.$

(*b*)
$$\begin{aligned} x+y+z&=a\\ x^2+y^2+z^2&=b\\ x^3+y^3+z^3&=c\\ xyz&=d \end{aligned}$$

(*c*)
$$\begin{aligned} x+y-z&=a\\ x^2+y^2-z^2&=b^2\\ x^3+y^3-z^3&=c^3\\ xyz&=d^3 \end{aligned}$$

(*d*) $x^2-ayz=y^2-bzx=z^2-cxy$

$$=\frac{1}{2}(x^2+y^2+z^2)\ (x, y, z)\neq(0, 0, 0)$$

(*e*)
$$\begin{aligned} y^2+z^2-x(y+z)&=a\\ z^2+x^2-y(z+x)&=b\\ x^2+y^2-z(x+y)&=c\\ x^3+y^3+z^3-3xyz&=d \end{aligned}$$

(*f*)
$$\begin{aligned} \frac{x}{y}+\frac{y}{z}+\frac{z}{x}&=a\\ \frac{x}{z}+\frac{y}{x}+\frac{z}{y}&=b\\ \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)&=c \end{aligned}$$

(*g*) x, y, z are the roots of $t^3-at^2+bt-c=0$ and are also the sides of a right-angled triangle.

(*h*)
$$\begin{aligned} y+z&=a(1-yz)\\ z+x&=b(1-zx)\\ x+y&=c(1-xy)\\ x+y+z-xyz&=d(1-xy-yz-zx) \end{aligned}$$

(*i*)
$$\begin{aligned} \frac{y}{z}+\frac{z}{y}&=a\\ \frac{x}{z}+\frac{z}{x}&=b\\ \frac{x}{y}+\frac{y}{x}&=c \end{aligned}$$

(*j*)
$$\begin{aligned} \frac{y}{z}-\frac{z}{y}&=a\\ \frac{z}{x}-\frac{x}{z}&=b\\ \frac{x}{y}-\frac{y}{x}&=c \end{aligned}$$

60. Let $p(x)=x^2+ax+b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer M such that

$$p(n)p(n+1)=p(M).$$

61. Let f be a bijective (*i.e.*, one-one, onto) function from $A=\{1, 2, 3, \ldots, n\}$ to itself. Show that there is a positive integer $M>1$ such that

$$(\underbrace{f\circ f\circ f\circ\cdots\circ f}_{n\text{ times}})(i)=f(i)$$

for each $i\in A$.

62. Show that there is a natural number n such that $n!$ when written in decimal notation (*i.e.*, in base 10) ends exactly in 1993 zeros.

63. Let $f(x)$ be a polynomial in x with integer coefficients and suppose that for five distinct integers a_1, a_2, a_3, a_4, a_5 we have

$$f(a_1) = f(a_2) = f(a_3) = f(a_4) = f(a_5) = 2.$$

Show that there does not exist an integer b such that $f(b) = 9$.

64. Determine all functions $f: R - \{0, 1\} \rightarrow R$ such that

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}.$$

65. Determine all pairs (x, y) of nonnegative integers x and y for which $(xy - 7)^2 = x^2 + y^2$.

66. Let a, b, c, d be any four real numbers not all equal to zero. Prove that the roots of the polynomial

$$f(x) = x^6 + ax^3 + bx^2 + cx + d$$

cannot all be real.

67. Let A denote a subset of

$$\{1, 11, 21, 31, \ldots, 541, 551\}$$

having the property that no two elements of A add up to 552. Prove that A cannot have more than 28 elements.

68. Let a, b, c denote the sides of a triangle. Show that the quantity

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

must be between the limits $\frac{3}{2}$ and 2. Can equality hold at either limit?

69. Let $m_1, m_2, m_3, \ldots, m_n$ be a rearrangement of the numbers $1, 2, \ldots, n$. Suppose that n is odd. Prove that the product

$$(m_1 - 1)(m_2 - 2) \ldots (m_n - n)$$

is an even integer.

70. Prove that the product of four consecutive natural numbers cannot be a perfect cube.

71. If a and b are positive real numbers and $a + b = 1$, prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq 12\frac{1}{2}.$$

72. Find the number of permutations of the n pairs of letters $a_1, a_1; a_2, a_2; a_3, a_3; \ldots a_n, a_n$ (each pair being identical), such that no two adjacent letters are identical.

73. Evaluate in 'closed' form.

(a) $1^3\binom{n}{1} + 2^3\binom{n}{2} + \ldots + n^3\binom{n}{n}$;

(b) $1^3 2\binom{n}{1} + 2^3 3\binom{n}{2} + \ldots + n^3(n+1)\binom{n}{n}$;

(c) If $S = \{1, 2, \ldots, n\}$ and $1 \leq r \leq n$

(i) $\displaystyle\sum_{\substack{A \subset S\\ |A| = r}} (\min A)$ (ii) $\displaystyle\sum_{\substack{A \subset S\\ |A| = r}} (\max A)$

(iii) $\displaystyle\sum_{\substack{A \subset S\\ |A| = r}} (\min A)^2$ (iv) $\displaystyle\sum_{\substack{A \subset S\\ |A| = r}} (\max A)^2$

74. If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

75. Let f_n denote the number of ways of tossing a coin n times such that successive heads never appear. Show that, taking $f_0 = 1$ and $f_1 = 2$,

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 2$$

Find the probability (in terms of f_n) that successive heads never appear when a coin is tossed n times, assuming that all possible outcomes of the n tosses are equally likely. What is the answer if $n = 8$?

76. A round-robin tournment is played amongst n-players. Each pair of players plays one game that culminates in a win for one player. A win earns 1 point; a loss 0 points. At the culmination of the tournament, player i has a_i points. $(0 \leq a_i \leq n-1)$. Suppose that $b = (b_1, b_2, \ldots, b_n)$ is a given n-tuple of nonnegative integers with $b_1 \leq b_2 \leq \ldots \leq b_n$. Prove that the n-tuple b could have arisen from a round-robin tournament as the set of final scores iff

(*i*) $\Sigma_{i=1}^{n} b_i = \binom{n}{2}$ (*ii*) $\Sigma_{i=1}^{k} b_i \geq \binom{k}{2}$ for each k, $1 \leq k \leq n$.

77. Let p be an interior point of a ΔABC. Show that at least one of the angles PAB, PBC, PCA is less than or equal to 30°.

78. The incircle of ΔABC touches BC at p. The line through p parallel to IA meets the incircle again at Q. The tangent to the incircle at Q meets AB, AC at C', B' respectively. Prove that $\Delta AB'C'$ is similar to ΔABC.

79. A permutation $a_1, a_2, \ldots a_n$ of $1, 2, \ldots, n$ is said to be *indecomposable* if n is the least positive integer j for which

$$\{a_1, a_2, \ldots, a_j\} = \{1, 2, \ldots, j\}$$

Let $f(n)$ be the number of indecomposable permutations of $1, 2, \ldots, n$. Find a recurrence relation for $f(n)$.

80. Show that the product of five consecutive positive integers is never a perfect square.

81. The incircle of ΔABC touches BC, CA at D, E respectively. Let BI meet DE at G. Show that AG is perpendicular to BG.

82. Let ABC be a triangle in plane Σ. Find the set of all points p (distinct from A, B, C) in the plane Σ such that the circumcircle of the triangles ABP, BCP, CAP have the same radius.

83. Prove that, if r be the inradius of ΔABC, the sum of the distances from a point p inside the triangle is at least $6r$.

84. Prove the following inequalities for a ΔABC:

(*a*) $3(bc + ca + ab) \leq (a + b + c)^2 < 4(bc + ca + ab)$

(*b*) $(a^2 + b^2 + c^2) \geq \frac{36}{35}\left(s^2 + \frac{abc}{s}\right)$

(*c*) $8(s - a)(s - b)(s - c) \leq abc$

(*d*) $abc < a^2(\Delta - a) + b^2(\Delta - b) + c^2(\Delta - c) \leq \frac{3}{2}abc$

(*e*) $\Sigma bc(b + c) \geq 48(s - b)(s - c)(s - a)$

(*f*) $\frac{2s}{abc} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

(*g*) $\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$

(*h*) $0 < \sin A + \sin B + \sin C \leq \frac{3}{2}\sqrt{3}$

(*i*) $\sin A + \sin B + \sin C \geq \sin 2A + \sin 2B + \sin 2C$

(*j*) $s \geq \frac{3}{2}\sqrt{6Rr}$.

85. Prove that a triangle is either acute, or right or obtuse-angled according as $a^2 + b^2 + c^2 - 8R^2$ is positive or zero or negative.

86. If $a^2 + b^2 > 5c^2$ in a ΔABC, show that c is the smallest side.

87. Let M be any point in the plane of a ΔABC. Find the minimum of $MA^2 + MB^2 + MC^2$.

88. Let S, I and O be the circumcentre, in-centre and the orthocentre of ΔABC. Prove that $SO \geq IO\sqrt{2}$.

89. If a polygon is inscribed in a circle and a second polygon is circumcircled by drawing tangents to the circle at the vertices of the first, prove that the product of the ⊥rs on the sides of the first polygon from a point of the circle, equals the product of the ⊥rs from the same point to the sides of the second polygon.

90. Prove that the sum of the ⊥rs from the vertices of a regular polygon of n sides to any line tangent to the circumscribed circle is equal to n times the radius.

91. For an acute angled triangle ABC prove that

$$s^2 \leq \frac{27R^2}{27R^2 - 8r^2}(2R + r)^2$$

with the usual notation.

92. In a ΔABC prove that

$$\Sigma \cos\left(\frac{B - C}{2}\right) \leq \frac{1}{\sqrt{3}}\left(\Sigma \sin A + \Sigma \cos\frac{A}{2}\right).$$

93. If $a, b > 0$ and $a + b = 1$, show that

$(11 + (1/a^4))^{1/3} + (11 + (1/b^4))^{1/3} \geq 6$.

94. In a triangle ABC, angle A is twice angle B. Show that $a^2 = b(b + c)$. Prove the converse.

95. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let T be the circumcentre of ΔAPB and H be the orthocentre of ΔCPD. Show that H, P, T are collinear.

96. Prove that if the Euler line of a triangle passes through a vertex then the triangle should either be a right angled triangle or on isosceles triangle or both.

97. If the Euler line of ΔABC is parallel to BC, Show that $\tan B \tan C = 3$.

98. Six different points are given on a circle. The orthocentre of the triangle formed by three of these points is joined to the centroid of the triangle formed by the other three points by a line segment. Prove that, when this is done for all possible triads of the six points chosen, the 20 line segments so formed are concurrent.

99. Prove that a straight line dividing the perimeter and the area, of the triangle in the same ratio, passes through the incentre.

100. Let P be the Fermat point of ΔABC. Prove that the Euler lines of Δs PAB, PBC, PCA are concurrent and the point of concurrence is G.

101. Prove that the three points on the circle whose pedal lines pass through the nine-point centre form an equilateral triangle.

102. Let A be one of the two points of two circles with centres X, Y respectively in the plane. The tangents at A to the two circles meet the circles again at B, C. Let point P be located so that $PXAY$ is a parallelogram. Show that P is also the circumcentre of ΔABC.

103. Triangle ABC is scalene with angle A having a measure greater than 90 degrees. Determine the set of points D that lie on the extended line BC, for which $|AD| = \sqrt{|BD|.|CD|}$ where $|XY|$ denotes the distance between X and Y.

104. Let ABC be an acute angled triangle. For any point p lying within this triangle, let D, E, F denote the feet of the ⊥rs from P onto the sides BC, CA, AB resply. Determine the set of all possible positions of the point P for which the ΔDEF is isosceles. For which position of P will ΔDEF be equilateral?

105. Given an angle QBP and a point L outside the angle, but in the same plane, show how to construct a straight line through L meeting BQ in A and BP in C such that ΔABC has a given perimeter.

106. Prove that the sum of the squares of the distances from any point on a circle to the vertices of a regular polygon of n sides incircled in the circle is constant and equals $2nR^2$.

107. Prove that the sum of the squares of all the connectors of the vertices of a regular polygon of n sides inscribed in a circle n^2R^2.

108. In a ΔABC the incircle Σ touches the sides BC, CA, AB at D, E, F resply. Let p be any point within Σ and let the segments AP, BP, CP meet Σ at X, Y, Z resply. Prove that DX, EY, FZ are concurrent.

109. Eliminate (x, y, z) from the equations:

(a) $ax^2 + by^2 + cz^2 = 0$

$ayz + bzx + cxy = 0$

$x^3 + y^3 + z^3 + \lambda xyz = 0$

(b) $(z + x - y)(x + y - z) = ayz$

$(x + y - z)(y + z - x) = bzx$

$(y + z - x)(z + x - y) = cxy$

(c) $y^2 + yz + z^2 = a;\ z^2 + zx + x^2 = b;$

$x^2 + xy + y^2 = c;\quad xy + yz + zx = 0.$

(d) $a^2 + x^2 = 2\lambda ax$

$x^2 + y^2 = 2\lambda xy$

$y^2 + z^2 = 2\lambda yz$

$z^2 + b^2 = 2\lambda zb$

(a, c, y, z, b are all unequal)

(e) $ax^2 + by^2 + cz^2$

$= ax + by + cz = xy + yz + zx = 0$

(f) $x^2 + y^2 + z^2 = x + y + z = 1$

$\frac{a}{x}(x - p) = \frac{b}{y}(y - q) = \frac{c}{z}(z - r)$

(g) $lx + my + nz = mx + ny + lz$

$= nx + ly + mz = h(x^2 + y^2 + z^2) = 1$

(h) $x(x - a) = yz;\ y(y - b) = zx;$

$z(z - c) = xy;\ x^2 + y^2 + z^2 = d^2.$

ANSWERS TO SELECTED QUESTIONS IN THE EXERCISES AND PROBLEMS

EXERCISE 2.2. (p. 26)

1. (*a*) 3. (*b*) 9. (*c*) 333. (*d*) 6.
2. (*a*) 270 (*b*) $n(n+1)$ (c) 3702.
6. (*i*) $x = 9 + 22k$, $y = -11 - 21k$.
   (*ii*) $x = 21 - 40k$, $y = 32 - 61k$.
   (*iii*) $x = 21 - 4k$, $y = 21 - 5k$.
   (*iv*) There exist solutions when $x = \ldots -6, -1, 4, 9 \ldots$. In particular, when $x = 4$, $y = 5 + 3k$, $z = -5 - 2k$.

10. 4 13. $a = b =$ any integer. 14. φ.

15. $a = \pm 10$, $b = \pm 100$; $a = \pm 20$, $b = \pm 50$;
$a = \pm 100$, $b = \pm 10$; $a = \pm 50$, $b = \pm 20$.

17. 1 if a is even and 2 if a is odd. 21. $(p+1)/2$, $(p-1)/2$.

EXERCISE 2.3. (p. 33)

2. $2^{15}3^{10}5^{6}$ 3. 12 4. 16
6. 249 7. {1} 8. 960
13. $\{252n \mid n$ is an odd integer$\}$.

PROBLEMS (Chapter 2) (p. 34)

16. (*i*) $(7125)\,10^n$ for $n \geq 0$
   (*ii*) There is no number which reduces by 58 when its leading digit is deleted.
18. 198 20. No integral solutions.
21. Solution set: $\{x.10^k \mid x = 10125, 2025, 30375, 405, 50625, 6075, 70875 : k \geq 0\}$.

EXERCISE 5.2. (p. 183)

1. 1, 5/3. 2. 6, – 3. 4. 3/2, – 4/3. 6. $-\sqrt{3}/2, -4/\sqrt{3}$
7. $\sqrt{5}/\sqrt{7}, \sqrt{7}/\sqrt{5}$ 8. $\sqrt{3}, \sqrt{3}/2$. 9. – 7, 1/6
10. 4, – 2 12. 1/3, – 4.

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$(y + z - x)(z + x - y) = cxy$

(c) $y^2 + yz + z^2 = a$; $z^2 + zx + x^2 = b$;
$x^2 + xy + y^2 = c$; $xy + yz + zx = 0$.

(d) $a^2 + x^2 = 2\lambda ax$
$x^2 + y^2 = 2\lambda xy$
$y^2 + z^2 = 2\lambda yz$
$z^2 + b^2 = 2\lambda zb$
(a, c, y, z, b are all unequal)

(e) $ax^2 + by^2 + cz^2$
$= ax + by + cz = xy + yz + zx = 0$

(f) $x^2 + y^2 + z^2 = x + y + z = 1$
$\frac{a}{x}(x - p) = \frac{b}{y}(y - q) = \frac{c}{z}(z - r)$

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$z(z - c) = xy$; $x^2 + y^2 + z^2 = d^2$.

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$a = \pm 100, b = \pm 10; a = \pm 50, b = \pm 20.$

17. 1 if a is even and 2 if a is odd.   21. $(p+1)/2, (p-1)/2$.

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21. Solution set: $\{x.10^k \mid x = 10125, 2025, 30375, 405, 50625, 6075, 70875; k \geq 0\}$.

EXERCISE 5.2. (p. 183)

1. 1, 5/3.   2. 6, – 3.   4. 3/2, – 4/3.   6. $-\sqrt{3}/2, -4/\sqrt{3}$

7. $\sqrt{5}/\sqrt{7}, \sqrt{7}/\sqrt{5}$ 8. $\sqrt{3}, \sqrt{3}/2$.   9. – 7, 1/6

10. 4, – 2   12. 1/3, – 4.

EXERCISE 5.3. (p. 187)

1. (*a*) $(1/4)(-1 \pm i\sqrt{7})$. (*c*) $-1 \pm i\sqrt{3}$.

(*d*) $(1/4)(-1 \pm \sqrt{5})$. (*e*) $-3 \pm \sqrt{3}$.

(*f*) $(1/4)(-5 \pm i\sqrt{7})$. (*g*) $-2, -1/4$.

(*h*) $(1/2\sqrt{3})(-2 \pm \sqrt{10})$. (*j*) $1, -6$.

2. (*a*) $a \geq -81/4$ (*b*) $|a| \geq 4$

(*c*) $a \leq 2$ (*d*) $a \leq -8$ or $a \geq 0$.

(*e*) For all real numbers a. (*f*) $|a| \leq 2$

3. ₹ 9/- **4.** $m, \dfrac{1-m^2}{2m}$

5. $1, -1/3$.

EXERCISE 5.4. (p. 191)

1. (*a*) $\alpha + \beta = -9, \alpha\beta = 8$ (*c*) $-3, 4/3$.

(*e*) $-7/6, -1/2$. (*g*) $6, -72$.

(*j*) $-3/2, 1/2$.

2. $\alpha^3 + \beta^3$	$\alpha/\beta + \beta/\alpha$	$\alpha^2\beta + \alpha\beta^2$
(*a*) $-7/2$	$-19/9$	$9/8$
(*b*) 464	30	16
(*c*) $-135\sqrt{3}/2$	16	$-9\sqrt{3}/2$
(*d*) $67\sqrt{5}/64$	$-43/24$	$-3\sqrt{5}/8$
(*e*) -3024	-6	432
(*g*) -43	$-29/14$	14

3. (*b*) $x^2 - 3\sqrt{2}x + 4$. (*e*) $x^2 - (1.8)x + 0.72$.

(*g*) $x^2 - 4x + 13$. (*j*) $x^2 + 2$.

4. $a = -2, c = -4$. **5.** The other root is 1, $b = -8$.

6. $x^2 - 2x - 2$. **7.** $b^3 + ac(a + c) - 3abc = 0$.

$x = 30$, the roots are $4 \pm i\sqrt{14}$. **10.** $x^2 + (31/36)x - (449/216)$.

11. $Q^2 + q^2 - p^P(Q + q) - 2qQ + qP^2 + Qp^2$.

EXERCISE 5.5. (p. 194)

1. (*b*) $\pm(\sqrt{2} + 1), \pm(\sqrt{2} - 1)$. (*d*) $1, \pm i$.

(*f*) $\sqrt{2}$ is a double root. (*h*) 0.

(*j*) $\pm\sqrt{5}, \pm\sqrt{2}$. (*l*) $(1/2)(5 \pm \sqrt{29}), (1/2)(3 \pm \sqrt{13})$.

2. 6, 3 **4.** 3 **5.** 6, 3

6. $4, 6; -6, -4$ **8.** $8, -1$

10. $k \geq 1$ or $k \leq -7/9$ **9.** Sides are 10 and 8 units.

11. $a = 9$ or $a = 15$.

EXERCISE 5.6.(p. 199)

1. All reals. **2.** $x > 3$ or $x < 1$. **4.** $6 - 4\sqrt{6} < x < 6 + 4\sqrt{6}$.

6. $x < (1/2)\,(3 - \sqrt{21})$ or $x > (1/2)\,(3 + \sqrt{21})$.

7. $-1 - \sqrt{13} < x < -1 + \sqrt{13}$. **8.** All reals. **9.** $-1 \leq x \leq 8/3$.

10. $x \leq 8 - \sqrt{73}$, $8 - \sqrt{57} \leq x \leq 8 + \sqrt{57}$ or $x \geq 8 + \sqrt{73}$.

11. $x = 2$ or $-2 \leq x \leq -1$. **12.** $-4 < x < 1$ or $x > 7$.

13. $-4 \leq x \leq 1/2$ or $3 \leq x \leq 5$. **15.** $-3/8 \leq x \leq -1/21$.

16. $|x| > 3$. **17.** $x < -1$ or $0 < x < 1/2$ or $x > 1$.

18. $x < -\sqrt{2}$ or $0 < x < 1$ or $\sqrt{2} < x < 2$.

19. $x \leq -2$ or $-1 \leq x \leq 1$. **20.** $x \leq -3/2$ or $x \geq 1/2$

21. $-3 - \sqrt{5} \leq x \leq -4$ or $-2 \leq x \leq 0$.

PROBLEMS (Chapter 5). (p. 199)

3. $n = 11$. **9.** $(w + a)\,\overline{u}$ is real where $u^2 = a^2 - b$.

10. u and v are real multiples of $a - c$ where $u = \sqrt{a^2 - 4b}$, $v = \sqrt{c^2 - 4d}$.

11. Any polynomial $ax^2 + bx + c$, where ib/a and c/a are real.

15. $x = 5, 10$.

EXERCISE 6.1. (p. 203)

2. $1° 43' 8''$ **3.** (*a*) $(\pi/9)^c$. (*b*) $(\pi/18)^c$.

4. 3,86,700 km. nearly.

EXERCISE 6.2. (p. 205)

2. (*a*) $\left\{(2n+1)\dfrac{\pi}{2} : n \in \mathbf{Z}\right\}$. (*b*) $\{n\pi : n \in \mathbf{Z}\}$.

(*c*) $\left\{(2n+1)\dfrac{\pi}{2} : n \in \mathbf{Z}\right\}$. (*d*) $\{n\pi : n \in \mathbf{Z}\}$.

17. $\sin\theta = \sqrt{1 - K^2}$, $\tan\theta = \left(\sqrt{1 - K^2}\right)\Big/K$,

$\sec\theta = 1/K$, $\operatorname{cosec}\theta = -1\Big/\left(\sqrt{1 - K^2}\right)$,

$\cot\theta = K\Big/\left(\sqrt{1 - K^2}\right)$.

18. $9/4$.

EXERCISE 6.3. (p. 208)

1. (*a*) 1. (*b*) 1. (*c*) $\cos^4 A$.

2. (*a*) $-1/2$. (*b*) $-1/2$. (*c*) $1/\sqrt{3}$. (*d*) -2. (*e*) $\sqrt{3}$.

(*f*) $-2\sqrt{3}$.

EXERCISE 6.4. (p. 214)

2. (*a*) $f(x) = \sqrt[3]{x}$. (*b*) $f(x) = \begin{cases} x/2, & x \geq 0 \\ x, & x < 0 \end{cases}$.

(*c*) Not invertible. (*d*) Not invertible.

(*e*) $f(x) = \tan^{-1} x$. (*f*) Not invertible.

(*g*) $f^{-1} = f$.

4. (*a*) $R - \{0\}$. (*b*) $R - \{-1, 0, 1\}$.

(*c*) $[-3, 3]$. (*d*) $(-\infty, -3) \cup (3, \infty)$.

(*e*) $\bigcup_{n \in Z} (2n\pi, (2n+1)\pi)$. (*f*) $R - \{n\pi / n \in Z\}$.

10. (*a*) n^m (*b*) $n!/(n-m)!$

(*c*) $2^m - 2$ for $n = 2$;

$3^m - 3 \cdot 2^m + 3$ for $n = 3$.

EXERCISE 6.5. (p. 223)

23. $\dfrac{4 - p^2 - q^2}{p^2 + q^2}, \dfrac{q^2}{p^2}$

28. $\cos 27° = \dfrac{\sqrt{5+\sqrt{5}} + \sqrt{3-\sqrt{5}}}{4}$

$\sin 27° = \dfrac{\sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}}{4}$

29.

	sine	*cosine*	*tangent*	*cotangent*	*secant*	*cosecant*
15°	$\dfrac{\sqrt{6}-\sqrt{2}}{4}$	$\dfrac{\sqrt{6}+\sqrt{2}}{4}$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{6}-\sqrt{2}$	$\sqrt{6}+\sqrt{2}$
75°	$\dfrac{\sqrt{6}+\sqrt{2}}{4}$	$\dfrac{\sqrt{6}-\sqrt{2}}{4}$	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{6}+\sqrt{2}$	$\sqrt{6}-\sqrt{2}$

EXERCISE 6.7. (p. 233)

22. $x = \pm 1/\sqrt{2}$.

23. $x = \sqrt{3}, -\sqrt{3} - 2$.

24. $x = \sqrt{3}/(2\sqrt{7})$.

25. $x = 0, 1/2$.

EXERCISE 6.8. (p. 239)

The values for θ, in Questions 1 – 12 are as follows.

1. $(2n)\,180° - 104°\,18'\,39''$, $\quad n \in Z$.
2. $n\pi$, $\quad n \in Z$.
3. $k\pi/n$; $\;2k\pi/n \pm \pi/3n$, $\quad k \in Z$.
4. $(2n+1)\pi/6$, $\quad n \in Z$.
5. $n\pi, n\pi \pm \pi/3$, $\quad n \times 180° \pm 35°15'$, $\quad n \in Z$.
6. $n\pi/2$, $\quad n \in Z$.
7. $n\pi + (-1)^n \dfrac{3\pi}{8} - \dfrac{\pi}{8}$, $\quad n \in Z$.
8. $(4n+1)\dfrac{\pi}{2}, (4n+1)\dfrac{\pi}{2} - \alpha, n \in Z$, where $\alpha = 33°\,41'\,24''$.
9. $n\pi + \dfrac{\pi}{6} \pm \dfrac{\pi}{5}, n \in Z$.
10. $\dfrac{n\pi}{3} + \dfrac{\pi}{12}, n \in Z$.
11. $n\pi \pm \dfrac{\pi}{4}, n\pi \pm \alpha, n \in Z$, where $\alpha = 73°\,13'\,17''$.
12. $\dfrac{n\pi}{2}, 2n\pi \pm \dfrac{2\pi}{3}, n \in Z$.
13. $(d-a)(d-c) \le b^2$.
14. $\theta = 2n\pi + \dfrac{\pi}{2}, 2n\pi + \dfrac{\pi}{4}, n \in Z$.
16. $\cos\theta = \cot A + \cot B + \cot C$.
17. Division by $\cos\theta$ is not allowed if $\cos\theta = 0$.
19. (*a*) $-\sqrt{2}, \sqrt{2}$ (*b*) $-2, 2$ (*c*) $-3, 7$.

    (*d*) $\dfrac{1}{2}\left[(a+c) - \sqrt{(a-c)^2 + b^2}\right], \dfrac{1}{2}\left[(a+c) + \sqrt{(a-c)^2 + b^2}\right]$.

    (*e*) $-\sqrt{1 + \sin^2\alpha}$; $\sqrt{1 + \sin^2\alpha}$.
20. $x = n\pi + \dfrac{\pi}{3}, y = n\pi + \dfrac{\pi}{6}, n \in Z$.

EXERCISE 6.9B. (p. 246)

3. $(m-n)\,AD^2 = m \cdot AC^2 - n \cdot AB^2 + \dfrac{mn}{(m-n)} \cdot BC^2$.

5. $h_a = 48/13$, $h_b = 12$, $h_c = 16/5$.

15. 50 m and 21 m

17. (*i*) $\dfrac{na^2}{4}\cot\dfrac{\pi}{n}$ (*ii*) $nr^2\tan\dfrac{\pi}{n}$ (*iii*) $\dfrac{nR^2}{2}\sin\dfrac{2\pi}{n}$

20. (*i*) $A = 60°\,42'$, $B = 42°\,39'$, $C = 76°\,39'$

(*ii*) $B = 64°\,6'$, $C = 39°\,54'$, $a = 116.5$

(*iii*) $C = 77°$, $b = 37.36$, $c = 59.13$

(*iv*) (*a*) $C = 63°\,27'$, $A = 64°\,33'$, $a = 42.40$;

(*b*) $C = 116°33'$, $A = 11°\,27'$, $a = 9.321$.

(*v*) $B = 35°\,10'$, $C = 104°\,50'$, $c = 168.4$.

(*vi*) No solution exists

(*vii*) $C = 90°$, $A = 15°$, $a = 2 - \sqrt{3}$

(*vii*) $B = 37°\,48'$, $C = 12°\,12'$, $c = 27.59$.

(*ix*) No solution exists

(*x*) $A = 21°\,47'$, $B = 38°\,13'$, $C = 120°$.

EXERCISE 6.10. (p. 261)

2. $\dfrac{4a(a+b)\sin^2\alpha + b^2}{2(a+b)\cos\alpha}$

3. If $\alpha = 1 = \beta$, height of tower $= (a\tan\alpha)/\sqrt{3}$. Otherwise it is equal to

$$a \times \sqrt{\frac{p^2+q^2+r^2-\sqrt{2p^2q^2+2p^2r^2+2q^2r^2-p^4-q^4-r^4}}{2(p^4+q^4+r^4-p^2q^2-p^2r^2-q^2r^2)}}\,,$$

where $p = \cot\alpha$, $q = \cot\beta$ and $r = \cot\gamma$.

EXERCISE 6.11. (P. 264)

1. $a^2 + b^2 = c^2 + d^2$.

3. $(x+y)^{2/3} + (x-y)^{2/3} = 2$.

5. $(\lambda^2 - 1)^2 = 27\,\lambda^2\sin^2\alpha\cdot\cos^2\alpha$.

7. $(a^2 + b^2 - 1)^2 = 4[(a-1)^2 + b^2]$.

9. $(b^2+1)^2 + 2a(b^2+1)(a+b) = 4(a+b)^2$.

11. $(x^2 + y^2 - 1)^2 = (y+1)^2 + x^2$.

13. $(x+y)^{2/5} + (x-y)^{2/5} = 2$.

15. $m^2 + m\cos\alpha = 2$.

17. $xy = (y - x)\tan\alpha$.

19. $\dfrac{x^2}{p} + \dfrac{y^2}{q} = \dfrac{1}{p} + \dfrac{1}{q}$.

20. $2x^3 + z = 3x(1+y)$.

PROBLEMS (Chapter 6) (p. 265)

12. $x = 2k\pi$ or $(2k - (1/2))\,\pi$, $k \in Z$, if n is odd; $x = k\pi$, $k \in Z$, if n is even
13. $x = (2k + 1)\pi/4$, $(2k + 1)\pi/6$, $k \in Z$
14. $[\pi/4, 7\pi/4]$
25. Angles : $\pi - 2A, \pi - 2B, \pi - 2C;$
    Sides : $a\,\lambda \cos A, b\,\lambda \cos B, c\,\lambda \cos C;$
    Area : $2\Delta\lambda^2 \cos A \cos B \cos C;$
    Circumradius : $\frac{1}{2}R\lambda;$
    Inradius : $R(1 + \cos A \cos B \cos C),$
    where $\lambda = (1 + \cos A + \cos B + \cos C)/2 \cos A \cos B \cos C$
27. $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = 0$

EXERCISE 7.2. (p. 301)

3. $y = (3/2)x$, $y = (1/6)x$
5. $y = 0$
9. $(c_2 - c_1)^2/(m_1 - m_2)$
11. $y + 2x = 7$
18. (*i*) $x - 3y - 8 = 0$; (*ii*) $3x - 16y - 30 = 0$
20. $\alpha = 15°$ or $75°$

EXERCISE 7.3. (p. 313)

1. $3x^2 + 3y^2 + 2x - 20y + 17 = 0$
2. $25x^2 + 25y^2 - 100x - 150y + 156 = 0$
9. $x^2 + y^2 = 0.$
17. $x^2 + y^2 + 3x + 3y = 0$
18. $4ax - y^2 = 3a^2$

PROBLEMS (Chapter 7). (p. 314)

14. $a(1 + \cos\alpha) + p = 0$. or $a(1 - \cos\alpha) + p = 0$
33. 32 square units.
43. $c^2x^2 - y^2(a^2 - b^2 + abc) - 2bcxy - ac^2x + a^2yc = 0,$
    where $(a, 0)$ is A and (b, c) is B

EXERCISE 8.1. (p. 327)

1. $x = 3, y = -2$
2. No solution.
3. $x = 5 + y$, y arbitrary
4. $x = 5, y = -1$
5. $x = 1 + y$, $z = 2 - 2y$, y arbitrary.
6. $x = -(11/3) + (13/3)w + (2/3)u$, $y = 5 - 2w - u$,
   $z = (7/3) - (8/3)w - (1/3)u$, w and u being arbitrary.

7. $x = 2, y = -1, z = -1$
8. $x = 13/9, y = 5/3, z = -5$
9. $x = 2 - y, z = -1$, y arbitrary
10. $x = 2, y = -1 - z$, z arbitrary
11. $x = 4, y = 3, z = 5$
12. No solution. 13. $x = -1, y = -1, z = -2$.
14. $x = -3z + 2w - 3, y = 5 - w$, w and z being arbitrary.
15. $x = -w + 14, y = z + 4$, w and z arbitrary.
16. $x = (-6/4)z + (14/4), y = (6/4)z - (10/4)$.
17. $x = -2, y = 3, z = 3/2, w = -1/2$.

EXERCISE 8.2. (p. 350)

1. (*a*) 17 (*b*) -45 (*c*) 25 (*d*) -4
2. $A_{11} = -5$, $A_{12} = 10$, $A_{13} = -5$,
$A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$,
$A_{31} = 4$, $A_{32} = -8$, $A_{33} = 4$,
$\det A = 0$
3. $A_{11} = 16$, $A_{12} = -64$, $A_{13} = -64$
$A_{14} = 48$, $\det A = -192$
4. $(a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$
5. $\lambda = 0, \dfrac{3 \pm \sqrt{33}}{2}$

PROBLEMS. (Chapter 8) (p. 351)

1. (*a*) $x = 0, y = 5/9, z = -5/9, w = -1$
(*b*) No solution.
2. If $k^2 \neq 1$, the system has one solution.
If $k = 1$, it has infinitely many solutions.
If $k = -1$, it has no solution.
7. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
9. $= 0$

EXERCISE 9.1. (p. 381)

1. 2160; 720; 144 3. 7^{10} ; $7! \times 7^3$ 5. 2^6
7. $\binom{20}{5} 2^5$ 9. 4536; 2296; 120
15. $\binom{n+k}{k}$ 17. $(n - 1)\,!$ 19. $\dfrac{8!}{(2!)^4}$

20. $\binom{n}{m}$ 22. 2^n ; $\binom{n}{m}$ 24. $6! \times \binom{7}{2}$; $5! \times \binom{5}{2}$

EXERCISE 9.3. (p. 387)

3. (*a*) Zero (*b*) No term independent of x.
(*c*) -672×3^6
5. 3

PROBLEMS (Chapter 9). (p. 387)

1. $\frac{n(n-1)}{4}$ 2. (*a*) 240 (*b*) 120
3. $2^{10} - 2^5$ 6. $\frac{n(n+1)\ldots(n+r-1)}{r!}$; $\binom{n}{r}$
13. 11!; 5! 6! 15. $9[1 + 9_1 + 9_2 + \ldots + 9_9]$
16. $\sum_{k=j}^{m} P(m, k)\binom{n}{k}$ where $P(m, k)$ is the number of ordered partitions of m into k parts.

EXERCISE 10.1. (p. 392)

1. $x^6 + 8x^4 + 9x^3 + 10x^2 + 12x + 13$;
$8x^{10} + 9x^9 + 12x^7 + 84x^6 + 90x^5 + 72x^4 + 201x^3 + 40x^2 + 108x + 36$
3. $x^9 + x^8 + x^7 + \ldots + x^2 + 2x$; $x^{10} - 1$
5. $\sqrt{7}x^4 + (8 + \sqrt{7})x^3 - 6x^2 + 12x + 4$;
$7x^7 + 2\sqrt{7}x^6 + (2\sqrt{7} - 48)x^5 + 16x^4 + 4\sqrt{7}x^3 - 24x^2 + 8x$
7. $7x^6 + (1/2)x^4 + 3x^3 + 3x^2/4 + 35/6$
$$\frac{7}{2}x^{10} + \frac{21}{4}x^8 + \frac{3}{2}x^7 + \frac{35}{6}x^6 + \frac{9}{4}x^5 + \frac{5}{2}x^4 + \frac{15}{6}x^3 + \frac{15}{4}x^2 + \frac{25}{6}.$$
9. $x^5 - x^4 + x^3 - x^2 + 2x$; $x^6 - 1$
11. $2\sqrt{2}x^2$; $-3x^6 + (2 - 2\sqrt{3})x^4 - x^2$
13. $10x^3 + 10x^2 + 3x + 4$;
$10x^6 + 19x^5 + 31x^4 + 41x^3 + 23x^2 + 6x + 2$

EXERCISE 10.2. (p. 397)

In the answers to Questions 1 and 2 below, the first mentioned expression is $q(x)$ and the second mentioned is $r(x)$.

1. (*b*) $x^5 + x - 1$; $x^3 + x^2 - x + 2$.
(*d*) $x^5 + 2^{1/6}x^4 + 2^{1/3}x^3 + 2^{1/2}x^2 + 4^{1/3}x + 2^{5/6}$; 0
(*f*) $4x^2 + 6x$; $11x + 9$

(*h*) $x - 7; 9x + 1$

(*j*) $-20x^2 + 7x + 13; -6x + 6$

(*r*) $x^3 + 2x^2 + 5x + 10; 21x + 21$

2. (*a*) $2x^2 - 11x + 15; 0$

(*c*) $5x^2 - 27x; 18$

(*e*) $x^9 + 2x^8 + 4x^7 - x^6 - 2x^5 - 4x^4 - 5x^3 - 10x^2 - 20x - 40; -76$

(*g*) $x^4 + 2x^2 + 3; 14$

(*i*) $x^2 - 3x + 18; 112$

(*k*) $4x^2 + 4x; 3x^2 + 4x + 2$

(*m*) $x^3 - 2x^2 - 6x + 12; -15$

(*o*) $x^3 + 4x^2 - 14x + 9; -15x - 2$

3. $(x - 3)^3$

6. $u^{10} + 10u^9 + 45u^8 + 120u^7 + 210u^6$
$+ 252u^5 + 210u^4 + 120u^3 + 45u^2 + 10u$, where $u = x - 1$

8. $u^4 + 3u^3 + 9u^2 + 27u + 17$ where $u = x - 3$

10. $(u^6 + 6u^5 + 15u^4 + 29u^3 + 42u^2 + 65u + 58)$
$\times (u^5 + 5u^4 + 10u^3 + 10u^2 - 2u)$ where $u = x - 1$

EXERCISE 10.3. (p. 404)

1. 'No' Answers: (*a*), (*b*), (*d*), (*l*), (*m*).
The remaining answers are 'yes'.

EXERCISE 10.4. (p. 406)

1. In each of the answers below, the first – mentioned is *the gcd* and the second is *the lcm*.

(*a*) $(x^2 + 2)(x - 1); (x^2 + 2)(x + 9)(x^5 - 1)(x^4 + 2)$

(*b*) $\left(x + \frac{1}{2}\right); \left(x^2 - \frac{16}{9}\right)\left(x + \frac{1}{2}\right)(x^2 - 3x + 2)(x + 4)$

(*c*) $(x - 3)(x - \sqrt{2}); \left(x - \frac{3}{\sqrt{2}}\right)(x^2 - 9)(x^2 - 2)$

(*d*) $(x + 1)\left(4x + \frac{1}{4}\right);$

$$(x^3 - 12x^2 + 47x - 60) \times \left(x^2 - \frac{5}{12}x - \frac{1}{6}\right) \times \left(x^2 - \frac{5}{9}x - \frac{4}{9}\right)$$

(*e*) $x^3 - x^2 + x - 1; x^4 - 1$

(*f*) $(x + \sqrt{2})(x^2 + 1)(x - 2); x(x + \sqrt{2})(x^2 + 1)(x - 2)^2$

(*g*) $(x + 3)\left(x + \frac{1}{\sqrt{3}}\right); (x + 1)(x + 3)\left(x + \frac{1}{\sqrt{3}}\right)^2$

(*h*) $x(x + 4)\left(x + \frac{1}{\sqrt{3}}\right); x^3\left(x^2 + \frac{4}{\sqrt{3}}x + 1\right) \times (x^2 + 4x + 3)(x + \sqrt{5})$

EXERCISE 10.5. (p. 413)

2. (*a*) $(x+1)(x+3)$; $(x+1)(x^2+2)$
   (*b*) $(x^2+1)(x^4+1)$; (x^4-1)
   (*c*) $(x+1)(x^3+2)$; $(x+1)(x^2-x+1)^2$
   (*d*) $(x+1)^2(x-1)$; $(x+1)^2(x+2)$

PROBLEMS (Chapter 10). (p. 414)

4. $l(x) = \left(x^2 + \frac{9}{4}x + \frac{3}{2}\right)$, $m(x) = \left(x^2 - \frac{7}{4}x + \frac{1}{2}\right)$.

11. $6x^5 - 15x^4 + 10x^3$

PROBLEMS (Chapter 11). (p. 433)

15. $x = 3/4$, $y = 1/2$, $z = 1/4$ 17. $2\sqrt{2}$

18. The expression has constant value equal to 2.

19. No 20. $(a+k)^n$

EXERCISE 12.1 (p. 444)

1. 582 2. 1806

3. 312 7. 66, 40, 02,144.

PROBLEMS (Chapter 12). (p. 454)

1. $10^n - 4.9^n + 6.8^n - 4.7^n + 6^n$

2. 120.

5. $D_{n/2}$ or $D_{(n-1)/2}$ according as n is even or odd.

PROBLEMS (Chapter 13). (p. 470)

4. (*a*) $\left[\binom{10}{3}\binom{8}{4} + \binom{10}{2}\binom{8}{5} + \binom{10}{1}\binom{8}{6} + \binom{8}{7}\right] + \binom{8}{7}$

   (*b*) $1 + \dfrac{\binom{8}{7} + \binom{10}{7}}{\binom{18}{7}}$

5. 0.665 6. 1/4 7. 0 8. No 9. 21/100

10. $1 - \dfrac{2^n}{3^n}$ 11. $\dfrac{4}{\binom{52}{13}}$

EXERCISE 15.1. (p. 487)

1. 1/6, 3/6, 9/6, 27/6, 81/6
3. All constant sequences.
4. 1, 2, 4, 8, 16, 32, 64, 128, 256
5. 4, 12, 36
6. 3, -6, 12, -24
7. -5, -13, -21, -29
8. -6, -3, 0, 3, ..., 42
11. 32

EXERCISE 15.2. (p. 490)

2. 275
3. 0
4. 20, 18, 16, 14, ..., -18
6. $n = 17$, $u_n = -24$
7. 9119700
8. 38
10. 4905
12. 55
13. 2, 4, 6, 8, ..., 20
14. 8825

EXERCISE 15.3. (p. 492)

1. 5, 15, 45, ..., 3645
5. 3, 7, 11 or 12, 7, 2
6. 3 or -3

EXERCISE 15.4. (p. 496)

1. $\dfrac{n(n+1)(2n+13)}{6}$
3. $\dfrac{n+1}{n+2}$
4. $\dfrac{n}{2n+1}$
5. $\dfrac{n(n+3)}{4(n+1)(n+2)}$
6. $\dfrac{8n+9}{12(2n+3)}$
15. $\dfrac{2n^2+2n+1}{4(2n+1)(2n+3)}$
18. $\dfrac{5(n-4)}{24(n+2)}$
23. 2926
27. $\sqrt{n+1}-1$

EXERCISE 15.6. (p. 504)

1. $n(n-1)\,2^{n-2}$
3. $\dfrac{2^{n+1}-1}{n+1}+2^n$
8. $n \cdot 2^{n-1}$
12. $\dfrac{n(n+1)}{2}$
16. $\dfrac{2^n-1}{n!}$
17. $\dfrac{n(n+1)}{30}\,(6n^3+9n^2+n-1)$

PROBLEMS (Chapter 15) (p. 506)

5. $\dfrac{(n-1)n\,(n+1)\,(3n+2)}{24}$

7. $\dfrac{n(n+1)}{4(n+2)}$

EXERCISE 16.1. (p. 512)

1. $\cos 14\alpha + i \sin 14\alpha$

3. modulus $= \left(\dfrac{1+\sin\theta+\cos\theta}{1+\cos\theta}\right)^3 . 2^{-(3/2)}$;

amplitude $= \pi/4$

7. (a) $\dfrac{\sin(n\,\theta/2)}{\sin(\theta/2)}\left(\sin\left(\dfrac{n+1}{2}\right)\theta\right)$

(b) $\dfrac{n}{2}+\dfrac{1}{2}\,\dfrac{\sin n\,\theta\cos(n+1)\,\theta}{\sin\theta}$

EXERCISE 16.2. (p. 515)

1. (a) $2^{1/6}\operatorname{cis}\left(\dfrac{\pi}{9}+\dfrac{2k\pi}{6}\right)$, $k = 0, 1, \ldots, 5$

(b) $2^{1/6}\operatorname{cis}\left(\dfrac{\pi}{12}+\dfrac{2k\pi}{3}\right)$, $k = 0, 1, 2$

5. $6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

6. $\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

PROBLEMS (Chapter 16). (p. 516)

3. (a) $\tan(2^n\theta) - \tan\theta$

5. $16x^4 - 8x^3 - 12x^2 + 4x + 1 = 0$

6. $x = y = z = \dfrac{a+b+c}{3}$

MISCELLANEOUS PROBLEMS (P. 518)

9. (a) $(bc + ca + ab)\,(bc + ca + ab - a^2 - b^2 - c^2)$

(b) $(a^2 + b^2 + c^2 - bc - ca - ab)$

$\times\,[(a^2 + b^2 + c^2 - ab - bc - ca)\,(ab + bc + ca)^2$

$-\,2(a^2 - bc)\,(b^2 - ca)\,(c^2 - ab)]$

45. (a) $a^4 + b^4 = c^2\,(a^2 + b^2)$

(b) $(bc + cd + ab)\,(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

(c) $a^2 + b^2 = 6ab$

(d) $(p + q)^{2/3} + (p - q)^{2/3} = 2$

(e) $(a + b)^{2/3} + (a - b)^{2/3} = 4c^{2/3}$

(f) $c^2(a+b-1)^2-c(a+b-1)(a^2-3ab+b^2-a-b)+ab=0$

(g) $a^2=c^2\pm bc$ or $a^2=b^2-c^2$

(h) $a^3+2c^3=3ab^2$

59. (a) $a^3+b^3+c^3=a^2(b+c)+b^2(c+a)+c^2(a+b)$

(b) $a^3-3ab+2c-bd=0$

(c) $(a^3-3ab^2+2c^3)(a^4-3a^2b^2+3b^4-4ac^3)+16d^2(a^2-b^2)^2=0$

(d) $a^2+b^2+c^2-2abc-1=0$

(e) $a^3+b^3+c^3-3abc=2a^2$

(f) $ab=c+1$

(g) $(a^2-2b)(a^2+b)^2=2(a^3-2ab+2c)^2$

(h) $2d(1-bc-ca-ab)=(1-d^2)(a+b+c-abc)$

(i) $a^2+b^2+c^2-abc-4=0$

(j) $2(b^2c^2+c^2a^2+a^2b^2-a^4-b^4-c^4)+a^2b^2c^2=0$

109. (a) $a^3+b^3+c^3=a^2(b+c)+b^2(c+a)+c^2(a+b)$

(b) $abc=(a+b+c-4)^2$

(c) $A^2+B^2+C^2+3(BC+CA+AB)=0$, where

$A=a^2-bc,\ B=b^2-ac,\ C=c^2-ab$

(d) $(a^2+b^2)=(\lambda^4-4\lambda^2+2)ab$

(e) $(a+b+c)^3-4(b+c)(c+a)(a+b)+5abc=0$

(f) $$\frac{1}{(a-b)cr+(a-c)bq}+\frac{1}{(b-c)ap+(b-a)cr}$$

$$+\frac{1}{(c-a)bq+(c-b)ap}+\frac{1}{bcqr+carp+abpq}$$

(g) $(l+m+n)^2=3h^2$

(h) $$(b^2-ca)^2(c^2-ab)^2+(c^2-ab)^2(a^2-bc)^2+(a^2-bc)^2(b^2-ca)^2$$
$$=d^2(a^3+b^3+c^3-3abc)^2.$$

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